ASYMPTOTICS FOR EXIT PROBLEM AND PRINCIPAL EIGENVALUE FOR A CLASS OF NON-LOCAL ELLIPTIC OPERATORS RELATED TO DIFFUSION PROCESSES WITH RANDOM JUMPS AND VANISHING DIFFUSION

ROSS G. PINSKY

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ABSTRACT. Let $D \subset R^d$ be a bounded domain and denote by $\mathcal{P}(D)$ the space of probability measures on D. Let

$$L = \frac{1}{2}\nabla \cdot a\nabla + b\nabla$$

be a second order elliptic operator. Let $\mu \in \mathcal{P}(D)$ and $\delta > 0$. Consider a Markov process $X(t)$ in D which performs diffusion in D generated by the operator δL and is stopped at the boundary, and which while running, jumps instantaneously, according to an exponential clock with spatially dependent intensity $V > 0$, to a new point, according to the distribution μ. The Markov process is generated by the operator $L_{\delta,\mu,V}$ defined by

$$L_{\delta,\mu,V}\phi \equiv \delta L\phi + V(\int_D \phi \, d\mu - \phi).$$

Let $\phi_{\delta,\mu,V}$ denote the solution to the Dirichlet problem

$$L_{\delta,\mu,V}\phi = 0 \text{ in } D;$$

$$\phi = f \text{ on } \partial D,$$

where f is continuous. The solution has the stochastic representation

$$\phi_{\delta,\mu,V}(x) = E_x f(X(\tau_D)).$$

One has that $\phi_{0,\mu,V}(f) \equiv \lim_{\delta\to 0} \phi_{\delta,\mu,V}(x)$ is independent of $x \in D$. We evaluate this constant in the case that μ has a density in a neighborhood of ∂D. We also study the asymptotic behavior as $\delta \to 0$ of the principal eigenvalue $\lambda_0(\delta,\mu,V)$ for the operator $L_{\delta,\mu,V}$, which generalizes previously obtained results for the case $L = \frac{1}{2}\Delta$.

1. Introduction and Statement of Results

Let $D \subset R^d$ be a bounded domain with $C^{2,\alpha}$-boundary ($\alpha \in (0,1]$) and let $\mathcal{P}(D)$ denote the space of probability measures on D. Let

$$L = \frac{1}{2}\nabla \cdot a\nabla + b\nabla \tag{1.1}$$

be a second order elliptic operator. Assume that the coefficients $a = \{a_{i,j}\}_{i,j=1}^n$ and $b = \{b_i\}_{i=1}^n$ are in $C^{1,\alpha}(\bar{D})$ and that $a(x)$ is positive definite for each $x \in \bar{D}$. Fix a measure $\mu \in \mathcal{P}(D)$ and fix $\delta > 0$. Consider a Markov process $X(t)$ in D which performs diffusion in D generated by the operator δL and is stopped at the boundary, and which while running, jumps instantaneously, according to an exponential clock with spatially dependent intensity V, to a new point, according to the distribution μ. That is, the probability that the process $X(\cdot)$ has not jumped by time t is given by $\exp(-\int_0^{t\wedge\tau_D} V(X(s))ds)$, where $\tau_D = \inf\{t \geq 0 : X(t) \notin D\}$ is the first exit time from D. From its new position after the jump, the process repeats the above behavior independently of what has transpired previously. We assume that $V > 0$ in $\bar{D}$ and that $V \in C^\alpha(\bar{D})$. Denote probabilities and expectations for the process starting from $x \in D$ by $P_x^{\delta,\mu,V}$ and $E_x^{\delta,\mu,V}$. We will call the process a jump-diffusion.

Let $L_{\delta,\mu,V}$ denote the operator defined by

$$L_{\delta,\mu,V}\phi \equiv \delta L\phi + V(\int_D \phi\ d\mu - \phi).$$

The operator $L_{\delta,\mu,V}$ generates the jump-diffusion $X(t)$, and consequently, $\phi(X(t\wedge\tau_D)) - \int_0^{t\wedge\tau_D} L_{\delta,\mu,V}\phi(X(s))ds$ is a martingale.

Let $\phi_{\delta,\mu,V}$ denote the solution to the Dirichlet problem

$$\begin{aligned} &L_{\delta,\mu,V}\phi = 0 \text{ in } D;\\ &\phi = f \text{ on } \partial D, \end{aligned} \tag{1.2}$$

where f is continuous. It follows that $\phi_{\delta,\mu,V}(X(t\wedge\tau_D))$ is a martingale; thus $\phi(x) = E_x^{\delta,\mu,V}\phi_{\delta,\mu,V}(X(t\wedge\tau_D))$, for all t. Letting $t \to \infty$ gives the stochastic representation

$$\phi_{\delta,\mu,V}(x) = E_x^{\delta,\mu,V} f(X(\tau_D)). \tag{1.3}$$

In this paper we investigate the behavior of $\phi_{\delta,\mu,V}$ as $\delta \to 0$; that is, in the small diffusion limit. Since $L_{0,\mu,V}\phi = V(x)(\int_D \phi d\mu - \phi(x))$, one expects that $\lim_{\delta\to 0}\phi_{\delta,\mu,V}(x)$ will be independent of $x \in D$, and we can prove this trivially via the stochastic representation in (1.3). We wish to calculate the constant

$$\phi_{0,\mu,V}(f) \equiv \lim_{\delta\to 0}\phi_{\delta,\mu,V}(x),\ x \in D. \tag{1.4}$$

Let $e_{x,\delta}(\cdot) \equiv P_x^{\delta,\mu,V}(X(\tau_D) \in \cdot)$ denote the exit distribution of the jump-diffusion. If we calculate the weak limit $\nu_{\mu,V}$ of $e_{x,\delta}$ as $\delta \to 0$, then this will give us $\phi_{0,\mu,V}(f) = \int_{\partial D} f d\nu_{\mu,V}$.

The above jump-diffusion process arises naturally in the context of a system of switching diffusions with a fast motion and a slow motion. Let $(\phi_{\delta,\mathrm{sys}}, \psi_{\delta,\mathrm{sys}})$ denote the solution to the system of boundary value problems given by

$$\begin{aligned}
&\delta L\phi + V(\psi - \phi) = 0 \text{ in } D;\\
&\frac{1}{\delta}\mathcal{L}\psi + \mathcal{V}(\phi - \psi) = 0 \text{ in } D;\\
&\phi = f \text{ on } \partial D;\ \nabla\psi \cdot n = 0 \text{ on } \partial D,
\end{aligned} \tag{1.5}$$

where L and V are as above, and $\mathcal{L}$ and $\mathcal{V}$ are another such pair. Let μ be the invariant probability measure for the diffusion generated by $\mathcal{L}$ with normal reflection. The above system of equations corresponds to a Markov process $(X(t), \kappa(t))$. The coordinate $\kappa(t)$ has two states—1 and 2. When $\kappa(\cdot) = 1$, $X(\cdot)$ performs δL diffusion in D until it reaches the boundary. When $\kappa(\cdot) = 2$, $X(\cdot)$ performs $\frac{1}{\delta}\mathcal{L}$ diffusion in D with reflection at the boundary. The process $\kappa(t)$ jumps from state 1 to state 2 with $X(\cdot)$-dependent intensity $V(X(t))$ and from state 2 to state 1 with intensity $\mathcal{V}(X(t))$. Let $E_{x,k}$ denote the expectation starting from (x,k). Let $\tau_{D,1} = \inf\{t \ge 0 : X(t) \in \partial D, \kappa(t) = 1\}$. Then $\phi_{\delta,\mathrm{sys}}(x) = E_{x,1} f(X(\tau_{D,1}))$ and $\psi_{\delta,\mathrm{sys}}(x) = E_{x,2} f(X(\tau_{D,1}))$. One can show that

$$\lim_{\delta\to 0} \phi_{\delta,\mathrm{sys}}(x) = \lim_{\delta\to 0} \psi_{\delta,\mathrm{sys}}(x) = \int_{\partial D} f(y)\nu_{\mu,V}(dy),$$

where $\nu_{\mu,V}$ is the limiting exit distribution for the jump-diffusion as defined above. The heuristic explanation for this is as follows. If one observes the process $X(\cdot)$ only when $\kappa(\cdot)$ is in state 1, one sees a process very similar to our jump-diffusion process. Indeed, when κ jumps from state 1 to state 2, $X(\cdot)$ starts to run under a very fast clock according to the reflected $\mathcal{L}$-diffusion. However, the time for the clock $\mathcal{V}$ to ring and return the $\kappa(\cdot)$ process to state 1 is on order unity; by this time the distribution of the fast-running $X(\cdot)$ process is almost at its invariant distribution μ. Thus, when $\kappa(\cdot)$ jumps back to state 1 and we begin observing the $X(\cdot)$ process again, it starts up from a position that is distributed almost like μ.

Returning to our jump-diffusion processs, we note that the constant in (1.4) depends very strongly on the behavior of μ near the boundary. In this paper, we treat the case that $\mathrm{supp}(\mu)\cap\partial D\neq\emptyset$ and that μ has a density in a neighborhood of the boundary. The density may vanish on the boundary. Let $\tilde{L}$ denote the formal adjoint of L:

$$\tilde{L}=\frac{1}{2}\nabla\cdot a\nabla-b\nabla-\nabla\cdot b.$$

Theorem 1. *Let $D\subset R^d$, $d\geq 1$, be a bounded domain with a $C^{2,\alpha}$-boundary ($\alpha\in(0,1]$) and let $\mu\in\mathcal{P}(D)$. Assume that $V>0$ on $\bar{D}$. Let $D^\epsilon=\{x\in D: dist(x,\partial D)<\epsilon\}$.*
Assume that for some $\epsilon>0$, the restriction of μ to D^ϵ possesses a density: $\mu(dx)|_{D^\epsilon}\equiv\mu(x)dx$. Assume that for some $k\geq 0$, the following conditions hold. If k is even, assume that $\mu\in C^k(\bar{D}^\epsilon)$; if k is odd, assume that $\mu\in C^{k+1}(\bar{D}^\epsilon)$. Assume that

$$\frac{d^\beta\mu}{dx^\beta}\equiv 0 \text{ on } \partial D,\quad \text{for all } |\beta|\leq k-1,\ \text{if } k\geq 1;$$
$$\frac{d^\beta\mu}{dx^\beta}\not\equiv 0 \text{ on } \partial D,\ \text{for some } |\beta|=k,\ \text{if } k\geq 0.$$

Assume that $V\in C^{2,\alpha}(\bar{D})$, if $k=0,2$, and that $V\in C^k(\bar{D})$, if $k\geq 4$ is even; assume that $V\in C^{k+1}(\bar{D})$, if k is odd.
Assume that $a_{i,j},b_i\in C^{1,\alpha}(\bar{D})$, if $k=0,2$, and that $a_{i,j},b_i\in C^{k-1}(\bar{D})$, if $k\geq 4$ is even; assume that $a_{i,j},b_i\in C^{1,\alpha}(\bar{D})$, if $k=1$, and that $a_{i,j},b_i\in C^k(\bar{D})$, if $k\geq 3$ is odd.
Let n denote the inward unit normal to D at ∂D. Let σ denote Lebesgue measure on ∂D.
If k is even, then the solution $\phi_{\delta,\mu,V}$ to (1.2) satisfies (1.4) with

$$\phi_{0,\mu,V}(f)=\frac{\int_{\partial D}f\sqrt{(n\cdot an)}V^{-\frac{k+1}{2}}\tilde{L}^{\frac{k}{2}}\mu d\sigma}{\int_{\partial D}\sqrt{(n\cdot an)}V^{-\frac{k+1}{2}}\tilde{L}^{\frac{k}{2}}\mu d\sigma}.$$

If k is odd, then the solution $\phi_{\delta,\mu,V}$ to (1.2) *satisfies* (1.4) *with*

$$\phi_{0,\mu,V}(f)=\frac{\int_{\partial D} fV^{-\frac{k+1}{2}}a\nabla(\tilde{L}^{\frac{k-1}{2}}\mu)\cdot n d\sigma}{\int_{\partial D} V^{-\frac{k+1}{2}}a\nabla(\tilde{L}^{\frac{k-1}{2}}\mu)\cdot n d\sigma}.$$

In particular then, if $k=0$, one has

$$\phi_{0,\mu,V}(f)=\frac{\int_{\partial D} f\sqrt{(n\cdot an)}\frac{\mu}{\sqrt{V}}d\sigma}{\int_{\partial D}\sqrt{(n\cdot an)}\frac{\mu}{\sqrt{V}}d\sigma},$$

and if $k=1$, one has

$$\phi_{0,\mu,V}(f)=\frac{\int_{\partial D} fV^{-1}a\nabla\mu\cdot n d\sigma}{\int_{\partial D} V^{-1}a\nabla\mu\cdot n d\sigma}.$$

Remark 1. Note that if $k=0$ or $k=1$, then $\phi_{0,\mu,V}(f)$ depends on the diffusion coefficient a, but not on the drift coefficient b, whereas for $k\ge 2$, $\phi_{0,\mu,V}(f)$ depends on a and b.

Remark 2. If μ has compact support, the behavior of $\phi_{0,\mu,V}(f)$ is completely different. In this case, $\phi_{0,\mu,V}(f)$ can be studied using Wentzell-Freidlin action functionals. Assuming the uniqueness of the minimum of a certain such functional, $\phi_{0,\mu,V}(f)$ will have the form $f(x_0)$ for some $x_0\in\partial D$.

Define the contraction semigroup

$$T_t^{\delta,\mu,V}f(x)=E_x^{\delta,\mu,V}(f(X(t));\tau_D>t),\ f\in C_0(\bar{D}),$$

where $C_0(\bar{D})$ is the space of continuous functions on $\bar{D}$ vanishing on ∂D. The infinitesimal generator of this semigroup is an extension of the operator $L_{\delta,\mu,V}$, defined on $C^2(\bar{D})\cap\{\phi:\phi,L_{\delta,\mu,V}\phi\in C_0(\bar{D})\}$ with the homogeneous Dirichlet boundary condition. The operator $T_t^{\delta,\mu,V}$ is compact (see [3] where the case of constant coefficients is considered); thus, the resolvent operator for $T_t^{\delta,\mu,V}$ is also compact, and consequently the spectrum $\sigma(L_{\delta,\mu,V})$ of $L_{\delta,\mu,V}$ consists exclusively of eigenvalues. By the Krein-Rutman theorem, one deduces that $-L_{\delta,\mu,V}$ possesses a principal eigenvalue, $\lambda_0(\delta,\mu,V)$; that is, $\lambda_0(\delta,\mu,V)$ is real and simple and satisfies $\lambda_0(\delta,\mu,V)=$

$\inf\{\mathrm{Re}(\lambda) : \lambda \in \sigma(-L_{\delta,\mu,V})\}$ [5]. It is known that $\lambda \in \sigma(-L_{\delta,\mu,V})$ if and only if $\exp(-\lambda t) \in \sigma(T_t^{\delta,\mu,V})$ [2]. Thus, since $||T_t^{\delta,\mu,V}|| < 1$, it follows that $\lambda_0(\delta,\mu,V) > 0$. We have

$$\sup_{f\in C_0(\bar{D}),||f||\leq 1} ||T_t^{\delta,\mu,V} f|| = \sup_{x\in D} P_x^{\delta,\mu,V}(\tau_D > t);$$

thus, a standard result [6] allows us to conclude that

$$\lim_{t\to\infty} \frac{1}{t} \log \sup_{x\in D} P_x^{\delta,\mu,V}(\tau_D > t) = -\lambda_0(\delta,\mu,V).$$

It is well known that this is equivalent to

$$\lim_{t\to\infty} \frac{1}{t} \log P_x^{\delta,\mu,V}(\tau_D > t) = -\lambda_0(\delta,\mu,V), \quad x \in D. \tag{1.6}$$

In the case that $L = \frac{1}{2}\Delta$, that is the case that the underlying motion is Brownian motion, the papers [3], [4] investigated the behavior of the principal eigenvalue $\lambda_0(\delta,\mu,V)$ as $\delta \to 0$. (Actually, in those papers, one finds the operator $\gamma L_{\frac{1}{\gamma},\mu,V}\phi = \frac{1}{2}\Delta\phi + \gamma V(\int_D \phi d\mu - \phi)$ with $\gamma \to \infty$.) The key calculations contained in Proposition 1 of this paper for the case of a general diffusion operator L generalize calculations in [4] for the operator $\frac{1}{2}\Delta$. Using the methods of [4] along with Proposition 1, one obtains the following generalization of the results in [4].

Theorem 2. *Let the assumptions of Theorem 1 be in effect for some $k \geq 0$. Then the principal eigenvalue $\lambda_0(\delta,\mu,V)$ of the operator $-L_{\delta,\mu,V}$ behaves asymptotically as follows:*
i. If k is even,

$$\lim_{\delta\to 0} \delta^{-\frac{k+1}{2}} \lambda_0(\delta,\mu,V) = \frac{\int_{\partial D} \sqrt{(n\cdot an)} V^{-\frac{k+1}{2}} \tilde{L}^{\frac{k}{2}} \mu d\sigma}{\sqrt{2}\int_D \frac{1}{V} d\mu}; \tag{1.7}$$

ii. If k is odd,

$$\lim_{\delta\to 0} \delta^{-\frac{k+1}{2}} \lambda_0(\delta,\mu,V) = \frac{\int_{\partial D} V^{-\frac{k+1}{2}} a\nabla(\tilde{L}^{\frac{k-1}{2}}\mu)\cdot n d\sigma}{2\int_D \frac{1}{V} d\mu}. \tag{1.8}$$

Remark 1. Note that if $k = 0$ or $k = 1$, then the leading asymptotic behavior of $\lambda_0(\delta, \mu, V)$ depends on the diffusion coefficient a, but not on the drift coefficient b, whereas for $k \geq 2$, it depends on a and b.

Remark 2. We note that if μ has compact support, then there exist constants $c_1, c_2 > 0$ such that $\exp(-c_2\delta^{-\frac{1}{2}}) \leq \lambda_0(\delta, \mu, V) \leq \exp(-c_1\delta^{-\frac{1}{2}})$, for small $\delta > 0$. This was proven in [3, 4] for the case $L = \frac{1}{2}\Delta$. The same type of proof works for general L.

Remark 3. As with the Dirichlet problem in Theorem 1, the eigenvalue problem in Theorem 2 can also be connected to a system of switching diffusions. If in the second equation in (1.5), one multiplies $\mathcal{V}$ by say $\frac{1}{\delta}^{\frac{1}{2}}$, then the principal eigenvalue for the system will have the same asymptotic behavior as $\delta \to 0$ as does the principal eigenvalue for our jump-diffusion. To see this heuristically, we note that the principal eigenvalue for the system is connected to the switching diffusions via the analog of (1.6). Now observe the $X(\cdot)$ process only when $\kappa(\cdot)$ is in state 1. The point is that for almost all of the time up to the exit time, the process is running with $\kappa(\cdot)$ in state 1. The reason for this is that the speed of the clock for jumping back from state 2 to state 1 is now very fast. However, since this clock speed is still slower than the clock of the diffusion while $\kappa(\cdot)$ is in state 2 ($\frac{1}{\delta^{\frac{1}{2}}}$ as opposed to $\frac{1}{\delta}$), the distribution of $X(\cdot)$ will almost be the invariant measure μ when $\kappa(\cdot)$ jumps back to state 1 and we begin observing the $X(\cdot)$ process again.

If $\mu \not\equiv 0$ on ∂D, then Theorem 2 gives

$$\lambda_0(\delta, \mu, V) \sim \frac{\int_{\partial D} \sqrt{(n \cdot an)} \frac{\mu}{V^{\frac{1}{2}}} d\sigma}{\sqrt{2} \int_D \frac{1}{V} dx} \delta^{\frac{1}{2}}, \text{ as } \delta \to 0. \tag{1.9}$$

Theorem 2 is proven under the assumption that $V > 0$ in $\bar{D}$. This condition is essential. Note that if V vanishes in a sub-domain $A \subset D$, then as long as the process $X(t)$ remains in A, it never jumps, and thus starting from a point in A, the probability that $X(t)$ does not exit D by time t is greater than the probability that a δL-diffusion process does not exit A

by time t; thus, in light (1.6) and the corresponding equation for the δL diffusion process, it follows that $\lambda_0(\delta,\mu,V)\le\delta\lambda_0^A(L)$, where $\lambda_0^A(L)>0$ is the principal eigenvalue for $-L$ in A. In particular, $\lambda_0(\delta,\mu,V)$ is on a smaller order than in (1.9).

Now consider the case that $V>0$ in D but $V\equiv 0$ on ∂D. On the one hand, since the process needs to *not* jump in order to exit D, allowing the jump mechanism to weaken at the boundary should help the process exit. Thus, if $\mu\not\equiv 0$ on ∂D, one might expect that $\lambda_0(\delta,\mu,V)$ will be on a larger order than $\delta^{\frac12}$. But on the other hand, if $V\equiv 0$ in D^ϵ, then by the argument in the previous paragraph, $\lambda_0(\delta,\mu,V)\le\lambda_0^{D^\epsilon}(L)\delta$. Thus, we expect that if V vanishes identically on the boundary to high enough order, then $\lambda_0(\delta,\mu,V)$ will be on a smaller order than $\delta^{\frac12}$.

Now let $V_\epsilon\equiv\epsilon+\hat V$,where $\hat V>0$ in D and $\hat V\equiv 0$ on ∂D, and substitute V_ϵ for V in the righthand side of (1.9). If $\hat V$ vanishes to the first order on ∂D, then the right hand side of (1.9) is on the order $(\epsilon^{\frac12}\log\epsilon)^{-\frac12}$; in particular, it converges to ∞. This suggests that if V vanishes to first order on ∂D, then $\lambda_0(\delta,\mu,V)$ will be on a larger order than $\delta^{\frac12}$. If $\hat V$ vanishes to second order on ∂D, then the right hand side of (1.9) stays bounded and bounded from 0 as $\epsilon\to 0$. This suggests that if V vanishes to second order, then $\lambda_0(\delta,\mu,V)$ will be on the order $\delta^{\frac12}$, as in (1.9). If $\hat V$ vanishes to third order on ∂D, then the right hand side of (1.9) goes to 0 as $\epsilon\to 0$. This suggests that if V vanishes to third order or higher, then $\lambda_0(\delta,\mu,V)$ will be on a smaller order than $\delta^{\frac12}$.

Open Question: *Consider the case that $\mu\not\equiv 0$ on ∂D, so that if V were strictly positive in $\bar D$, then* (1.9) *would hold. Assume that $V>0$ in D and that V vanishes identically on ∂D to the order k, $k\ge 1$. At what order does $\lambda_0(\delta,\mu,V)$ approach 0 when $\delta\to 0$?*

In section 2 we present several auxiliary results which then allow for a quick proof of Theorem 1. The proof of one of the auxiliary results is deferred to section 3.

2. Auxiliary Results and Proof of Theorem 1

In this section we present three lemmas and one proposition, from which the theorem will follow quickly. We begin however with a useful construction of the process $X(\cdot)$ up to its exit time from ∂D. On a common probability space with probability measure $\mathcal{P}^\delta$, let $\{Y_n\}_{n=0}^\infty$ be an independent sequence of diffusion processes, where each $Y_n(\cdot)$ is a diffusion corresponding to the operator δL and stopped upon reaching the boundary, and where $Y_0(0) = x \in D$, and the distribution of $Y_n(0)$ for $n \geq 1$ is μ. (We have purposely suppressed the dependence of $\mathcal{P}^\delta$ on x. Note that Y_n does not depend on x for $n \geq 1$.) Let $\mathcal{F}_{t,n} = \sigma(Y_n(s), 0 \leq s \leq t)$ denote the filtration up to time t for the process $Y_n(\cdot)$. Denote the exit time of $Y_n(\cdot)$ from D by $\tau_{D,n}$. Let J_n be a stopping time for $Y_n(\cdot)$ satisfying $\mathcal{P}^\delta(J_n > t|\mathcal{F}_{t,n}) = \exp(-\int_0^{t\wedge\tau_{D,n}} V(Y_n(s))ds)$. Now define by induction:

$$X(t) = Y_0(t), \text{ for } 0 \leq t < J_0 \wedge \tau_{D,0};$$

$$\text{if } J_{n-1} < \tau_{D,n-1}, \text{ then}$$

$$X(t) = Y_n(t - \sum_{k=0}^{n-1} J_k), \text{ for } \sum_{k=0}^{n-1} J_k \leq t < \sum_{k=0}^{n-1} J_k + J_n \wedge \tau_{D,n}.$$

Recall that $e_{\delta,x}(\cdot) = P_x^{\delta,\mu,V}(X(\tau_D) \in \cdot)$ denotes the exit distribution of the process $X(\cdot)$ from D starting from $X(0) = x$. Let J denote the first jump time of $X(\cdot)$. Since the distribution of $\tau_{D,0}$ converges to the point mass at ∞ as $\delta \to 0$, we have $\lim_{\delta\to 0} \mathcal{P}^\delta(\tau_{D,0} < J_0) = 0$; equivalently, $\lim_{\delta\to 0} P_x^{\delta,\mu,V}(\tau_D < J) = 0$. Since $P_x^{\delta,\mu,V}(X(\tau_D) \in \cdot|J < \tau_D)$ is independent of $x \in D$, we conclude that the weak limit of $e_{\delta,x}$ (which will be shown to exist) is independent of $x \in D$. Indeed, from the above considerations and the above construction of $X(\cdot)$, we have

$$\text{w} - \lim_{\delta\to 0} e_{\delta,x} = \text{w} - \lim_{\delta\to 0} e_\delta, \tag{2.1}$$

where

$$e_\delta(\cdot) \equiv \mathcal{P}^\delta(Y_1(\tau_{D,1}) \in \cdot|\tau_{D,1} < J_1).$$

(Note that the 1 appearing in four places on the righthand side above can be replaced by any $n \geq 2$ without changing the value of the expression.)

For each $k = 0, 1, \cdots$, let $e_0^k(\cdot)$ denote the probability measure on ∂D with density $e_0^k(x)$ given by

$$e_0^k(x) = (\int_{\partial D} \sqrt{(n \cdot an)} V^{-\frac{k+1}{2}} \tilde{L}^{\frac{k}{2}} \mu d\sigma)^{-1} \sqrt{(n \cdot an)(x)} V^{-\frac{k+1}{2}}(x) \tilde{L}^{\frac{k}{2}} \mu(x),$$

$k \geq 0, k$ even;

$$e_0^k(x) = (\int_{\partial D} V^{-\frac{k+1}{2}} a \nabla (\tilde{L}^{\frac{k-1}{2}} \mu) \cdot n d\sigma)^{-1} \ V^{-\frac{k+1}{2}}(x) a(x) (\nabla (\tilde{L}^{\frac{k-1}{2}} \mu) \cdot n)(x),$$

$k \geq 1, k$ odd.

From (1.3) and (2.1), to prove the theorem we need to prove that if μ satisfies the conditions of the theorem for a particular $k \geq 0$, then e_δ converges weakly to e_0^k.

From now on we assume that μ satisfies the conditions of the theorem for a particular $k \geq 0$. Fix $m \geq 1$ and let $\{A_j\}_{j=1}^{m+1}$ be a partition of $\bar{D}$ into $m+1$ disjoint connected sets satisfying the following conditions: (i) A_j has a nonempty interior, for all j; (ii) $A_j \cap \partial D$ has a nonempty interior in the relative topology of ∂D, for all $j \neq m+1$; (iii) $e_0^k(A_1 \cap \partial D) > 0$; (iv) $\mathrm{dist}(A_{m+1}, \partial D) > 0$. To prove the theorem, it is enough to show that

$$\lim_{\delta \to 0} e_\delta(A_1 \cap \partial D) = e_0^k(A_1 \cap \partial D). \tag{2.2}$$

Let $u_{\delta,V}$ denote the solution to

$$\begin{aligned} &\delta L u - V u = 0 \text{ in } D; \\ &u = 1 \text{ on } \partial D. \end{aligned} \tag{2.3}$$

Let $\mathcal{E}^\delta$ denote expectations corresponding to $\mathcal{P}^\delta$. As is well-known, $u_{\delta,V}$ has the stochastic representation

$$u_{\delta,V}(x) = \mathcal{E}^\delta(\exp(-\int_0^{\tau_{D,1}} V(Y_1(t)))|Y_1(0) = x).$$

Lemma 1.

$$\mathcal{P}^\delta(Y_1(0) \in A_j, \tau_{D,1} < J_1) = \int_{A_j} u_{\delta,V} \ d\mu, \ j = 1, \cdots, m+1. \tag{2.4}$$

Proof. For $x \in D$, we have

$$\mathcal{P}^\delta(\tau_{D,1} < J_1 | Y_1(0) = x) = \int_0^\infty \mathcal{P}^\delta(J_1 > t, \tau_{D,1} = dt | Y_1(0) = x) =$$
$$\int_0^\infty \mathcal{E}^\delta \mathcal{P}^\delta(J_1 > t, \tau_{D,1} = dt | \mathcal{F}_{t,1}, Y_1(0) = x) =$$
$$\int_0^\infty \mathcal{E}^\delta(\exp(-\int_0^{t\wedge\tau_{D,1}} V(Y_1(s))ds)1_{dt}(\tau_{D,1})|Y_1(0) = x) =$$
$$\mathcal{E}(\exp(-\int_0^{\tau_{D,1}} V(Y_1(t))dt)|Y_1(0) = x) = u_{\delta,V}(x).$$

The result follows from this. □

Lemma 2. *There exists a constant $c = c(a, b, d, V) > 0$ depending on the coefficients a, b of L, on V and on the dimension d such that*

$$u_{\delta,V}(x) \le c\exp(-\frac{dist(x, \partial D)}{c\delta^{\frac12}}).$$

In particular then by Lemma 1, for some $c_1 > 0$,

$$\mathcal{P}^\delta(Y_1(0) \in A_{m+1}, \tau_{D,1} < J_1) \le c_1\exp(-\frac{dist(A_{m+1}, \partial D)}{c_1\delta^{\frac12}}). \tag{2.5}$$

Proof. Let $V_{\min} = \min_{x\in\bar D} V(x)$. From the stochastic representation of $u_{\delta,V}$, we have for any $t > 0$,

$$\begin{aligned} u_{\delta,V}(x) &\le \exp(-tV_{\min}) + \mathcal{P}^\delta(\tau_{D,1} \le t|Y_1(0) = x) \le \\ &\exp(-tV_{\min}) + \mathcal{P}^\delta(\max_{0\le s\le t} |Y_1(s) - x| \ge \mathrm{dist}(x, \partial D)|Y_1(0) = x). \end{aligned} \tag{2.6}$$

Since L has been written in divergence form, the drift of Y_1 is $\delta(b + \frac12\nabla\cdot a)$. Let $B = \max_{x\in\bar D} |b(x) + \frac12\nabla\cdot a(x)|$ and let $A = \max_{|v|=1}\max_{x\in\bar D}(v, a(x)v)$. From [5, Theorem 2.2-ii], it follows that

(2.7)
$$\mathcal{P}^\delta(\max_{0\le s\le t} |Y_1(s) - x| \ge \lambda|Y_1(0) = x) \le 2d\exp(-\frac{(\lambda - \delta Bt)^2}{2d\delta At}), \text{ for } \lambda > \delta Bt.$$

Letting $\lambda = \mathrm{dist}(x, \partial D)$ and $t = (\mathrm{dist}(x, \partial D))\delta^{-\frac12}$ in the above inequality and substituting the resulting estimate on the right hand side of (2.6), we obtain

$$u_{\delta,V}(x) \le \exp(-\delta^{-\frac12}\mathrm{dist}(x, \partial D)V_{\min}) + 2d\exp(-\delta^{-\frac12}\mathrm{dist}(x, \partial D)\frac{(1 - \delta^{\frac12}B)^2}{2dA}),$$

for small $\delta > 0$, from which the lemma follows. $\square$

The key result for proving Theorem 1 is the following proposition, whose proof is postponed to section 3.

Proposition 1. *Let the assumptions of Theorem 1 be satisfied for some $k \geq 0$. Let $j \in \{1, \cdots, m\}$. If k is even, then*

$$\lim_{\delta \to 0} \delta^{-\frac{k+1}{2}} \int_{A_j} u_{\delta,V} d\mu = \frac{1}{\sqrt{2}} \int_{A_j \cap \partial D} \sqrt{(n \cdot an)} V^{-\frac{k+1}{2}} \tilde{L}^{\frac{k}{2}} \mu d\sigma.$$

If k is odd, then

$$\lim_{\delta \to 0} \delta^{-\frac{k+1}{2}} \int_{A_j} u_{\delta,V} d\mu = \frac{1}{2} \int_{A_j \cap \partial D} V^{-\frac{k+1}{2}} a \nabla (\tilde{L}^{\frac{k-1}{2}} \mu) \cdot n d\sigma.$$

Lemma 3. *Let μ satisfy the assumptions in Theorem 1 for some $k \geq 0$. Let $j \in \{1, \cdots, m\}$ be such that $e_0^k(A_j \cap \partial D) > 0$. Then*

$$\lim_{\delta \to 0} \mathcal{P}^\delta(Y_1(\tau_{D,1}) \in A_j \cap \partial D | Y_1(0) \in A_j, \tau_{D,1} < J_1) = 1.$$

Proof. Define

$$u_{\delta,V,j}(x) = \mathcal{E}^\delta (1_{\partial D - A_j}(Y_1(\tau_{D,1})) \exp(-\int_0^{\tau_{D,1}} V(Y_1(t))) | Y_1(0) = x).$$

An argument just like that used in the proof of Lemma 1 shows that

$$\mathcal{P}^\delta(Y_1(0) \in A_j, \tau_{D,1} < J_1, Y_1(\tau_{D,1}) \in \partial D - A_j) = \int_{A_j} u_{\delta,V,j} \, d\mu. \tag{2.8}$$

An argument just like that used in the proof of Lemma 2 shows that

$$u_{\delta,V,j}(x) \leq c \exp(-\frac{\text{dist}(x, \partial D - A_j)}{c\delta^{\frac{1}{2}}}). \tag{2.9}$$

Let $N > 0$ be a positive integer and let $A_j^{\frac{1}{N}} = \{x \in A_j : \text{dist}(x, \partial A_j) < \frac{1}{N}\}$. By Lemma 1 and (2.8), we have

$$\begin{aligned} &\mathcal{P}^\delta(Y_1(\tau_{D,1}) \in \partial D - A_j | Y_1(0) \in A_j, \tau_{D,1} < J_1) = \\ &\frac{\int_{A_j^{\frac{1}{N}}} u_{\delta,V,j} d\mu + \int_{A_j - A_j^{\frac{1}{N}}} u_{\delta,V,j} d\mu}{\int_{A_j} u_{\delta,V} d\mu}. \end{aligned} \tag{2.10}$$

Proposition 1 of course also holds with A_j replaced by $A_j^{\frac{1}{N}}$. Using the fact that $u_{\delta,V,j} \le u_{\delta,V}$, applying Proposition 1 with A_j and with $A_j^{\frac{1}{N}}$, and using (2.9), we obtain

(2.11)

$$\limsup_{\delta\to 0} \mathcal{P}^\delta(Y_1(\tau_{D,1}) \in \partial D - A_j | Y_1(0) \in A_j, \tau_{D,1} < J_1) \le \frac{e_0^k(A_j^{\frac{1}{N}} \cap \partial D)}{e_0^k(A_j \cap \partial D)}.$$

Letting $N \to \infty$ completes the proof of the lemma. □

We can now prove (2.2), which will complete the proof of Theorem 1. Recall that by assumption $e_0^k(A_1) > 0$. Thus, by Lemmas 1 and 3, it follows that

(2.12)

$$\mathcal{P}^\delta(Y_1(0) \in A_1, Y_1(\tau_D) \in A_1 \cap \partial D, \tau_{D,1} < J_1) = \big(1+o(1)\big) \int_{A_1} u_{\delta,V} d\mu, \text{ as } \delta \to 0.$$

By Lemmas 1 and 3 and Proposition 1, it follows that

(2.13)

$$\mathcal{P}^\delta(Y_1(0) \in A_j, Y_1(\tau_D) \in A_1 \cap \partial D, \tau_{D,1} < J_1) = o(\delta^{\frac{k+1}{2}}), \text{ for } j \in \{2, \cdots, m\}.$$

Using (2.12), (2.13) and Lemma 2, we have

$$\begin{aligned}
& e_\delta(A_1 \cap \partial D) = \mathcal{P}^\delta(Y_1(\tau_{D,1}) \in A_1 \cap \partial D | \tau_{D,1} < J_1) = \\
& \frac{\mathcal{P}^\delta(Y_1(\tau_D) \in A_1 \cap \partial D, \tau_{D,1} < J_1)}{\mathcal{P}^\delta(\tau_{D,1} < J_1)} = \\
& \frac{\sum_{j=1}^{m+1} \mathcal{P}^\delta(Y_1(0) \in A_j, Y_1(\tau_D) \in A_1 \cap \partial D, \tau_{D,1} < J_1)}{\sum_{j=1}^{m+1} \mathcal{P}^\delta(Y_1(0) \in A_j, \tau_{D,1} < J_1)} = \\
& \frac{\big(1+o(1)\big) \int_{A_1} u_{\delta,V}\ d\mu + o(\delta^{\frac{k+1}{2}})}{\sum_{j=1}^{m} \int_{A_j} u_{\delta,V}\ d\mu + o(\delta^{\frac{k+1}{2}})}, \text{ as } \delta \to 0.
\end{aligned} \tag{2.14}$$

Now from (2.14) and Proposition 1, we have

$$\lim_{\delta\to 0} e_\delta(A_1 \cap \partial D) = e_0^k(A_1 \cap \partial D). \tag{2.15}$$

□

3. Proof of Proposition 1

For the proof of the proposition in the case of even k, we will need the following lemma.

Lemma 4. *Let n denote the unit inward normal to D at ∂D. One has*

$$\lim_{\delta\to 0}\delta^{\frac{1}{2}}(n\cdot a\nabla u_{\delta,V})(x)=-\sqrt{2V(x)(n\cdot an)(x)},\ \textit{uniformly over } x\in\partial D.$$

Proof. The proof of the result in the case that $L=\frac{1}{2}\Delta$ was given in [3]. In the proof, it was shown that everything could be reduced to local considerations. In particular, it was enough to prove that the above equation holds pointwise under the assumption that the boundary had constant curvature. We can thus make the same assumptions here, and we can also assume that L and V have constant coefficients. More specifically, note that L is given in non-divergence by

$$\begin{aligned}L=\frac{1}{2}\sum_{i,j=1}^{d}a_{i,j}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial a_{i,j}}{\partial x_i}(x)\frac{\partial}{\partial x_j}+\sum_{j=1}^{d}b_j(x)\frac{\partial}{\partial x_j}\equiv\\ \frac{1}{2}\sum_{i,j=1}^{d}a_{i,j}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{d}B_j(x)\frac{\partial}{\partial x_j}.\end{aligned}\tag{3.1}$$

Thus for the proof we may assume that $L=\frac{1}{2}\sum_{i,j=1}^{d}a_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{d}B_j\frac{\partial}{\partial x_j}$, where $a_{i,j}$ and B_j are constant. Similar to what was done in [3], for zero curvature, we take $D=\{x\in R^d:0<x_1<1\}$ and consider $(n\cdot a\nabla u_{\delta,V})(0)$; for curvature $R>0$, we assume that $D=A_{\frac{R}{2},R}\equiv\{x\in R^d:\frac{R}{2}<|x|<R\}$ and consider $(n\cdot a\nabla u_{\delta,V})(x)$, for some x with $|x|=R$; and for negative curvature $-R<0$, we assume that $D=A_{R,2R}\equiv\{x\in R^d:R<|x|<2R\}$ and consider $(n\cdot a\nabla u_{\delta,V})(x)$, for some x with $|x|=R$. We will consider the cases of zero curvature and positive curvature; the case of negative curvature being handled similarly to the case of positive curvature.

We begin with the case of zero curvature. Let $\{e_j\}_{j=1}^d$ denote the standard basis vectors. Let H_1 denote the hyperplane $x_1=0$. The interior unit normal to D on $\partial D\cap H_1$ is constant and equal to e_1; that is,

$n=e_1$. Let y denote the projection of $n\cdot a$ onto H_1. Then $y+(n\cdot an)e_1=n\cdot a$. Since $u_{\delta,V}=1$ on H_1, we have

(3.2)

$$(n\cdot a\nabla u_{\delta,V})(0)=\lim_{t\to 0+}\frac{u_{\delta,V}(tn\cdot a)-u_{\delta,V}(0)}{t}=\lim_{t\to 0+}\frac{u_{\delta,V}(tn\cdot a)-u_{\delta,V}(ty)}{t}=$$
$$\lim_{t\to 0^+}\frac{u_{\delta,V}(ty+t(n\cdot an)e_1)-u_{\delta,V}(ty)}{t}=n\cdot an\lim_{t\to 0^+}\frac{\partial u_{\delta,V}}{\partial x_1}(ty+s_t(n\cdot an)e_1)=$$
$$(n\cdot an)(n\cdot\nabla u_{\delta,V})(0),$$

where $0<s_t<t$. Since $u_{\delta,V}$ depends only on x_1 and since $n=e_1$, we can reduce the calculation of $(n\nabla u_{\delta,V})(0)$ to a one-dimensional problem. So we write $u_{\delta,V}=u_{\delta,V}(x)$ with $0<x<1$. Now u solves the constant coefficient equation $\frac{1}{2}\delta a_{1,1}u''_{\delta,V}+\delta B_1u'_{\delta,V}-Vu=0$ with the boundary condition $u_{\delta,V}(0)=u_{\delta,V}(1)=1$. The quantity $(n\cdot\nabla u_{\delta,V})(0)$ above is now given by $u'_{\delta,V}(0)$. One can solve this explicitly and check that $\lim_{\delta\to 0}\delta^{\frac{1}{2}}u'_{\delta,V}(0)=-\sqrt{\frac{2V}{a_{1,1}}}$. Substituting this in (3.2) and noting that $a_{1,1}=n\cdot an$, we obtain $\lim_{\delta\to 0}(n\cdot a\nabla u_{\delta,V})(0)=-\sqrt{2(n\cdot an)V}$.

Now we turn to the case that the curvature is $R>0$. We let $D=A_{\frac{R}{2},R}$ and consider the boundary point Re_1. We need to evaluate $\lim_{\delta\to 0}(n\cdot a\nabla u_{\delta,V})(Re_1)$. We first reduce the calculation to the calculation of the normal derivative, similar to (3.2). Note that the inward unit normal $n=n(Re_1)$ at Re_1 satisfies $n=-e_1$. For small $t>0$, let z_t denote the point on $|x|=R$ which is closest to $Re_1+tn\cdot a$. Define the vector w_t by $z_t+w_t=Re_1+tn\cdot a$. (Note that Re_1, z_t and w_t take on the roles played by 0, ty and $t(n\cdot an)e_1$ respectively in the case of zero curvature.) Of course $\lim_{t\to 0^+}|z_t-Re_1|=0$. Since the curvature is positive, we have $|w_t|<(n\cdot an)t$; however $\lim_{t\to 0+}\frac{|w_t|}{t}=n\cdot an$. Note also that the direction $\frac{w_t}{|w_t|}$ of w_t approaches the direction of n as $t\to 0^+$. Thus, since $u_{\delta,V}=1$ on

$|x| = R$, we have

$$
\begin{aligned}
&(n\cdot a\nabla u_{\delta,V})(Re_1) = \lim_{t\to 0^+} \frac{u_{\delta,V}(Re_1 + tn\cdot a) - u_{\delta,V}(Re_1)}{t} = \\
&\lim_{t\to 0+} \frac{u_{\delta,V}(z_t+w_t) - u_{\delta,V}(z_t)}{t} = \lim_{t\to 0^+} \frac{|w_t|}{t} \frac{u_{\delta,V}(z_t+w_t) - u_{\delta,V}(z_t)}{|w_t|} = \\
&(n\cdot an)(n\cdot \nabla u_{\delta,V})(Re_1).
\end{aligned} \tag{3.3}
$$

We now consider $(n\cdot\nabla u_{\delta,V})(Re_1)$. Let (r,θ) with $\theta\in S^{d-1}$ denote polar coordinates. We rewrite the constant coefficient operator L in polar form. Of course now the operator will no longer have constant coefficients; however by the localization mentioned above, we may consider instead the constant coefficient operator obtained by evaluating the coefficients at Re_1. Call the resulting operator L. We have $L = \frac{1}{2}(n\cdot an)\frac{d^2}{dr^2} + B\frac{d}{dr} +$ terms involving differentiation with respect to θ and maybe also r, where B is a certain constant whose form is irrelevant for our purposes. Now $u_{\delta,V}$ solves $\delta L u_{\delta,V} - V u_{\delta,V} = 0$ for $\frac{R}{2} < r < R$, and $u_{\delta,V} = 1$ at $r = \frac{R}{2}$ and $r = R$. It follows that $u_{\delta,V}$ is a function of r alone. Thus $u_{\delta,V}$ satisfies the one-dimensional equation $\frac{1}{2}\delta(n\cdot an)u''_{\delta,V} + \delta B u'_{\delta,V} - V u_{\delta,V} = 0$ for $\frac{R}{2} < r < R$ and $u(\frac{R}{2}) = u(R) = 1$, and $(n\cdot\nabla u_{\delta,V})(Re_1)$ becomes $-u'_{\delta,V}(R)$. We have thus reduced the problem to the previous case of zero curvature, and conclude that $\lim_{\delta\to 0}(n\cdot a\nabla u_{\delta,V})(Re_1) = -\sqrt{2(n\cdot an)V}$.

□

Proof of Proposition 1. Let $\mu_0(\cdot)$ be an arbitrary probability measure on $\bar{D}$ which has a density $\mu_0(x)$ which satisfies the same smoothness assumptions in $\bar{D}$ that the density μ satisfies in D^ϵ, and which satisfies the same vanishing conditions on ∂D that the density μ satisfies there. An easy argument then shows that to prove the proposition, it suffices to prove it with A_j replaced by $\bar{D}$, $d\mu$ replaced by $d\mu_0$ and $\mu(x)$ replaced by $\mu_0(x)$. We will first prove the proposition for the case $k=1$, which is easier than the case $k=0$. We then show how to go from the case $k=1$ to the case $k=3$, from which it will be clear how to proceed for odd k. After that we will prove the proposition for

$k=0$ and then we show how to go from the case $k=0$ to the case $k=2$, from which it will be clear how to proceed for even k.

In light of the above paragraph, we consider $\int_D u_{\delta,V}\mu_0 dx$. Since $k=1$, μ_0 vanishes on ∂D, but $\nabla\mu_0$ does not vanish identically on ∂D. Using (2.3) and the fact that μ_0 vanishes on ∂D, and recalling that n denotes the inward unit normal, integration by parts gives

$$\begin{aligned}&\delta^{-1}\int_D u_{\delta,V}\mu_0 dx=\int_D Lu_{\delta,V}\frac{\mu_0}{V}dx=\\ &\int_D u_{\delta,V}\tilde L\frac{\mu_0}{V}dx+\frac{1}{2}\int_{\partial D}a\nabla(\frac{\mu_0}{V})\cdot nd\sigma.\end{aligned}\tag{3.4}$$

(Note that by assumption, μ_0 and V are C^2-functions so there is no problem with the integration by parts.) By Lemma 2, $u_{\delta,V}$ converges to 0 boundedly pointwise on D. Also, since μ_0 vanishes on ∂D, we have $\nabla(\frac{\mu_0}{V})\cdot n=\frac{1}{V}\nabla\mu_0\cdot n$ on ∂D. Thus, letting $\delta\to 0$ in (3.4), we obtain

$$\lim_{\delta\to0}\delta^{-1}\int_D u_{\delta,V}\mu_0 dx=\frac{1}{2}\int_{\partial D}V^{-1}a\nabla\mu_0\cdot nd\sigma.$$

We now turn to the case $k=3$. In the case $k=3$, μ_0 and all its derivatives up to order 2 vanish on ∂D; in particular, the last term on the right hand side of (3.4) is 0. Thus, using (2.3) again, integrating by parts and using the fact that the second order derivatives of μ_0 vanish on ∂D, we have from (3.4),

$$\begin{aligned}&\delta^{-2}\int_D u_{\delta,V}\mu_0 dx=\delta^{-1}\int_D u_{\delta,V}\tilde L\frac{\mu_0}{V}dx=\int_D(Lu_{\delta,V})\frac{1}{V}\tilde L\frac{\mu_0}{V}dx=\\ &\int_D u_{\delta,V}\tilde L\frac{1}{V}\tilde L\frac{\mu_0}{V}dx+\int_{\partial D}a\nabla(\frac{1}{V}\tilde L\frac{\mu_0}{V})\cdot nd\sigma.\end{aligned}\tag{3.5}$$

(Note that by assumption, μ_0 and V are C^4-functions and $a_{i,j}$ and b_i are C^3-functions, so there is no problem with the integration by parts.) Using Lemma 2 again and the fact that μ_0 and all its derivatives up to order 2 vanish on ∂D, we obtain

$$\lim_{\delta\to0}\delta^{-2}\int_D u_{\delta,V}\mu_0 dx=\int_{\partial D}V^{-2}a\nabla(\tilde L\mu_0)\cdot nd\sigma.$$

The same technique is used repeatedly to handle larger values of odd k, the smoothness requirements in the statement of Theorem 1 being the smoothness required to implement the integration by parts.

Now we turn to the case $k=0$. Let w solve the equation

$$\begin{aligned} &\tilde{L}\frac{w}{V}=0 \text{ in } D;\\ &w=\mu_0 \text{ on } \partial D. \end{aligned} \tag{3.6}$$

Note that by the smoothness assumptions on a, b, V, μ_0, it follows that w is the solution to an elliptic equation with $C^\alpha(\bar{D})$-coefficients and continuous boundary data. Thus, $w \in C^{2,\alpha}(D)\cap C(\bar{D})$.

We will show below that

$$\lim_{\delta\to 0}\delta^{-\frac{1}{2}}\int_D u_{\delta,V}(\mu_0-w)dx=0. \tag{3.7}$$

Thus, it is enough to show that

$$\lim_{\delta\to 0}\delta^{-\frac{1}{2}}\int_D u_{\delta,V}w dx=\frac{1}{\sqrt{2}}\int_{\partial D}\sqrt{(n\cdot an)}\frac{\mu_0}{\sqrt{V}}d\sigma. \tag{3.8}$$

Using (2.3) and (3.6), and integrating by parts, we have

$$\delta^{-\frac{1}{2}}\int_D u_{\delta,V}w dx=\delta^{\frac{1}{2}}\int_D(Lu_{\delta,V})\frac{w}{V}dx=-\frac{\delta^{\frac{1}{2}}}{2}\int_{\partial D}\frac{\mu_0}{V}a\nabla u_{\delta,V}\cdot n d\sigma, \tag{3.9}$$

where we have used the fact that

$$\frac{1}{2}\int_{\partial D}a\nabla(\frac{w}{V})\cdot n d\sigma-\int_{\partial D}\frac{w}{V}b\cdot n d\sigma=\int_D\tilde{L}\frac{w}{V}dx=0$$

by (3.6). (Actually, since w is not necessarily C^2 up to the boundary, in the above integrals one should replace D by $D-\bar{D}^\epsilon$ and ∂D by $\partial(D-\bar{D}^\epsilon)$ and then let $\epsilon\to 0$.) Letting $\delta\to 0$ in (3.9), and using Lemma 4, we obtain (3.8).

It remains to prove (3.7). By Lemma 2, we have

$$|\delta^{-\frac{1}{2}}\int_{D-D^\epsilon}u_{\delta,V}(\mu_0-w)dx|\le\sup_{x\in D}(\mu_0(x)+w(x))|D|\delta^{-\frac{1}{2}}c\exp(-\frac{\epsilon}{c\delta^{\frac{1}{2}}}). \tag{3.10}$$

We also have

$$|\delta^{-\frac{1}{2}}\int_{D^\epsilon}u_{\delta,V}(\mu_0-w)dx|\le\sup_{x\in D^\epsilon}|\mu_0(x)-w(x)|(\delta^{-\frac{1}{2}}\int_D u_{\delta,V}dx). \tag{3.11}$$

Now (3.8) holds for every μ_0 in a wide class; in particular, it holds for μ_0 which are uniformly positive on $\bar{D}$. In such a case, it follows by the maximum principal that w is uniformly positive on $\bar{D}$. (The principal eigenvalue for $\tilde{L}$ coincides with that of L, and is consequently negative. Thus the generalized maximum principal holds: $\tilde{L}v = 0$ in D and $v > 0$ on ∂D guarantees that $v > 0$ on $\bar{D}$. Apply this with $v = \frac{w}{V}$.) By considering (3.8) with such a uniformly positive w, it follows that $\delta^{-\frac{1}{2}} \int_D u_{\delta,V} dx$ is bounded as $\delta \to 0$. Using this, the proof of (3.7) now follows from (3.10), (3.11) and the fact that $\lim_{\epsilon\to 0} \sup_{x\in D^\epsilon} |\mu_0(x) - w(x)| = 0$.

We now turn to the case $k = 2$. Since μ_0 and all its derivatives up to order one vanish on ∂D, we can write (3.4) as

$$\delta^{-\frac{3}{2}} \int_D u_{\delta,V} \mu_0 dx = \delta^{-\frac{1}{2}} \int_D u_{\delta,V} \tilde{L} \frac{\mu_0}{V} dx. \tag{3.12}$$

As with the case $k = 0$, we define an auxiliary function w. Let w solve the equation

$$\begin{aligned} &\tilde{L}\frac{w}{V} = 0 \text{ in } D; \\ &w = \tilde{L}\frac{\mu_0}{V} \text{ on } \partial D. \end{aligned} \tag{3.13}$$

(By assumption, μ_0 and its first order partial derivatives vanish on ∂D, but not all of its second order partial derivatives vanish on ∂D. It then follows from the maximum principal that $\tilde{L}\frac{\mu_0}{V} \gneq 0$ on ∂D.) The same argument used to show (3.7) shows that

$$\lim_{\delta\to 0} \delta^{-\frac{1}{2}} \int_D u_{\delta,V} (\tilde{L}\frac{\mu_0}{V} - w)\mu_0 dx = 0. \tag{3.14}$$

In light of (3.12) and (3.14), it is enough to prove that

$$\lim_{\delta\to 0} \delta^{-\frac{1}{2}} \int_D u_{\delta,V} w dx = \frac{1}{\sqrt{2}} \int_{\partial D} \sqrt{(n \cdot an)} V^{-\frac{3}{2}} \tilde{L}\mu_0 d\sigma. \tag{3.15}$$

Using (2.3), integrating by parts and using (3.13), we have

(3.16)

$$\delta^{-\frac{1}{2}} \int_D u_{\delta,V} w dx = \delta^{\frac{1}{2}} \int_D (Lu_{\delta,V}) \frac{w}{V} dx = -\frac{\delta^{\frac{1}{2}}}{2} \int_{\partial D} \frac{1}{V} (\tilde{L}\frac{\mu_0}{V}) a\nabla u_{\delta,V} \cdot n d\sigma,$$

where we have used the fact that

$$\frac{1}{2}\int_{\partial D} a\nabla(\frac{w}{V})\cdot n d\sigma - \int_{\partial D}\frac{w}{V} b\cdot n d\sigma = \int_D \tilde{L}\frac{w}{V}dx = 0$$

by (3.13). Since μ_0 and all its first order partial derivatives vanish on ∂D, we have $\tilde{L}\frac{\mu_0}{V} = \frac{1}{V}\tilde{L}\mu_0$ on ∂D. Using this and Lemma 4, and letting $\delta \to 0$ in (3.16), we obtain (3.15). The same technique is used repeatedly to handle larger values of even k, the smoothness requirements in the statement of Theorem 1 being the smoothness required to implement the integration by parts. □

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Department of Mathematics, Technion—Israel Institute of Technology, Haifa, 32000, Israel

URL: http://www.math.technion.ac.il/~pinsky/