AN IDENTITY INVOLVING STIRLING NUMBERS OF BOTH KINDS AND ITS CONNECTION TO RIGHT-TO-LEFT MINIMA OF CERTAIN SET PARTITIONS

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Abstract. For $1 \leq k \leq n$, $n \in \mathbb{N}$, let $S(n,k)$ denote the Stirling numbers of the second kind and let $s(n,k) = (-1)^{n-k}|s(n,k)|$ denote the Stirling numbers of the first kind, where $|s(n,k)|$ denote the unsigned Stirling numbers of the first kind. Extend the definitions to all $n,k \in \mathbb{N}$ by defining all of these Stirling numbers to be equal to zero if $k > n$. Let $S = \{S(n,k)\}$ (respectively, $s = \{s(n,k)\}$, $|s| = \{|s(n,k)|\}$) denote the infinite matrix whose nk-th entry is $S(n,k)$ (respectively $s(n,k), |s(n,k)|$). A classical result states that S and s are inverses of one another: $Ss = sS = Id$. In this note, as a byproduct of a result concerning the statistics of right-to-left minima in certain set partitions, we calculate explicitly $|s|S$, via a completely combinatorial method.

1. Introduction and Statement of Results

Let S_n denote the set of permutations of $[n]$. For a permutation $\sigma = (\sigma_1, \cdots, \sigma_n) \in S_n$, the entry σ_i is called a *right-to-left minimum* if $\sigma_i < \sigma_j$, for all $j = i+1, \cdots n$. Similarly, one can define a right-to-left maximum and a left-to-right minimum or maximum. These concepts play an important role in the study of pattern-avoiding permutations; see for example [2], [3]. For $j \in [n]$, the number of permutations in S_n with exactly j right-to-left minima is equal to the unsigned Stirling number of the first kind $|s(n,j)|$ [1]. (Recall that $|s(n,j)|$ is also equal to the number of permutations in S_n containing exactly j cycles.)

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One may think of a permutation $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n)$ as a list of the numbers in [n]. Let $k \in [n]$. Consider now placing the elements of $[n]$ into k nonempty lists. (Of course, for $k = 1$, what we obtain is a permutation of S_n.) We will refer to such an object as a *set of lists*, and we denote the collection of sets of lists for a particular n and k by $\mathcal{SL}(n, k)$.

It is easy to calculate the number of elements in $\mathcal{SL}(n, k)$. To do so, we first define the collection of *lists of lists* $\mathcal{LL}(n, k)$. An element of $\mathcal{LL}(n, k)$ is simply an element of $\mathcal{SL}(n, k)$ with the order of the k sets taken into account. Thus, for example, there are 12 elements in $\mathcal{LL}(3, 2)$: 1,2/3; 1,3/2; 1/2,3; 2,1/3; 3,1/2; 1/3,2; 3/1,2; 2/1,3; 2,3/1; 3/2,1; 2/3,1; 3,2/1. while there are 6 elements in $\mathcal{SLP}(3, 2)$: $1, 2/3$; $1, 3/2$; $1/2, 3$; $2, 1/3$; $3, 1/2$; $1/3, 2$. See [6] for more on sets of lists and lists of lists.

Proposition 1. $|\mathcal{SL}(n, k)| = \frac{n!}{k!}\binom{n-1}{k-1}$.

Proof. To prove the proposition, it suffices to prove that $|\mathcal{LL}(n, k)| = n!\binom{n-1}{k-1}$. Choose a permutation $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n)$ of $[n]$. Choose $k - 1$ numbers $\{j_i\}_{i=1}^{k-1}$ from $[n - 1]$. This generates an element of $\mathcal{LL}(n, k)$; namely $\sigma_1, \cdots, \sigma_{j_1}/\sigma_{j_1+1}, \cdots, \sigma_{j_2}/\cdots/\sigma_{j_{k-1}+1}, \cdots, \sigma_n$. Clearly, as one runs over all permutations σ of $[n]$ and all subsets $\{j_i\}_{i=1}^{k-1}$ of length $k - 1$ from $[n - 1]$, one generates every element of $\mathcal{LL}(n, k)$ exactly once. □

Remark. The numbers $\frac{n!}{k!}\binom{n-1}{k-1}$ are called (unsigned) Lah numbers. See entry A048854 in the OEIS (On-line Encyclopedia of Integers sequences).

Consider an element $\nu \in \mathcal{SL}(n, k)$. We say that the number $j \in [n]$ is a right-to-left minimum for ν if j is a right-to-left minimum with respect to the numbers in the list to which it belongs. For example, consider the element $\nu_0 \in \mathcal{SL}(9, 4)$ given by $4, 1/2/7, 3, 6, 5/8, 9$. Then the numbers 1,2,3,5,8,9 are right-to-left minima for ν. For $\nu \in \mathcal{SL}(n, k)$, define $M_{R/L}(\nu)$ to be the number of right-to-left minima it possesses; thus, for example, $M_{R/L}(\nu_0) = 6$, where ν_0 is as above. Clearly, $M_{R/L}(\nu) \in \{k, k + 1, \cdots, n\}$, for $\nu \in \mathcal{SL}(n, k)$. Let

$$\mathcal{SL}_j(n, k) = \{\nu \in \mathcal{SL}(n, k) : M_{R/L}(\nu) = j\}, \text{ for } j = k, \cdots n.$$

In this note we calculate $|\mathcal{SL}_j(n,k)|$ in a completely combinatorially fashion. Let $S(n,k)$ denote the Stirling number of the second kind, which counts the number of ways to partition the set $[n]$ into k non-empty, disjoint subsets.

Theorem 1.

$$|\mathcal{SL}_j(n,k)| = |s(n,j|)S(j,k), \; j = k, \cdots n. \tag{1.1}$$

From the theorem and Proposition 1 we obtain the following corollary. Define $|s(n,k)| = S(n,k) = 0$, for $k > n$. Let $S = \{S(n,k)\}$ $(|s| = \{|s(n,k)|\})$ denote the infinite matrix whose nk-th entry is $S(n,k)$ $(|s(n,k)|)$.

Corollary 1.

$$|s|S = \left\{ \frac{n!}{k!}\binom{n-1}{k-1} 1_{n\geq k} \right\};$$

that is,

$$\sum_{j=1}^{\infty} |s(n,j)|S(j,k) = \frac{n!}{k!}\binom{n-1}{k-1}, \; 1 \leq k \leq n. \tag{1.2}$$

Remark 1. Let $s(n,k) = (-1)^{n-k}|s(n,k)|$ denote the corresponding signed Stirling number of the first kind, and let $s = \{s(n,k)\}$. Corollary 1 complements the following well-known result [8].

$$sS = Ss - Id;$$

that is

$$\sum_{j=1}^{\infty} s(n,j)S(j,k) = \sum_{j=1}^{\infty} S(n,j)s(j,k) = \begin{cases} 1, & \text{if } n = k; \\ 0, & \text{if } n \neq k. \end{cases}$$

Remark 2. We were unable to find the identity (1.2) in the published journal literature, however we did find it implicitly in an unpublished arXiv article by Donald Knuth [5] and in the Mathematics Stack Exchange online [7]. Knuth's method is via so-called convolution matrices. The proof in [7] is via repeated use of generating functions. In contrast, our proof of the identity is completely combinatorial, and presents a natural combinatorial structure that the formula is enumerating.

Remark 3. A complicated proof of Theorem 1 using certain two-parameter generating functions appears in [4].

Remark 4. In light of Corollary 1, it is natural to consider the quantity $\sum_{j=1}^{\infty} S(n,j)|s(j,k)|$. We can give a combinatorial description of this quantity, however we don't know how to evaluate it. Let $\mathcal{LL}(n) = \cup_{k=1}^{n} \mathcal{LL}(n,k)$ denote the total collection of lists of lists of the numbers in $[n]$. Let $\nu \in \mathcal{LL}(n)$; so $\nu \in \mathcal{LL}(n,k)$, for some k. Label each of the k lists by the smallest element in the list. Call any one of the k lists a right-to-left minimum if its label is smaller than the labels on all of the lists that appear to its right. Denote the total number of right-to-left minima by $\hat{M}_{R/L}(\nu)$. For example, consider $\nu \in \mathcal{LL}(n,k)$ given by 3,2,7/6,4/9,1/5,8. Then the list 3,2,7 gets the label 2, the list 6,4 gets the label 4, the list 9,1 gets the label 1, and the list 5,8 gets the label 5. Thus, the lists 9,1 and 5,8 are right-to-left minima and $\hat{M}_{R/L}(\nu) = 2$. We have

$$\sum_{j=1}^{\infty} S(n,j)|s(j,k)| = |\{\nu \in \mathcal{LL}(n) : \hat{M}_{R/L}(\nu) = k\}|.$$

The explanation for this is as follows. For each $j \in \{k, \cdots, n\}$, there are $S(n,j)$ different ways to arrange the numbers in $[n]$ into j nonempty subsets. Given such a partition, we label the j subsets as above, according to their smallest element. These j labels can be arranged into $j!$ different permutations, of which $|s(j,k)|$ of them will have k right-to-left minima.

2. Proof of Theorem 1

Note that from the definitions, it follows that

$$|\mathcal{SL}_n(n,k)| = S(n,k). \tag{2.1}$$

Using this along with the fact that $|s(n,n)| = 1$, the case $j = n$ in (1.1) follows immediately. Assume now that $k \le j \le n-1$. An arbitrary element ν of $\mathcal{SL}_j(n,k)$ can be built as follows. Choose $n-j$ integers $2 \le a_1 < \cdots < a_{n-j} \le n$ to be the integers that are not right-to-left minima. Now choose an element of $\mathcal{SL}_j(j,k)$. By (2.1), there are $S(j,k)$ possible elements to choose from. Relabel the integers $1, 2, \cdots, j$ appearing in this element of $\mathcal{SL}_j(j,k)$ by the integers in $[n] - \{a_i\}_{i=1}^{n-j}$, where 1 is replaced by the smallest integer in $[n] - \{a_i\}_{i=1}^{n-j}$, 2 is replaced by the second smallest integer in $[n] - \{a_i\}_{i=1}^{n-j}$, etc. Call this object $\hat{\nu}$. Finally, enter the $n-j$ integers $\{a_i\}_{i=1}^{n-j}$ into $\hat{\nu}$, one at a time, according to the order $a_1, a_2, \cdots, a_{n-j}$. The resulting element ν belongs to $\mathcal{SL}(n,k)$. In order that ν indeed belong to $\mathcal{SL}_j(n,k)$, we must enter these $n-j$ integers in such a way that none of them will be right-to-left minima. Therefore, a_1 needs to be entered immediately to the left of one of the integers $1, \cdots, a_1 - 1$ (and in the same subset as that integer). Once a_1 has been entered, then a_2 must be entered immediately to the left of one of the integers $1, \cdots, a_2 - 1$ (and in the same subset as that integer), etc. Thus, there are $\prod_{i=1}^{n-j}(a_i - 1)$ different ways to enter the integers $\{a_i\}_{i=1}^{n-j}$. As an example, let $n = 9, k = 3, j = 6$, let $a_1 = 3, a_2 = 6, a_3 = 7$, and let the other integers be arranged as 149/28/5. Then $a_1 = 3$ can be entered in $a_1 - 1 = 2$ possible positions: to the left of 1 or to the left of 2. Let's say we choose the latter possibility; then we have 149/328/5. Now $a_2 = 6$ can be entered in $a_2 - 1 = 5$ possible positions: to the left of 1,2,3,4 or 5. Let's say we choose the first of these possibilities; then we have 6149/328/5. Finally, $a_3 = 7$ can be entered in $a_3 - 1 = 6$ possible positions: to the left of 1,2,3,4,5 or 6.

From the above argument, we conclude that

$$|\mathcal{SL}_j(n,k)| = S(j,k) \sum_{\{a_i\}_{i=1}^{n-j} \subset [n]} \prod_{i=1}^{n-j}(a_i - 1). \tag{2.2}$$

Define

$$A_{n,j} = \sum_{\{a_i\}_{i=1}^{n-j} \subset [n]} \prod_{i=1}^{n-j}(a_i - 1), \ j = 1, \cdots, n$$

(by convention, $A_{n,n} = 1$). In light of (2.2), to complete the proof, we need to show that

$$A_{n,j} = |s(n,j)|. \tag{2.3}$$

Using the convention that $|s(n,0)| = 0$, recall [8] that the unsigned Stirling numbers of the first kind are uniquely determined by the conditions

$$\begin{aligned} &|s(n,n)| = 1, \ n \geq 1; \\ &|s(n+1,j)| = n|s(n,j)| + |s(n,j-1)|, \ n \geq j \geq 1. \end{aligned} \tag{2.4}$$

We now show that the $\{A_{n,j}\}$ also satisfy the recursive equation in (2.4) and the boundary condition there, thereby proving (2.3).

Since $A_{n,n} = 1$, the boundary condition in (2.4) is satisfied. For the recursive equation in (2.4), we break up the sum defining $A_{n+1,j}$ into two parts, one involving those $\{a_i\}_{i=1}^{n+1-j}$ for which $a_{n+1-j} \neq n+1$, and one involving those $\{a_i\}_{i=1}^{n+1-j}$ for which $a_{n+1-j} = n+1$:

$$A_{n+1,j} = \sum_{\{a_i\}_{i=1}^{n+1-j} \subset [n+1]} \prod_{i=1}^{n+1-j} (a_i - 1) = \sum_{\{a_i\}_{i=1}^{n+1-j} \subset [n]} \prod_{i=1}^{n+1-j} (a_i - 1) +$$

$$\sum_{\{a_i\}_{i=1}^{n-j} \subset [n]} \big(\prod_{i=1}^{n-j} (a_i - 1) \big) \cdot n = A_{n,j-1} + nA_{n,j}.$$

Thus $\{A_{n,j}\}$ satisfies the recursive equation in (2.4). □

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