DIFFUSIVE SEARCH WITH SPATIALLY DEPENDENT RESETTING

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Abstract. We consider a stochastic search model with resetting for an unknown stationary target $a \in \mathbb{R}$ with known distribution μ. The searcher begins at the origin and performs Brownian motion with diffusion constant D. The searcher is also armed with an exponential clock with spatially dependent rate $r = r(\cdot)$, so that if it has failed to locate the target by the time the clock rings, then its position is reset to the origin and it continues its search anew from there. Denote the position of the searcher at time t by $X(t)$. Let $E_0^{(r)}$ denote expectations for the process $X(\cdot)$. The search ends at time $T_a = \inf\{t \geq 0 : X(t) = a\}$. The expected time of the search is then $\int_{\mathbb{R}} (E_0^{(r)} T_a)\, \mu(da)$. Ideally, one would like to minimize this over all resetting rates r. We obtain quantitative growth rates for $E_0^{(r)} T_a$ as a function of a in terms of the asymptotic behavior of the rate function r, and also a rather precise dichotomy on the asymptotic behavior of the resetting function r to determine whether $E_0^{(r)} T_a$ is finite or infinite. We show generically that if $r(x)$ is of the order $|x|^{2l}$, with $l > -1$, then $\log E_0^{(r)} T_a$ is of the order $|a|^{l+1}$; in particular, the smaller the asymptotic size of r, the smaller the asymptotic growth rate of $E_0^{(r)} T_a$. The asymptotic growth rate of $E_0^{(r)} T_a$ continues to decrease when $r(x) \sim \frac{D\lambda}{x^2}$ with $\lambda > 1$; now the growth rate of $E_0^{(r)} T_a$ is more or less of the order $|a|^{\frac{1+\sqrt{1+8\lambda}}{2}}$. Note that this exponent increases to ∞ when λ increases to ∞ and decreases to 2 when λ decreases to 1. However, if $\lambda = 1$, then $E_0^{(r)} T_a = \infty$, for $a \neq 0$. Our results suggest that for many distributions μ supported on all of $\mathbb{R}$, a near optimal (or optimal) choice of resetting function r in order to minimize $\int_{\mathbb{R}^d} (E_0^{(r)} T_a) \mu(da)$ will be one which decays quadratically as $\frac{D\lambda}{x^2}$ for some $\lambda > 1$. We also give explicit, albeit rather complicated, variational formulas for $\inf_{r \gneqq 0} \int_{\mathbb{R}^d} (E_0^{(r)} T_a) \mu(da)$. For distributions μ with compact support, one should set $r = \infty$ off of the support. We also discuss this case.

2010 *Mathematics Subject Classification.* 60J60.

Key words and phrases. random target, diffusive search, resetting, optimization .

1. Introduction and Statement of Results

A number of recent papers have considered a stochastic search model for a stationary target $a \in R^d$, which might be random and have a known distribution attached to it, whereby a searcher sets off from a fixed point, say the origin, and performs Brownian motion with diffusion constant D. The searcher is also armed with a (possibly space dependent) exponential resetting time, so that if it has failed to locate the target by the time the clock rings, then its position is reset to the origin and it continues its search anew from there. One may be interested in several statistics, the most important one being the expected time to locate the target. (In dimension one, the target is considered "located" when the process hits the point a, while in dimensions two and higher, one chooses an $\epsilon_0 > 0$ and the target is said to be "located" when the process hits the ϵ_0-ball centered at a.) Without the resetting, this expected time is infinite. When the resetting rate is constant, the expected time to locate the target is finite. See, for example, [2, 3, 4, 5]. For related models with resetting, see [6, 8, 9] as well as the references in all of the above articles. Also see [7] and [13] for a related problem without resetting, which is motivated by the above resetting problem.

Here is a more formal mathematical definition of the model. Let $a \in \mathbb{R}^d$ denote an unknown stationary target with known probability distribution $\mu \in \mathcal{P}(\mathbb{R}^d)$, the space of probability measures on $\mathbb{R}^d$. The process $X(t)$ on $\mathbb{R}^d$ is defined as follows. Let $r = r(x) \gneqq 0$ be a continuous function on $\mathbb{R}^d$. This function will serve as the resetting rate. The process starts from $0 \in \mathbb{R}^d$ and performs Brownian motion with diffusion constant D, until a random exponential clock rings. The conditional probability that the clock has not rung by time $t > 0$, given that the path up to time t is $\{X(s), 0 \leq s \leq t\}$, is equal to $\exp(-\int_0^t r\big(X(s)\big)ds)$. When the clock rings, the process is instantaneously reset to its initial position 0, and continues its search afresh with an independent resetting clock, and the above scenario is repeated, etc. We define the process so that it is left continuous. From the above description, it follows that $X(\cdot)$ is a Markov process whose generator

$\mathcal{L}$ satisfies

$$\mathcal{L}u(x) = \frac{D}{2}\Delta u(x) + r(x)\big(u(0) - u(x)\big). \tag{1.1}$$

(For more details on such types of constructions, see [12].)

If $d = 1$, let $T_a = \inf\{t \geq 0 : X(t) = a\}$, while if $d \geq 2$, fix $\epsilon_0 > 0$ and define $T_a = \inf\{t \geq 0 : |X(t) - a| \leq \epsilon_0\}$. Denote probabilities and expectations for the process starting at x by $P_x^{(r)}$ and $E_x^{(r)}$ respectively. Since the unknown target $a \in \mathbb{R}^d$ has distribution μ, the expected search time is then given by $\int_{\mathbb{R}^d} (E_0^{(r)} T_a)\mu(da)$. Ideally, one would like to minimize this expression over all resetting rates r.

For most choices of r, it is not possible to write down a completely explicit expression for $E_0^{(r)} T_a$. If $r > 0$ is constant, then one can calculate $E_0^{(r)} T_a$ explicitly in terms of appropriate Bessel functions [4]. When $d = 1$, this simplifies [3] and one has

$$E_0^{(r)} T_a = \frac{e^{\sqrt{\frac{2r}{D}}\,|a|} - 1}{r},\ a \in \mathbb{R}. \tag{1.2}$$

The only other case we've seen worked out explicitly in the literature is the case that $d = 1$ and $r(x)$ is equal to 0 for $|x| < a_0$ and equal to a constant $r > 0$ for $x \geq a_0$, where $a_0 > 0$ [3].

In this paper, we consider the one-dimensional case. In Theorem 1, for each $r \gneqq 0$, we obtain an explicit formula for $E_0^{(r)} T_a$ in terms of a positive function ϕ that solves $\frac{D}{2}\phi'' - r\phi = 0$. It is intuitively clear that $E_0^{(r)} T_a$ is not monotone in r. And indeed, one can see this explicitly when r is constant—it follows from (1.2) that $E_0^{(r)} T_a$ approaches ∞ both as $r \to 0$ and as $r \to \infty$. As will be seen, our explicit formula for $E_0^{(r)} T_a$ in terms of the function ϕ is in fact the quotient of two functions, each of which is known to be monotone in r. This fact, along with the explicit formula for each of these two functions in the quotient, will allow us to obtain in Theorems 2 and 3 quantitative growth rates for $E_0^{(r)} T_a$ as a function of a in terms of the asymptotic behavior of the rate function r, and also in Theorem 3 a rather precise dichotomy on the asymptotic behavior of the resetting function r which determines whether $E_0^{(r)} T_a$ is finite or infinite. We also consider the case that the target distribution μ is compactly supported, in which case it is advantageous for the searcher to be instantaneously reset as soon as its position has left the support of μ.

We begin with a proposition that supplies us with several options for the above-mentioned function ϕ.

Proposition 1. *Let $r \gneqq 0$ be a continuous function on $\mathbb{R}$. Then there exist strictly positive functions $\{\phi_i\}_{i=1}^3$, all satisfying*

$$\frac{D}{2}\phi''(x) - r(x)\phi(x) = 0,\ x \in \mathbb{R}, \tag{1.3}$$

and such that

$$\begin{aligned}
&\int_{-\infty} \phi_1^{-2}(x)dx = \infty, \quad \int^{+\infty} \phi_1^{-2}(x)dx < \infty,\\
&\int_{-\infty} \phi_2^{-2}(x)dx < \infty, \quad \int^{+\infty} \phi_2^{-2}(x)dx = \infty,\\
&\int_{-\infty}^{+\infty} \phi_3^{-2}(x)dx < \infty.
\end{aligned} \tag{1.4}$$

Furthermore, if $r(x)$ is an even function, then $\phi_3(x)$ can be chosen to be even.

For $a > 0$, let $u_{+,a}$ denote the solution to the equation

$$\begin{aligned}
&\frac{D}{2}u'' - r(x)u = 0,\ -\infty < x < a;\\
&u(a) = 1;\\
&0 \le u \le 1 \text{ and } u \text{ maximal.}
\end{aligned} \tag{1.5}$$

(The condition that u be maximal means that any other solution bounded above and below by 1 and 0 is smaller or equal to $u_{+,a}$. The existence of such a solution will follow from the proof of Proposition 2. In fact, from the proof of Proposition 3, one can infer that there is a unique bounded solution to the equation $\frac{D}{2}u'' - r(x)u = 0,\ -\infty < x < a$, with $u(a) = 1$.) Similarly, for $a<0$, let $u_{-,a}$ denote the solution to

$$
\begin{aligned}
&\frac{D}{2}u''-r(x)u=0,\ \ a<x<\infty;\\
&u(a)=1;\\
&0\le u\le 1 \text{ and } u \text{ maximal.}
\end{aligned}
\tag{1.6}
$$

For $a>0$, let $v_{+,a}$ denote the solution to

$$
\begin{aligned}
&\frac{D}{2}v''-r(x)v=-1,\ \ -\infty<x<a;\\
&v(a)=0;\\
&v\ge 0 \text{ and } v \text{ minimal.}
\end{aligned}
\tag{1.7}
$$

(The condition that v be minimal means that any other nonnegative solution is greater or equal to v. The existence of such a solution will follow from the proof of Proposition 2.) Similarly, for $a<0$ let $v_{-,a}$ denote the solution to

$$
\begin{aligned}
&\frac{D}{2}v''-r(x)v=-1,\ \ a<x<\infty;\\
&v(a)=0;\\
&v\ge 0 \text{ and } v \text{ minimal.}
\end{aligned}
\tag{1.8}
$$

Remark. It follows from the maximum principle that the functions $u_{+,a}, u_{-,a}$, $v_{+,a}$, $v_{-,a}$ are decreasing in their dependence on r. (For more explanation, see the remark at the end of the proof of Proposition 2.) Furthermore, if r is sufficiently small, then $v_{+,a}$ on $(-\infty,a)$ and $v_{-,a}$ on (a,∞) will be equal to infinity. The proof of Theorem 3-i shows that this occurs if $r(x)\le\frac{D}{\gamma+x^2}$ for some $\gamma>0$ and sufficiently large $|x|$.

Proposition 2. *Let $r\gneqq 0$ be a continuous function on $\mathbb{R}$. Then for $a>0$,*

$$
E_0^{(r)}T_a=\begin{cases}\infty, & \text{if } v_{+,a}\equiv\infty;\\ \frac{v_{+,a}(0)}{u_{+,a}(0)}, & \text{otherwise,}\end{cases}
$$

and for $a<0$,

$$
E_0^{(r)}T_a=\begin{cases}\infty, & \text{if } v_{-,a}\equiv\infty;\\ \frac{v_{-,a}(0)}{u_{-,a}(0)}, & \text{otherwise.}\end{cases}
$$

Proposition 3. *i. Let ϕ_3 be as in Proposition 1. Then*

$$u_{+,a}(x)=\frac{\phi_3(x)}{\phi_3(a)}\frac{\int_{-\infty}^x\phi_3^{-2}(y)dy}{\int_{-\infty}^a\phi_3^{-2}(y)dy},\ x\le a; \tag{1.9}$$

$$u_{-,a}(x)=\frac{\phi_3(x)}{\phi_3(a)}\frac{\int_x^{\infty}\phi_3^{-2}(y)dy}{\int_a^{\infty}\phi_3^{-2}(y)dy},\ x\ge a; \tag{1.10}$$

$$v_{+,a}(x)=2\phi_3(x)\frac{\int_{-\infty}^x dy\,\phi_3^{-2}(y)\int_x^a dt\phi_3^{-2}(t)\int_y^t\phi_3(z)dz}{\int_{-\infty}^a\phi_3^{-2}(y)dy},\ x\le a; \tag{1.11}$$

$$v_{-,a}(x)=2\phi_3(x)\frac{\int_x^{\infty} dy\,\phi_3^{-2}(y)\int_a^x dt\phi_3^{-2}(t)\int_t^y\phi_3(z)dz}{\int_a^{\infty}\phi_3^{-2}(y)dy},\ x\ge a. \tag{1.12}$$

ii. Let ϕ_1 be as in Proposition 1. Then

$$u_{+,a}(x)=\frac{\phi_1(x)}{\phi_1(a)},\ x\le a; \tag{1.13}$$

$$u_{-,a}(x)=\frac{\phi_1(x)}{\phi_1(a)}\frac{\int_x^{\infty}\phi_1^{-2}(y)dy}{\int_a^{\infty}\phi_1^{-2}(y)dy},\ x\ge a; \tag{1.14}$$

$$v_{+,a}(x)=2\phi_1(x)\int_x^a dy\,\phi_1^{-2}(y)\int_{-\infty}^y\phi_1(z)dz,\ x\le a; \tag{1.15}$$

$$v_{-,a}(x)=2\phi_1(x)\frac{\int_x^{\infty} dy\,\phi_1^{-2}(y)\int_a^x dt\phi_1^{-2}(t)\int_t^y\phi_1(z)dz}{\int_a^{\infty}\phi_1^{-2}(y)dy},\ x\ge a. \tag{1.16}$$

Remark 1. Of course, formulas similar to (1.13)-(1.16) can be given in terms of ϕ_2.

Remark 2. As noted in the remark after (1.8), when r is sufficiently small, $v_{+,a}$ and $v_{-,a}$ are infinite. For $v_{+,a}$, one sees from (1.11) that the infiniteness is equivalent to $\int_{-\infty} dy\phi_3^{-2}(y)\int_y^0\phi_3(z)dz=\infty$, and from (1.15) that it is equivalent to $\int_{-\infty}\phi_1(y)dy=\infty$. Similar equivalences hold for the infiniteness of $v_{-,a}$ from (1.12) and (1.16).

As an immediate corollary of Propositions 2 and Proposition 3 we obtain the following explicit representation of $E_0^{(r)}T_a$.

Theorem 1. *i. Let ϕ_3 be as in Proposition 1. Then*

(1.17)
$$E_0^{(r)}T_a = \begin{cases} \frac{2\phi_3(a)}{\int_{-\infty}^0 \phi_3^{-2}(x)dx} \int_{-\infty}^0 dx\, \phi_3^{-2}(x) \int_0^a dy\, \phi_3^{-2}(y) \int_x^y \phi_3(z)dz, \ a>0; \\ \frac{2\phi_3(a)}{\int_0^\infty \phi_3^{-2}(x)dx} \int_0^\infty dx\, \phi_3^{-2}(x) \int_a^0 dy\, \phi_3^{-2}(y) \int_y^x \phi_3(z)dz, \ a<0. \end{cases}$$

ii. Let ϕ_1 be as in Proposition 1. Then

(1.18)
$$E_0^{(r)}T_a = \begin{cases} 2\phi_1(a) \int_0^a dy\, \phi_1^{-2}(y) \int_{-\infty}^y \phi_1(z)dz, \ a>0; \\ \frac{2\phi_1(a)}{\int_0^\infty \phi_1^{-2}(x)dx} \int_0^\infty dx\, \phi_1^{-2}(x) \int_a^0 dy\, \phi_1^{-2}(y) \int_y^x \phi_1(z)dz, \ a<0. \end{cases}$$

Remark 1. In light of Remark 1 after Proposition 3, a formula analogous to (1.18) holds with ϕ_2 in place of ϕ_1.

Remark 2. In the case that $r(x) = r > 0$ is constant, letting $\phi_3(x) = \exp(\sqrt{\frac{2}{D}r}\, x) + \exp(-\sqrt{\frac{2}{D}r}\, x)$ and $\phi_1(x) = \exp(\sqrt{\frac{2}{D}r}\, x)$, one can check that (1.17) and (1.18) yield (1.2).

Using Theorem 1 with Propositions 2 and 3, along with the fact that the functions $u_{+,a}, u_{-,a}, v_{+,a}, v_{-,a}$ are decreasing in r, and choosing test functions ϕ_3 appropriately, we will prove the following quantitative estimates on $E_0^{(r)}T_a$ in terms of the behavior of the resetting rate r.

Theorem 2. *Let $l > -1$. If*

(1.19)
$$c_1(\gamma_1+x^2)^l \le r(x) \le c_2(\gamma_2+x^2)^l, \text{ for all } x\in\mathbb{R}, \text{ where } c_2 \ge c_1 > 0 \text{ and } \gamma_2,\gamma_1 > 0,$$

then there exist $K_2 > K_1 > 0$ and $M_1, M_2 > 0$ such that

$$|a| M_1 e^{K_1(1+a^2)^{\frac{l+1}{2}}} \le E_0^{(r)}T_a \le M_2 e^{K_2(1+a^2)^{\frac{l+1}{2}}}, \text{ for all } a\in\mathbb{R}. \tag{1.20}$$

Remark. It follows that if $c_1(\gamma_1+x^2)^l \le r(x) \le c_2(\gamma_2+x^2)^l$, for some $l > -1$, then a necessary condition for the finiteness of $\int_{\mathbb{R}} E_0^{(r)}T_a \mu(da)$ is that all the moments of μ are finite.

Theorem 3. *i. If $r(x) \le \frac{D}{\gamma+x^2}$, for some $\gamma > 0$ and for sufficiently large $|x|$, then $E_0^{(r)}T_a = \infty$, for all $a \in \mathbb{R} - \{0\}$;*
ii. If $r(x) \ge \frac{D\lambda}{\gamma+x^2}$, for some $\lambda > 1$ and some $\gamma > 0$, and for sufficiently large $|x|$, then $E_0^{(r)}T_a < \infty$, for all $a \in \mathbb{R}$.

iii. For any $\lambda > 1$, there exists an $r(x)$ satisfying $r(x) \sim \frac{D\lambda}{x^2}$ as $x \to \infty$, and such that $E_0^{(r)} T_a \sim C|a|^{\frac{1+\sqrt{1+8\lambda}}{2}}$, as $|a| \to \infty$, for some $C > 0$.
iv. If

$$\frac{D\lambda_1}{\gamma_1 + x^2} \le r(x) \le \frac{D\lambda_2}{\gamma_2 + x^2}, \text{ for all } x, \text{ where } 1 < \lambda_1 \le \lambda_2 \text{ and } \gamma_1, \gamma_2 > 0,$$

then for any $\epsilon > 0$, there exist $C_1, C_2 > 0$ such that

$$C_1 |a|^{\frac{1+\sqrt{1+8\lambda_1}}{2} - \epsilon} \le E_0^{(r)} T_a \le C_2 |a|^{\frac{1+\sqrt{1+8\lambda_2}}{2} + \epsilon}, \text{ for } |a| \ge 1. \tag{1.21}$$

Remark. We expect that part (iv) also holds with $\epsilon = 0$.

Theorems 2 and 3 show generically that if $r(x)$ is of the order $|x|^{2l}$, with $l > -1$, then $\log E_0^{(r)} T_a$ is of the order $|a|^{l+1}$; in particular, the smaller the asymptotic size of r, the smaller the asymptotic growth rate of $E_0^{(r)} T_a$. The asymptotic growth rate of $E_0^{(r)} T_a$ continues to decrease when $r(x) \sim \frac{D\lambda}{x^2}$ with $\lambda > 1$; now the growth rate of $E_0^{(r)} T_a$ is more or less of the order $|a|^{\frac{1+\sqrt{1+8\lambda}}{2}}$. Note that this exponent increases to ∞ when λ increases to ∞ and decreases to 2 when λ decreases to 1. However, if $\lambda = 1$, then $E_0^{(r)} T_a = \infty$, for all $a \neq 0$.

Theorem 3 shows that the dependence of $E_0^{(r)} T_a$ on r is very sensitive when r has quadratic decay. If one uses regularly varying resetting rates r, Theorem 3 shows that if μ is such that its pth moment ($p > 0$) is finite if and only if $p < p_0$, then if $p_0 \le 2$, $\int_{\mathbb{R}} (E_0^{(r)} T_a)\, \mu(da)$ will be infinite for all r, and if $p_0 > 2$, then $\int_{\mathbb{R}} (E_0^{(r)} T_a)\, \mu(da)$ will be finite only for a very narrow window of these rates; namely, for resetting rates $r(x)$ that satisfy $r(x) \sim \frac{D\lambda}{x^2}$, where $1 < \lambda < \frac{(2p_0-1)^2-1}{8}$.

Theorems 2 and 3 would seem to suggest that for many distributions μ supported on all of $\mathbb{R}$, a near-optimal (or optimal) r for which $\int_{\mathbb{R}} (E_0^{(r)} T_a)\, \mu(da)$ will be close to minimal (or minimal) will be one with quadratic decay. We have some numerical work that doesn't quite bear this out, however it doesn't disprove this thesis. We consider the case that the target distribution μ is a two-sided symmetric exponential distribution: $\mu((x,\infty)) = \frac{1}{2} e^{-\beta x}$, for $x > 0$ and some $\beta > 0$, and $\mu((-\infty,-x)) = \mu((x,\infty))$, for $x \ge 0$. We compare the expected time to locate the target in the case that the optimal *constant* resetting rate is used to the case that certain *quadratically decaying* resetting rates are used. By (1.2), the expected time to locate the target with constant resetting rate $r > 0$ is

$$\int_0^\infty \frac{e^{\sqrt{\frac{2r}{D}}\,a} - 1}{r} \beta e^{-\beta a} da.$$

A standard calculation reveals that this expression is minimized when $r = \frac{D\beta^2}{8}$, and that the minimum value is $\frac{8}{D\beta^2}$. Recall that the expected distance to the target is $\frac{1}{\beta}$; thus the optimal expected time to locate the target is $\frac{8}{D}$ times the square of the expected distance to the target.

Now consider $r(x) = \frac{m(m-1)}{2} D \frac{|x|^{m-2}}{\gamma + |x|^m}$, where $m > 2$ and $\gamma > 0$. (For any $\gamma > 0$, and with $\lambda = \frac{m(m-1)}{2}$, this function is an example of the function r appearing in Theorem 3-iii.) The function $\phi_3(x) = \gamma + |x|^m$ satisfies $\frac{D}{2}\phi_3''(x) - r(x)\phi_3(x) = 0$. Thus $E_0^{(r)} T_a$ is given by (1.17) with this choice of ϕ_3. Since r is symmetric, $E_0^{(r)} T_a$ is symmetric in a, and thus the expected time to locate the target is $\int_0^\infty (E_0^{(r)} T_a) \beta e^{-\beta a} da$. We substitute in this integral the expression for $E_0^{(r)} T_a$ in (1.17). We want to minimize the resulting quantity as γ varies over $(0, \infty)$ and m varies over $(2, \infty)$. My colleague, Nir Gavish, found that for $\beta \geq 0.04$, the infimum value as a function of β can be approximated by the function $\frac{8.14 + 12.42 \exp(-35.66\beta)}{D\beta^2}$, with an error of less than 1 percent in the numerator. This is slightly worse than what we obtained using the optimal constant resetting rate. Of course, it is still possible that the infimum of $\int_0^\infty (E_0^{(r)} T_a) \beta e^{-\beta a} da$ over all r of the form $r(x) = \frac{c_1}{c_2 + x^2}$, with $c_1, c_2 > 0$ is less than $\frac{8}{D\beta^2}$. And we certainly expect that the infimum over *all* r that exhibit quadratic decay will be less than $\frac{8}{D\beta^2}$.

The discussion above, as well as our results, have been geared in particular to the case that the support of the target distribution is all of $\mathbb{R}$. If the support of the target distribution is, say, $[-L_1, L_2]$, where $L_1, L_2 > 0$, then there is no reason to search outside of this interval, and thus as soon as the searcher reaches $-L_1$ or L_2, its position should be reset to 0. This is equivalent to setting $r \equiv \infty$ off of $[-L_1, L_2]$. We discuss this situation in section 7.

We end this presentation of results by noting that Proposition 1 and Theorem 1 furnish explicit, albeit rather complicated, variational formulas for $\inf_{r\gneqq 0}\int_{\mathbb{R}}(E_0^{(r)}T_a)\mu(da)$. Assume that μ has mass both in $(0,\infty)$ and in $(-\infty,0)$, and for convenience, assume that the origin is not an atom of the distribution μ. Then μ can be written in the form

$$\mu=(1-p)\mu_-+p\mu_+,\ \text{where } p\in(0,1),\ \text{and } \mu_+ \text{ and } \mu_-$$
$$\text{are probability measures on } (0,\infty) \text{ and } (-\infty,0) \text{ respectively.}$$

Corollary 1. *i.*

(1.22)
$$\inf_{0\lneqq r\in C(\mathbb{R})}\int_{-\infty}^{\infty}E_0^{(r)}T_a\,\mu(da)=\inf_{\substack{\phi\in C^2(\mathbb{R}),\phi>0,\phi''\gneqq 0\\ \int_{-\infty}^{\infty}\phi^{-2}(x)dx<\infty}}$$
$$\Big[\frac{2p}{\int_{-\infty}^{0}\phi^{-2}(x)dx}\int_0^\infty\mu_+(da)\phi(a)\int_{-\infty}^0 dx\,\phi^{-2}(x)\int_0^a dy\,\phi^{-2}(y)\int_x^y dz\,\phi(z)+$$
$$\frac{2(1-p)}{\int_0^\infty\phi^{-2}(x)dx}\int_{-\infty}^0\mu_-(da)\phi(a)\int_0^\infty dx\,\phi^{-2}(x)\int_a^0 dy\,\phi^{-2}(y)\int_y^x dz\,\phi_3(z)\Big].$$

ii.

(1.23)
$$\inf_{0\lneqq r\in C(\mathbb{R})}\int_{-\infty}^{\infty}E_0^{(r)}T_a\,\mu(da)=\inf_{\substack{\phi\in C^2(\mathbb{R}),\phi>0,\phi''\gneqq 0\\ \int_{-\infty}\phi^{-2}(x)dx=\infty}}$$
$$\Big[2p\int_0^\infty\mu_+(da)\phi(a)\int_0^a dy\,\phi^{-2}(y)\int_{-\infty}^y\phi(z)dz+$$
$$\frac{2(1-p)}{\int_0^\infty\phi^{-2}(x)dx}\int_{-\infty}^0\mu_-(da)\phi(a)\int_0^\infty dx\,\phi^{-2}(x)\int_a^0 dy\,\phi^{-2}(y)\int_y^x dz\,\phi(z)\Big].$$

Remark. If $\phi>0,\phi''\gneqq 0$ and $\int_{-\infty}\phi^{-2}(x)dx=\infty$, then necessarily $\int^\infty\phi^{-2}(x)dx<\infty$ (see the proof of Proposition 1), so there is no need to include this last condition in part (ii).

Consider the case that μ is symmetric; that is, the case that $p=\frac{1}{2}$ and $\mu_+(A)=\mu_-(-A)$, for $A\subset(0,\infty)$. Then presumably,

$$\inf_{0\lneqq r\in C(\mathbb{R})}\int_{-\infty}^{\infty}E_0^{(r)}T_a\,\mu(da)=\inf_{\substack{0\lneqq r\in C(\mathbb{R})\\ r\text{ is even}}}\int_{-\infty}^{\infty}E_0^{(r)}T_a\,\mu(da),$$

although we don't have a proof.

Corollary 2. *Assume that μ is symmetric.*

i.

(1.24)
$$\inf_{\substack{0\lneq r\in C(\mathbb{R})\\ r \text{ is even}}}\int_{-\infty}^{\infty}E_0^{(r)}T_a\,\mu(da)=\inf_{\substack{\phi\in C^2(\mathbb{R}),\phi>0,\phi''\gneq 0,\phi \text{ is even}\\ \int_{-\infty}^{\infty}\phi^{-2}(x)dx<\infty}}$$
$$\Big[\frac{2}{\int_{-\infty}^{0}\phi^{-2}(x)dx}\int_0^{\infty}\mu_+(da)\phi(a)\int_{-\infty}^{0}dx\,\phi^{-2}(x)\int_0^a dy\,\phi^{-2}(y)\int_x^y dz\,\phi(z)\Big].$$

ii.

$$\inf_{\substack{0\lneq r\in C(\mathbb{R})\\ r \text{ is even}}}\int_{-\infty}^{\infty}E_0^{(r)}T_a\,\mu(da)=\inf_{\substack{\phi\in C^2(\mathbb{R},\phi>0,\phi''\gneq 0,\frac{\phi''}{\phi} \text{ is even}\\ \int_{-\infty}\phi^{-2}(x)dx=\infty}}\tag{1.25}$$
$$\Big[2\int_0^{\infty}\mu_+(da)\phi(a)\int_0^a dy\,\phi^{-2}(y)\int_{-\infty}^{y}\phi(z)dz\Big].$$

Remark. In part (ii), ϕ cannot be even because, as noted in the remark following Corollary 1, the conditions $\phi>0,\phi''\gneq 0$ and $\int_{-\infty}\phi^{-2}(x)dx=\infty$ lead automatically to $\int^{\infty}\phi^{-2}(x)dx<\infty$.

We prove Propositions 1-3 in sections 2-4 respectively, and Theorems 2 and 3 in sections 5 and 6 respectively. In section 7 we discuss the case in which the target distribution is supported on a finite interval.

2. Proof of Proposition 1

The proof is an application of the criticality theory of second order elliptic operators—see [11, chapter 4]. The operator $L:=\frac{D}{2}\frac{d^2}{dx^2}-r(x)$ on $\mathbb{R}$ with $r\gneq 0$ is subcritical, and thus there exists a positive L-harmonic function ϕ (that is, $\phi>0$ and $L\phi=0$). Denote by L^{ϕ} the operator which is the h-transform of L by the function ϕ; $L^{\phi}u:=\frac{1}{\phi}L(\phi u)$. It follows that $L^{\phi}=\frac{D}{2}\frac{d^2}{dx^2}+D\frac{\phi'}{\phi}\frac{d}{dx}$. Subcriticality is preserved by h-transforms, so L^{ϕ} is also subcritical. For one-dimensional operators, in the subcritical case, the cone of positive harmonic functions is two-dimensional. Furthermore, one of them is minimal at $-\infty$ and the other one is minimal at $+\infty$. Denote by $\hat{\phi}_1$ and $\hat{\phi}_2$ positive L^{ϕ}-harmonic functions, with $\hat{\phi}_1$ minimal at $-\infty$ and $\hat{\phi}_2$ minimal at $+\infty$. Define $\phi_1=\phi\hat{\phi}_1$ and $\phi_2=\phi\hat{\phi}_2$. Then ϕ_1 and ϕ_2 are L-harmonic; indeed, $0=(L^{\phi})(\hat{\phi}_i)=\frac{1}{\phi}L(\phi\hat{\phi}_i)$, $i=1,2$. Since the zeroth order term in L^{ϕ}

vanishes, L^ϕ is the generator of a diffusion process. For diffusion generators, subcriticality is equivalent to transience. By the Martin boundary theory ([11, chapter 7], the h-transformed operators $(L^\phi)^{\hat\phi_1}$ and $(L^\phi)^{\hat\phi_2}$ are transient to $+\infty$ and $-\infty$ respectively. These operators are given by

$$(L^\phi)^{\hat\phi_1}=\frac{D}{2}\frac{d^2}{dx^2}+D\frac{\phi'}{\phi}\frac{d}{dx}+D\frac{\hat\phi_1'}{\hat\phi_1}\frac{d}{dx}$$

and

$$(L^\phi)^{\hat\phi_2}=\frac{D}{2}\frac{d^2}{dx^2}+D\frac{\phi'}{\phi}\frac{d}{dx}+D\frac{\hat\phi_2'}{\hat\phi_2}\frac{d}{dx}.$$

The transience to $+\infty$ $(-\infty)$ of a diffusion generator $\frac{D}{2}\frac{d^2}{dx^2}+Db(x)\frac{d}{dx}$ is equivalent to $\int_{-\infty}\exp(-\int_0^x 2b(y)dy)dx=\infty$ and $\int^\infty\exp(-\int_0^x 2b(y)dy)dx<\infty$ $(\int_{-\infty}\exp(-\int_0^x 2b(y)dy)dx<\infty$ and $\int^\infty\exp(-\int_0^x 2b(y)dy)dx=\infty)$. Since $b=\frac{\phi'}{\phi}+\frac{\hat\phi_i'}{\hat\phi_i}$ for $(L^\phi)^{\hat\phi_i}$, $i=1,2$, it follows that ϕ_1 and ϕ_2 satisfy (1.4). Define $\phi_3=\phi_1+\phi_2$. Then ϕ_3 is also L-harmonic and it satisfies (1.4). If r is an even function, then $\phi_3(-x)$ is also L-harmonic. Thus, $\bar\phi_3(x):=\phi_3(x)+\phi_3(-x)$ is L-harmonic, satisfies (1.4) (for ϕ_3) and is even. $\square$

3. Proof of Proposition 2

We will prove the proposition for $a>0$; the same type of proof works for $a<0$. For $n>0$, let $\mathcal{T}_t^{(n)}$ be the semigroup defined by

$$\mathcal{T}_t^{(n)}f(x)=E_x^{(r)}(f(X(t));T_a\wedge T_{-n}>t):=E_x^{(r)}(f(X(t))1_{\{T_a\wedge T_{-n}>t\}}),\ x\in[-n,a],$$

for bounded continuous f. Its generator is $\mathcal{L}$ as in (1.1) with the zero Dirichlet boundary condition at $x=a$ and at $x=-n$. Let $w_n(x,t)=\mathcal{T}_t^{(n)}1(x)$. Then $w_n(x,t)=P_x^{(r)}(T_a\wedge T_{-n}>t)$ and it solves

$$\begin{aligned}&\frac{\partial}{\partial t}w_n=\mathcal{L}w_n=\frac{D}{2}w_n''+r(x)\big(w_n(0,t)-w_n(x,t)\big),\ x\in(-n,a);\\ &w_n(x,0)=1,\ x\in(-n,a);\quad w_n(a,t)=w_n(-n,t)=0,\ t>0.\end{aligned}\tag{3.1}$$

Let

$$A_n(x,s)=\int_0^\infty\exp(-st)w_n(x,t)dt,\ x\in[-n,a],\ s>0.$$

Then

$$\frac{D}{2}\frac{d^2A_n}{dx^2}(x,s)+r(x)\big(A_n(0,s)-A_n(x,s)\big)=\mathcal{L}A_n(x,s)=$$
$$\int_0^\infty \exp(-st)\mathcal{L}w_n(x,t)dt=\int_0^\infty \exp(-st)\frac{\partial}{\partial t}w_n(x,t)dt=$$
$$-1+s\int_0^\infty \exp(-st)w_n(x,t)dt=-1+sA_n(x,s),\ \text{for } x\in(-n,a).$$

Letting $s\to 0$, we find that $A_n(x):=A_n(x,0)$ satisfies

$$\begin{aligned}&\frac{D}{2}A_n''(x)-r(x)A_n(x)=-1-r(x)A_n(0),\ x\in(-n,a);\\ &A_n(a)=A_n(-n)=0.\end{aligned}\tag{3.2}$$

Note that

$$A_n(x)=\int_0^\infty w_n(x,t)dt=\int_0^\infty P_x^{(r)}(T_a\wedge T_{-n}>t)dt=E_x^{(r)}T_a\wedge T_{-n};$$

thus

$$E_0^{(r)}T_a\wedge T_{-n}=A_n(0).\tag{3.3}$$

For $c>0$, let $B_{n,c}(x)$ solve the equation

$$\begin{aligned}&\frac{D}{2}B_{n,c}''-r(x)B_{n,c}(x)=-1-cr(x),\ x<a;\\ &B_{n,c}(a)=B_{n,c}(-n)=0.\end{aligned}\tag{3.4}$$

We look for $c_n>0$ satisfying $B_{n,c_n}(0)=c_n$. It then follows that $A_n(x)=B_{n,c_n}(x)$, and in particular,

$$A_n(0)=c_n.\tag{3.5}$$

Let $v_{n,+,a}$ solve the equation

$$\begin{aligned}&\frac{D}{2}v_{n,+,a}''-r(x)v_{n,+,a}=-1,\ x\in(-n,a);\\ &v_{n,+,a}(a)=v_{n,+,a}(-n)=0,\end{aligned}\tag{3.6}$$

and let $u_{n,+,a}$ solve the equation

$$\begin{aligned}&\frac{D}{2}u_{n,+,a}''-r(x)u_{n,+,a}=0,\ x\in(-n,a);\\ &u_{n,+,a}(a)=u_{n,+,a}(-n)=1.\end{aligned}\tag{3.7}$$

(We note that by the maximum principle, $0 \le u_{n,+,a} \le 1$.) Then $B_{n,c} = v_{+,a} + c(1-u_{n,+,a})$, and thus the equation $B_{n,c_n}(n) = c_n$ is solved by $c_n = \frac{v_{n,+,a}(0)}{u_{n,+,a}(0)}$. Thus, from (3.3) and (3.5),

$$E_0^{(r)} T_a \wedge T_{-n} = A_n(0) = c_n = \frac{v_{n,+,a}(0)}{u_{n,+,a}(0)}. \tag{3.8}$$

By the maximum principle $\lim_{n\to\infty} u_{n,+,a} = u_{+,a}$ and $\lim_{n\to\infty} v_{n,+,a} = v_{+,a}$, where $u_{+,a}$ and $v_{+,a}$ are given by (1.5) and (1.7). Thus, letting $n \to \infty$ in (3.8) gives $E_0^{(r)} T_a = \infty$, if $v_{+,a} \equiv \infty$; otherwise it gives

$$E_0^{(r)} T_a = \frac{v_{+,a}(0)}{u_{+,a}(0)}.$$

□

Remark. In the remark before Proposition 2, it was stated that it follows from the maximum principle that the functions $u_{+,a}, u_{-,a}, v_{+,a}, v_{-,a}$ are decreasing in their dependence on r. To see this, consider, for example, $u_{+,a}$. Let $r_2 \ge r_1 \ge 0$. Denote by $u_{n,+,a,i}$ the solution to (3.7) with $r = r_i$, $i = 1, 2$. Then $w_n := u_{n,+,a,1} - u_{n,+,a,2}$ satisfies

$$\begin{aligned} &\frac{D}{2} w_n'' - r_1 w_n \le 0, \ x \in (-n, a); \\ &w_n(a) = w_n(-n) = 0. \end{aligned}$$

Thus, from the maximum principle, $w_n \ge 0$. Letting $n \to \infty$, we conclude that $u_{+,a,1} \ge u_{+,a,2}$, where $u_{+,a,i}$ solves (1.5) with $r = r_i$.

4. Proof of Proposition 3

We'll prove the formulas for $u_{+,a}$ and $v_{+,a}$. The proofs for $u_{-,a}$ and $v_{-,a}$ are similar. Let $L := \frac{D}{2}\frac{d^2}{dx^2} - r(x)$.

The solution $u_{+,a}$ to (1.5) is obtained as

$$u_{+,a} = \lim_{N\to\infty} u_{+,a,N},$$

where $u_{a,+,N}$ satisfies $Lu_{a,+,N} = 0$ in $(-N, a)$ with boundary condition $u_{+,a,N}(a) = u_{+,a,N}(-N) = 1$. Let ϕ_i be as in Proposition 1, with either $i = 1$ or $i = 3$. As noted in section 2, the h-transform of L by ϕ_i, denoted by L^{ϕ_i}, is defined by $L^{\phi_i} u = \frac{1}{\phi_i} L(\phi_i u)$, and when written out, one obtains

$L^{\phi_i}=\frac{D}{2}\frac{d^2}{dx^2}+D\frac{\phi_i'}{\phi_i}\frac{d}{dx}$. Write $u_{+,a,N}$ in the form $u_{+,a,N}=\phi_i\bar u_{+,a,N}$. Since $Lu_{+,a,N}=0$, we have

$$0=L^{\phi_i}\bar u_{+,a,N}=\frac{D}{2}\bar u''_{+,a,N}+D\frac{\phi_i'}{\phi_i}\bar u'_{+,a,N}=\frac{D}{2}\frac{1}{\phi_i^2}\big(\phi_i^2\bar u'_{+,a,N}\big)',$$

and since $u_{+,a,N}(a)=u_{+,a,N}(-N)=1$, we have $\bar u_{+,a,N}(a)=\frac{1}{\phi_i(a)}$ and $\bar u_{+,a,N}(-N)=\frac{1}{\phi_i(-N)}$. Solving by integrating twice and using the boundary condition, we obtain

$$\bar u_{+,a,N}(x)=\frac{1}{\phi_i(a)}-\big(\frac{1}{\phi_i(a)}-\frac{1}{\phi_i(-N)}\big)\frac{\int_x^a\phi_i^{-2}(y)dy}{\int_{-N}^a\phi_i^{-2}(y)dy}.$$

Choose first $i=3$. Since $\int_{-\infty}\phi_i^{-2}(y)dy<\infty$, it follows that there exists a sequence $N_k\overset{k\to\infty}{\to}\infty$ such that $\lim_{k\to\infty}\phi_3(-N_k)=\infty$. Thus substituting N_k for N above and letting $k\to\infty$, we conclude that $u_{+,a}$ is given by (1.9). (Retroactively, it then follows from the uniqueness of the solution to (1.5) that in fact $\lim_{N\to\infty}\phi_3(-N)=\infty$.) Now choose $i=1$. Letting $N\to\infty$, we conclude that $u_{+,a}$ is given by (1.13).

The solution $v_{+,a}$ to (1.7) is obtained as

$$\lim_{N\to\infty}v_{+,a,N},$$

where $v_{+,a,N}$ solves $Lv_{+,a,N}=-1$ in $(-N,a)$ with boundary condition $v_{+,a,N}(a)=v_{+,a,N}(-N)=0$. As was done above in solving for $u_{+,a}$, we make an h-transform with ϕ_i, $i=1,3$. Similar to the above, we write $v_{+,a,N}$ in the form $v_{+,a,N}=\phi_i\bar v_{+,a,N}$, and then obtain the equation

$$-\frac{1}{\phi_i}=\frac{D}{2}\bar v_{+,a,N}+D\frac{\phi_i'}{\phi_i}\bar v'_{+,a,N}=\frac{D}{2}\frac{1}{\phi_i^2}\big(\phi_i^2\bar v'_{+,a,N}\big)',$$

with the boundary condition $\bar v_{+,a,N}(a)=\bar v_{+,a,N}(-N)=0$. Solving by integrating twice and using the boundary condition, we obtain

$$\bar v_{+,a,N}(x)=c_N\int_x^a\phi_i^{-2}(y)dy-2\int_x^a dy\phi_i^{-2}(y)\int_y^a\phi_i(z)dz,$$

where

$$c_N=\frac{2\int_{-N}^a dy\phi_i^{-2}(y)\int_y^a\phi_i(z)dz}{\int_{-N}^a dy\phi_i^{-2}(y)}.$$

Choosing $i=1$, we have $\lim_{N\to\infty} c_N = 2\int_{-\infty}^a \phi_1(y)dy$, and thus

$$v_{+,a}(x) = \lim_{N\to\infty} v_{+,a,N}(x) = \phi_1(x) \lim_{N\to\infty} \bar v_{+,a,N}(x) =$$
$$2\phi_1(x)\int_x^a dy\, \phi_1^{-2}(y)\int_{-\infty}^y \phi_1(z)dz,$$

as in (1.15).

Choosing $i=3$, we have

$$v_{+,a}(x) = \lim_{N\to\infty} v_{+,a,N}(x) = \phi_3(x) \lim_{N\to\infty} \bar v_{+,a,N}(x) = \frac{2\phi_3(x)}{\int_{-\infty}^a \phi_3^{-2}(y)dy}\times$$
$$\Big[\big(\int_{-\infty}^a dy\phi_3^{-2}(y)\int_y^a \phi_3(z)dz\big)\big(\int_x^a \phi_3^{-2}(t)dt\big) - \tag{4.1}$$
$$\big(\int_x^a dy\phi_3^{-2}(y)\int_y^a \phi_3(z)dz\big)\big(\int_{-\infty}^a \phi_3^{-2}(t)dt\big)\Big].$$

Write the first term of the square brackets in (4.1) as

$$\big(\int_{-\infty}^a dy\phi_3^{-2}(y)\int_y^a \phi_3(z)dz\big)\big(\int_x^a \phi_3^{-2}(t)dt\big) =$$
$$\big(\int_{-\infty}^x dy\phi_3^{-2}(y)\int_y^a \phi_3(z)dz\big)\big(\int_x^a \phi_3^{-2}(t)dt\big) +$$
$$\big(\int_x^a dy\phi_3^{-2}(y)\int_y^a \phi_3(z)dz\big)\big(\int_x^a \phi_3^{-2}(t)dt\big),$$

and write the second term there as

$$\big(\int_x^a dy\phi_3^{-2}(y)\int_y^a \phi_3(z)dz\big)\big(\int_{-\infty}^a \phi_3^{-2}(t)dt\big) =$$
$$\big(\int_x^a dy\phi_3^{-2}(y)\int_y^a \phi_3(z)dz\big)\big(\int_{-\infty}^x \phi_3^{-2}(t)dt\big) + \tag{4.2}$$
$$\big(\int_x^a dy\phi_3^{-2}(y)\int_y^a \phi_3(z)dz\big)\big(\int_x^a \phi_3^{-2}(t)dt\big).$$

Notice that the final line from each of the preceding two displays is the same. Thus, the expression in the square brackets in (4.1) is equal to

$$\begin{aligned}&\big(\int_{-\infty}^x dy\phi_3^{-2}(y)\int_y^a \phi_3(z)dz\big)\big(\int_x^a \phi_3^{-2}(t)dt\big) - \\ &\big(\int_x^a dt\phi_3^{-2}(t)\int_t^a \phi_3(z)dz\big)\big(\int_{-\infty}^x \phi_3^{-2}(y)dy\big).\end{aligned} \tag{4.3}$$

(The second term above is the middle line of (4.2), but we have switched the roles of the variables of integration t and y.) The expression in (4.3) can be written as

$$\int_{-\infty}^{x} dy\,\phi_3^{-2}(y)\int_x^a dt\phi_3^{-2}(t)\int_y^t \phi_3(z)dz.$$

Substituting this for the expression in the square brackets in (4.1), we conclude that $v_{+,a}$ is given by (1.11). □

5. Proof of Theorem 2

We will prove the estimate for $a > 0$; the same type of proof works for $a < 0$. Fix $l > -1$ and assume that r satisfies (1.19). That is, let

$$r_-(x) = c_1(\gamma_1 + x^2)^l, \qquad r_+(x) = c_2(\gamma_2 + x^2)^l,$$

where $0 < c_1 < c_2$ and $\gamma_1, \gamma_2 > 0$, and where $r_-(x) \le r(x) \le r_+(x)$, for all $x \in \mathbb{R}$. We will need to compare the solutions $u_{+,a}$ and $v_{+,a}$ of (1.5) and (1.7) for different choices of the function r, so we will denote them here by $u_{+,a,r}$ and $v_{+,a,r}$. By Proposition 2 and the fact that $u_{+,a,r}(0)$ and $v_{+,a,r}(0)$ are decreasing in their dependence on r, it follows that

$$\frac{v_{+,a,r_+}(0)}{u_{+,a,r_-}(0)} \le E_0^{(r)}T_a \le \frac{v_{+,a,r_-}(0)}{u_{+,a,r_+}(0)}. \tag{5.1}$$

Define

$$\phi_3(x) = e^{\psi(x)}, \text{ where } \psi(x) = \lambda(\gamma + x^2)^{\frac{l+1}{2}}, \text{ with } \gamma, \lambda > 0. \tag{5.2}$$

Let

$$\begin{aligned} &\hat r(x) := \frac{D}{2}\frac{\phi_3''(x)}{\phi_3(x)} = \frac{D}{2}\big((\psi'(x))^2 + \psi''(x)\big) = \\ &\frac{D}{2}\lambda(l+1)(\gamma + x^2)^{\frac{l-3}{2}}\big[(l+1)\lambda x^2(\gamma + x^2)^{\frac{l+1}{2}} + \gamma + lx^2\big]. \end{aligned} \tag{5.3}$$

We will show that one can choose γ and λ so that the corresponding $\hat r$, which we will denote by $\hat r_-$, satisfies $0 \le \hat r_- \le r_-$, and we will show that one can choose γ and λ so that the corresponding $\hat r$, which we will denote by $\hat r_+$, satisfies $\hat r_+ \ge r_+$. *Note that since* (5.3) *does not depend on a, the γ and λ that will be chosen for $\hat r_-$ and for $\hat r_+$ will not depend on a.* Since $u_{+,a,r}(0)$ and $v_{+,a,r}(0)$ are decreasing in their dependence on r, it will then follow from (5.1) that

$$\frac{v_{+,a,\hat r_+}(0)}{u_{+,a,\hat r_-}(0)} \le E_0^{(r)}T_a \le \frac{v_{+,a,\hat r_-}(0)}{u_{+,a,\hat r_+}(0)}. \tag{5.4}$$

In the case that γ and λ have been chosen to construct $\hat{r}_-$, denote the function ϕ_3 above by $\phi_{3,-}$, and in the case that γ and λ have been chosen to construct $\hat{r}_+$, denote the function ϕ_3 above by $\phi_{3,+}$. We will then be able to complete the proof of the theorem using (5.4) along with Proposition 3, which gives $u_{+,a,\hat{r}_+}$ and $v_{+,a,\hat{r}_+}$ explicitly in terms of $\phi_{3,+}$, and $u_{+,a,\hat{r}_-}$ and $v_{+,a,\hat{r}_-}$ explicitly in terms of $\phi_{3,-}$.

We begin with finding γ and λ to construct $\hat{r}_-$ in the case $l \in (-1,0)$. This is the most delicate case. The term in the square brackets on the right hand side of (5.3) will clearly be positive for all x if $(l+1)\lambda\gamma^{\frac{l+1}{2}} + l \ge 0$; thus, in particular, it will be positive for all x if $\gamma = \gamma(\lambda) := (\frac{-l}{(l+1)\lambda})^{\frac{2}{l+1}}$. Thus, from (5.3), the inequality $\hat{r}_- \ge 0$ will hold for any $\lambda > 0$, if we choose $\gamma = \gamma(\lambda)$. We now show that if λ is chosen sufficiently small, and $\gamma = \gamma(\lambda)$, then $r_- \ge \hat{r}_-$. We have

(5.5)

$$r_-(x) - \hat{r}_-(x) = \frac{D}{2}\lambda(l+1)(\gamma(\lambda)+x^2)^{\frac{l-3}{2}} \times$$
$$\Big[\frac{2c_1}{D\lambda(l+1)}(\gamma(\lambda)+x^2)^{\frac{3-l}{2}}(\gamma_1+x^2)^l - (l+1)\lambda x^2(\gamma(\lambda)+x^2)^{\frac{l+1}{2}} - \gamma(\lambda) - lx^2\Big].$$

Thus, it remains to show that for sufficiently small λ, the expression in the square brackets in (5.5) is nonnegative for all x. Since for $a,b>0$, one has $(a+b)^m \le a^m + b^m$ if $m \in [0,1]$ and $(a+b)^m \ge a^m + b^m$ if $m \ge 1$, the expression in the square brackets will be positive if

$$\frac{2c_1}{D\lambda(l+1)}\big(\gamma_1+x^2\big)^l\big(\gamma(\lambda)^{\frac{3-l}{2}}+x^{3-l}\big)-(l+1)\lambda x^2\big(\gamma(\lambda)^{\frac{l+1}{2}}+x^{l+1}\big)-\gamma(\lambda)-lx^2 \ge 0.$$

Since $\gamma(\lambda)^{\frac{l+1}{2}} = \frac{-l}{\lambda(l+1)}$, the above inequality can be rewritten as

$$\frac{2c_1}{D\lambda(l+1)}\big(\gamma_1+x^2\big)^l\big(\gamma(\lambda)^{\frac{3-l}{2}}+x^{3-l}\big) - (l+1)\lambda x^{3+l} - \gamma(\lambda) \ge 0. \tag{5.6}$$

Noting that the first term in (5.6) behaves asymptotically like $\frac{2c_1}{D\lambda(l+1)}x^{3+l}$ as $x \to \infty$ and noting that $\gamma(\lambda) \to \infty$ as $\lambda \to 0$ and that $\frac{3-l}{2} > 1$, it is easy to see that (5.6) holds for all x.

We now find γ and λ to construct $\hat{r}_-$ in the case $l \ge 0$. From (5.3), we automatically have $\hat{r}_- \ge 0$. We fix γ arbitrarily and consider small λ. We have as in (5.5),

(5.7)
$$r_-(x)-\hat{r}_-(x)=\frac{D}{2}\lambda(l+1)(\gamma+x^2)^{\frac{l-3}{2}}\times$$
$$\Big[\frac{2c_1}{D\lambda(l+1)}(\gamma+x^2)^{\frac{3-l}{2}}(\gamma_1+x^2)^l-(l+1)\lambda x^2(\gamma+x^2)^{\frac{l+1}{2}}-\gamma-lx^2\Big].$$

The term in the square brackets on the right hand side of (5.7), when evaluated at $x=0$, is equal to $\frac{2c_1}{D\lambda(l+1)}\gamma^{\frac{3-l}{2}}\gamma_1^l-\gamma$, and for large $|x|$ behaves asymptotically like $\big(\frac{2c_1}{D\lambda(l+1)}-(l+1)\lambda\big)x^{l+3}$. From this and the general form of the term in the square brackets, it is clear that the right hand side of (5.7) is positive for all x if λ is chosen sufficiently small.

We now find γ and λ to construct $\hat{r}_+$ for any $l>-1$. From (5.3), one has $\hat{r}(0)=\frac{D}{2}\lambda(l+1)\gamma^{\frac{l-1}{2}}$ and $\hat{r}(x)\sim\frac{D}{2}\lambda^2(l+1)^2x^{2l}$ as $x\to\infty$. It is clear from this and the general form of (5.3) that if one fixes γ arbitrarily and lets λ be sufficiently large, then $\hat{r}_+(x)\ge r_+(x)$ for all x.

We now turn to estimating $u_{+,a,\hat{r}_\pm}(0)$ and $v_{+,a,\hat{r}_\pm}(0)$, using Proposition 3. From (1.9) with $\phi_{3,+}$ or $\phi_{3,-}$ in place of ϕ_3, it is clear that $u_{+,a,\hat{r}_\pm}(0)$ satisfy the estimates

$$\begin{aligned}&u_{+,a,\hat{r}_\pm}(0)\sim\frac{C_\pm}{\phi_{3,+}(a)},\ \text{as}\ a\to\pm\infty,\ \text{for some}\ C_\pm>0;\\ &\lim_{a\to0^+}u_{+,a,\hat{r}_\pm}(0)=1.\end{aligned}\tag{5.8}$$

Now consider (1.11) with $\phi_{3,+}$ or $\phi_{3,-}$ in place of ϕ_3. Since $\int_{-\infty}^\infty\phi_{3,\pm}^{-2}(y)dy<\infty$, the denominator $\int_{-\infty}^a\phi_{3,\pm}^{-2}(y)dy$ of the fraction in (1.11) is bounded as $a\to\infty$. Write the numerator of that fraction with $x=0$ as

$$\int_{-\infty}^0dy\,\phi_{3,\pm}^{-2}(y)\int_0^adt\phi_{3,\pm}^{-2}(t)\int_y^t\phi_{3,\pm}(z)dz=\int_{-\infty}^0dy\,\phi_{3,\pm}^{-2}(y)\int_0^adt\phi_{3,\pm}^{-2}(t)\int_0^t\phi_{3,\pm}(z)dz+$$
$$\int_{-\infty}^0dy\,\phi_{3,\pm}^{-2}(y)\int_0^adt\phi_{3,\pm}^{-2}(t)\int_y^0\phi_{3,\pm}(z)dz.$$

Since the functions $\phi_{3,\pm}$ are increasing on $[0,\infty)$ and decreasing on $(-\infty,0]$, we have

$$\int_{-\infty}^0dy\,\phi_{3,\pm}^{-2}(y)\int_0^adt\phi_{3,\pm}^{-2}(t)\int_0^t\phi_{3,\pm}(z)dz\le\big(\int_{-\infty}^0\phi_{3,\pm}^{-2}(y)dy\big)\big(\int_0^\infty t\phi_{3,\pm}^{-1}(t)dt\big)<\infty;$$
$$\int_{-\infty}^0dy\,\phi_{3,\pm}^{-2}(y)\int_0^adt\phi_{3,\pm}^{-2}(t)\int_y^0\phi_{3,\pm}(z)dz\le\big(\int_{-\infty}^0y\phi_{3,\pm}^{-1}(y)dy\big)\big(\int_0^\infty\phi_{3,\pm}^{-2}(t)dt\big)<\infty.$$

We conclude from this that $v_{+,a,\hat{r}_\pm}(0)$ are bounded as $a\to\infty$. It is also clear from (1.11) that $v_{+,a,\hat{r}_\pm}(0)\sim ca$ as $a\to 0^+$, for some $c>0$. Using these facts along with (5.8), (5.4) and (5.2), we conclude that (1.20) holds.

□

6. Proof of Theorem 3

We will prove the theorem for $a>0$; the same type of proof works for $a<0$. As in the proof of Theorem 2, since we will need to compare with different choices of r, we denote the solutions $u_{+,a}$ and $v_{+,a}$ of (1.5) and (1.7) by $u_{+,a,r}$ and $v_{+,a,r}$.

Parts (i) and (ii). By Proposition 2, $E_0^{(r)}T_a=\infty$ if and only if $v_{+,a}(0)=\infty$. Define

$$\phi_3(x)=\gamma+x^2,\ \gamma>0.$$

Let

$$\hat{r}(x):=\frac{D}{2}\frac{\phi_3''(x)}{\phi_3(x)}=\frac{D}{\gamma+x^2}.$$

Then $v_{+,a,\hat{r}}$ is given as in (1.11) with ϕ_3 as above. The numerator of the fraction in (1.11) with $x=0$ satisfies

$$\int_{-\infty}^0 dy\,\phi_3^{-2}(y)\int_0^a dt\phi_3^{-2}(t)\int_y^t\phi_3(z)dz\ge$$
$$\big(\int_{-\infty}^0 dy\,\phi_3^{-2}(y)\int_y^0\phi_3(z)dz\big)\big(\int_0^a dt\phi_3^{-2}(t)\big)=\infty;$$

thus, $v_{+,a,\hat{r}}(0)=\infty$. Since $v_{+,a,r}(0)$ is decreasing in its dependence on r, we conclude that $E_0^{(r)}T_a=\infty$, if $r(x)\le\frac{D}{\gamma+x^2}$, for some $\gamma>0$ and for all x.

Now $v_{+,a,r}$ can be written as $v_{+,a,r}(x)=\int_{-\infty}^a G_{a,r}(x,y)dy$, where $G_{a,r}(x,y)$ is the Green's function for the operator $\frac{D}{2}\frac{d^2}{dx^2}-r(x)$ on $(-\infty,a)$ [11]. If $r_1,r_2\gneqq 0$ and r_1-r_2 is compactly supported, then there exists a constant $c\in(0,1)$ such that $c\le\frac{G_{a,r_1}(x,y)}{G_{a,r_2}(x,y)}\le\frac{1}{c}$, for all $x,y\in(-\infty,a)$ [10]. It follows from this that the finiteness or infiniteness of $v_{+,a,r}$ is not affected by compactly supported changes in r. Thus, we conclude that $E_0^{(r)}T_a=\infty$, if $r(x)\le\frac{D}{\gamma+x^2}$, for some $\gamma>0$ and for all sufficiently large $|x|$. This proves part (i).

We turn to part (ii). Let $\lambda > 1$ and $\gamma > 0$. Define $\phi_3(x) = \gamma_1 + |x|^m$, where $\gamma_1 > 0$ and $m > 2$ is chosen so that $\lambda_0 := \frac{m(m-1)}{2}$ satisfies $\lambda_0 < \lambda$. Let

$$\hat{r}(x) := \frac{D}{2}\frac{\phi_3''(x)}{\phi_3(x)} = \frac{D\lambda_0|x|^{m-2}}{\gamma_1 + |x|^m}.$$

Then $v_{+,a,\hat{r}}$ is given as in (1.11) with ϕ_3 as above. This shows that $v_{+,a,\hat{r}}(0) < \infty$. If γ_1 is chosen sufficiently large, then $\frac{D\lambda}{\gamma+x^2} \geq \hat{r}(x)$. Thus $v_{+,a,\frac{D\lambda}{\gamma+x^2}}(0) < \infty$, since $v_{+,a,r}$ is decreasing in its dependence on r. Since the finiteness or infiniteness of $v_{+,a,r}$ is not affected by compactly supported changes in r, we conclude that $v_{+,a,r}(0) < \infty$ if $r(x) \geq \frac{D\lambda}{\gamma+x^2}$ for sufficiently large $|x|$, and thus also $E_0^{(r)}T_a < \infty$ for such r.

Part (iii). Fix $\lambda > 1$ and choose $m = \frac{1+\sqrt{1+8\lambda}}{2} > 2$ so that $\frac{m(m-1)}{2} = \lambda$. Define $\phi_3(x) = 1 + |x|^m$ and let $r(x) := \frac{D}{2}\frac{\phi_3''(x)}{\phi_3(x)} = \frac{D\lambda|x|^{m-2}}{1+|x|^m}$. Then $r(x) \sim \frac{D\lambda}{|x|^2}$ as $|x| \to \infty$. Using (1.9) and (1.11) with ϕ_3 as above, it follows that $u_{+,a,r}(0) \sim \frac{C_0}{\phi_3(a)}$ as $a \to \infty$, for some $C_0 > 0$, and that $v_{+,a,r}$ is bounded as $a \to \infty$. Thus, it follows from Proposition 2 that $E_0^{(r)}T_a \sim C|a|^{\frac{1+\sqrt{1+8\lambda}}{2}}$, for some $C > 0$.

Part (iv). Fix $\lambda_1, \lambda_2, \gamma_1, \gamma_2$ as in the statement of the theorem. Fix $\epsilon > 0$, and define $m = \frac{1+\sqrt{1+8\lambda_1}}{2} - \epsilon$ and $M = \frac{1+\sqrt{1+8\lambda_2}}{2} + \epsilon$. Assume that ϵ is sufficiently small so that $m > 2$. We will show that $c_1, c_2 > 0$ can be chosen so that $\phi_{3,-}(x) := c_1 + c_2x^2 + |x|^m$ satisfies

$$r_-(x) := \frac{D}{2}\frac{\phi_{3,-}''(x)}{\phi_{3,-}(x)} \leq \frac{D\lambda_1}{\gamma_1 + x^2}, \tag{6.1}$$

and that $c_1, c_2 > 0$ can be chosen so that $\phi_{3,+}(x) := c_1 + c_2x^2 + |x|^M$ satisfies

$$r_+(x) := \frac{D}{2}\frac{\phi_{3,+}''(x)}{\phi_{3,+}(x)} \geq \frac{D\lambda_2}{\gamma_2 + x^2}. \tag{6.2}$$

It will then follow from Proposition 2 and the fact that $u_{+,a,r}$ and $v_{+,a,r}$ are decreasing in their dependence on r that

$$\frac{v_{+,a,r_+}(0)}{u_{+,a,r_-}(0)} \leq E_0^{(r)}T_a \leq \frac{v_{+,a,r_-}(0)}{u_{+,a,r_+}(0)} \tag{6.3}$$

The functions $u_{+,a,r_\pm}$ are given by (1.9) with ϕ_3 replaced by $\phi_{3,\pm}$ from above, and the functions $v_{+,a,r_\pm}$ are given by (1.11) with ϕ_3 replaced by $\phi_{3,\pm}$. One finds that $v_{+,a,r_\pm}(0)$ are bounded as $a\to\infty$ and that $u_{+,a,r_\pm}(0)\sim\frac{C_\pm}{\phi_{3,\pm}(a)}$ as $a\to\infty$, for constants $C_\pm>0$. Using this with (6.3) proves (1.21).

It remains to find a pair c_1,c_2 for $\phi_{3,-}$ and a pair c_1,c_2 for $\phi_{3,+}$. Define $\delta_1,\delta_2>0$ by $\frac{m(m-1)}{2}=\lambda_1-\delta_1$ and $\frac{M(M-1)}{2}=\lambda_2+\delta_2$. We begin with $\phi_{3,-}$. We have

$$\frac{D}{2}\frac{\phi''_{3,-}}{\phi_{3,-}}=\frac{Dc_2+D(\lambda_1-\delta_1)|x|^{m-2}}{c_1+c_2x^2+|x|^m}.$$

Thus, the inequality (6.1) we wish to satisfy can be written as

$$\begin{aligned}&D\lambda_1c_1+D\lambda_1c_2x^2+D\lambda_1|x|^m\ge\\&Dc_2\gamma_1+Dc_2x^2+D(\lambda_1-\delta_1)\gamma_1|x|^{m-2}+D(\lambda_1-\delta_1)|x|^m.\end{aligned}\tag{6.4}$$

It is clear that we can choose c_2 sufficiently large so that $D\lambda_1c_2x^2+D\lambda_1|x|^m\ge Dc_2x^2+D(\lambda_1-\delta_1)\gamma_1|x|^{m-2}+D(\lambda_1-\delta_1)|x|^m$. Once such a c_2 is chosen, it is clear that c_1 can be chosen sufficiently large so that (6.4) holds.

We now find a pair c_1,c_2 for $\phi_{3,+}$. We have

$$\frac{D}{2}\frac{\phi''_{3,+}}{\phi_{3,+}}=\frac{Dc_2+D(\lambda_2+\delta_2)|x|^{M-2}}{c_1+c_2x^2+|x|^M}.$$

Thus, the inequality (6.2) we wish to satisfy can be written as

$$\begin{aligned}&D\lambda_2c_1+D\lambda_2c_2x^2+D\lambda_2|x|^M\le\\&Dc_2\gamma_2+Dc_2x^2+D(\lambda_2+\delta_2)\gamma_2|x|^{M-2}+D(\lambda_2+\delta_2)|x|^M.\end{aligned}\tag{6.5}$$

It is clear that we can choose c_2 sufficiently small so that

$$\inf_{x\in\mathbb{R}}\Big(Dc_2\gamma_2+Dc_2x^2+D(\lambda_2+\delta_2)\gamma_2|x|^{M-2}+D(\lambda_2+\delta_2)|x|^M-\\ D\lambda_2c_2x^2-D\lambda_2|x|^M\Big)>0.$$

Once c_2 has been chosen, it is clear that c_1 can be chosen sufficiently small so that (6.5) holds. □

7. When μ is supported on $[-L_1,L_2]$

Let $L_1,L_2>0$ and assume that the support of the target distribution is $[-L_1,L_2]$. In this case, there is no reason to search outside of the above interval, and thus as soon as the searcher reaches $-L_1$ or L_2, its position should be reset to 0. This is equivalent to setting $r\equiv\infty$ off of $[-L_1,L_2]$. On $[-L_1,L_2]$ we only need assume now that $r\ge0$, not that $r\gneqq0$. We'll use the notation $P_x^{(r;[-L_1,L_2])}, E_x^{(r;[-L_1,L_2])}$ for probabilities and expectations for this process starting from $x \in [-L_1, L_2]$. As in section 3, for fixed $a \in (0, L_2]$ let $\mathcal{T}_t$ be the semigroup defined by

$$\mathcal{T}_t f(x) = E_x^{(r;[-L_1,L_2])}(f(X(t)); T_a > t) := E_x^{(r;[-L_1,L_2])}(f(X(t)) 1_{\{T_a > t\}}),$$
$$x \in (-L_1, a],$$

for bounded continuous f, with a parallel definition in the case $a \in [-L_1, 0)$. Its generator is $\mathcal{L}$ as in (1.1) with the zero Dirichlet boundary condition at $x = a$ and with the additional boundary condition requiring that the value of the function at $-L_1$ be equal to the value of the function at 0. (For boundary conditions for semigroups corresponding to processes that jump from the boundary to an interior point, see for example [1].)

Let $w(x,t) = \mathcal{T}_t 1(x)$. Then $w(x,t) = P_x^{(r;[-L_1,L_2])}(T_a > t)$ and it solves

$$\begin{aligned} & w_t = \mathcal{L}w = \frac{D}{2} w'' + r(x)\big(w(0,t) - w(x,t)\big),\ x \in (-L_1, a); \\ & w(x,0) = 1,\ x \in (-L_1, a); \\ & w(a,t) = 0,\ w(-L_1,t) = w(0,t),\ t > 0. \end{aligned} \tag{7.1}$$

In the present case, the statement of the result analogous to Proposition 2 from section 3 looks exactly the same, the only difference being that the functions $u_{\pm,a}$ and $v_{\pm,a}$ that satisfied the equations (1.5)-(1.8) will now be called $u_{\pm,a;-L_1,L_2}$ and $v_{\pm,a;-L_1,L_2}$ and they satisfy the equations below instead, (7.2)-(7.5). The only difference in the proof is that since the interval we work with is bounded—$(-L_1, a)$, if $a > 0$ and (a, L_2) if $a < 0$, we don't need to truncate to a bounded domain as was done in the proof of Proposition 2.

For $a \in (0, L_2]$, let $u_{+,a;-L_1,L_2}$ denote the solution to the equation

$$\begin{aligned} & \frac{D}{2} u'' - r(x) u = 0,\ x \in (-L_1, a); \\ & u(a) = 1; \\ & u(-L_1) = u(0). \end{aligned} \tag{7.2}$$

For $a \in [-L_1, 0)$, let $u_{-,a;-L_1,L_2}$ denote the solution to

$$\begin{aligned}
&\frac{D}{2}u'' - r(x)u = 0,\ x \in (a, L_2);\\
&u(a) = 1;\\
&u(L_2) = u(0).
\end{aligned} \tag{7.3}$$

For $a \in (0, L_2]$, let $v_{+,a;-L_1,L_2}$ denote the solution to

$$\begin{aligned}
&\frac{D}{2}v'' - r(x)v = -1,\ x \in (-L_1, a);\\
&v(a) = 0;\\
&v(-L_1) = v(0).
\end{aligned} \tag{7.4}$$

For $a \in [-L_1, 0)$ let $v_{-,a;-L_1,L_2}$ denote the solution to

$$\begin{aligned}
&\frac{D}{2}v'' - r(x)v = -1,\ x \in (a, L_2);\\
&v(a) = 0;\\
&v(L_2) = v(0).
\end{aligned} \tag{7.5}$$

We record the result that corresponds to Proposition 2.

Proposition 4. *Let $r \ge 0$ be a continuous function on $[-L_1, L_2]$. Then*

$$E_0^{(r;[-L_1,L_2])}T_a = \begin{cases} \frac{v_{+,a;-L_1,L_2}(0)}{u_{+,a;-L_1,L_2}(0)}, & 0 < a \le L_2;\\ \frac{v_{-,a;-L_1,L_2}(0)}{u_{-,a;-L_1,L_2}(0)}, & -L_1 \le a < 0. \end{cases} \tag{7.6}$$

Similar to Proposition 1, for any $r \ge 0$, one can find a positive solution ϕ to $\frac{D}{2}\Delta\phi - r(x)\phi = 0$ in $[-L_1, L_2]$. Using such a function ϕ in the manner that we used ϕ_i, $i = 1, 2, 3$, one can proof a result parallel to Proposition 3 that gives the solutions $u_{\pm,a;-L_1,L_2}$ and $v_{\pm,a;-L_1,L_2}$ explicitly in terms of ϕ. As in Theorem 1, one then obtains an explicit formula for $E_0^{(r;[-L_1,L_2])}T_a$ in terms of this function ϕ. The formulas are a bit more complicated than those appearing in Proposition 3, so we refrain from writing them down.

However, we will consider in detail the case that $r(x) = r \ge 0$ is constant on $[-L_1, L_2]$. Assume first that $r > 0$. In this case it is easy to see from the equations that $v_{+,a;-L_1,L_2} = \frac{1}{r}(1 - u_{+,a;-L_1,L_2})$. Thus, from (7.6), $E_0^{(r;[-L_1,L_2])}T_a = \frac{1}{r}(\frac{1}{u_{+,a;-L_1,L_2}(0)} - 1)$, for $a \in (0, L_2]$. Solving for

$u_{+,a;-L_1,L_2}$ from (7.2) and substituting $x=0$, and making similar calculations for $u_{-,a;-L_1,L_2}$, we obtain

(7.7)

$$E_0^{(r;[-L_1,L_2])}T_a=\begin{cases}\frac{1}{r}\Big[\frac{\sinh\sqrt{\frac{2r}{D}}(a+L_1)-\sinh\sqrt{\frac{2r}{D}}a-\sinh\sqrt{\frac{2r}{D}}L_1}{\sinh\sqrt{\frac{2r}{D}}L_1}\Big], & 0<a\le L_2\\ \frac{1}{r}\Big[\frac{\sinh\sqrt{\frac{2r}{D}}(|a|+L_2)-\sinh\sqrt{\frac{2r}{D}}|a|-\sinh\sqrt{\frac{2r}{D}}L_2}{\sinh\sqrt{\frac{2r}{D}}L_2}\Big], & -L_1\le a<0.\end{cases}$$

Making similar calculations when $r=0$, or simply taking the limit of the above expression when $r\to 0$, one obtains

$$E_0^{(r;[-L_1,L_2])}T_a=\begin{cases}\frac{a(a+L_1)}{D}, & 0<a\le L_2;\\ \frac{|a|(|a|+L_2)}{D}, & -L_1\le a<0.\end{cases} \tag{7.8}$$

From now on, consider the symmetric case, $L_1=L_2=A$, for some $A>0$. Consider the uniform target distribution on $[-A,A]$. The expected distance to the target is then AvgDist:$=\frac{A}{2}$. The expected time to locate the target is

$$\frac{1}{2A}\int_{-A}^{A}E_0^{(r;[-A,A])}T_a\,da=\frac{1}{A}\int_0^A E_0^{(r;[-A,A])}T_a\,da.$$

Substituting from (7.7) and performing the integration, we obtain

$$\frac{1}{2A}\int_{-A}^{A}E_0^{(r;[-A,A])}T_a\,da=$$
$$\frac{1}{rA\sinh(\sqrt{\frac{2r}{D}}A)}\sqrt{\frac{D}{2r}}\Big(\cosh(\sqrt{\frac{2r}{D}}(2A)-2\cosh(\sqrt{\frac{2r}{D}}A)+1\Big)-\frac{1}{r}.$$

Letting $x=\sqrt{\frac{2r}{D}}A$, we have

$$\frac{1}{2A}\int_{-A}^{A}E_0^{(r;[-A,A])}T_a\,da=\frac{2A^2}{D}\Big(\frac{\cosh(2x)-2\cosh(x)+1}{x^3\sinh(x)}-\frac{1}{x^2}\Big).$$

The minimum of the function in parentheses above is obtained at $x=0$, the value of the function there being $\frac{5}{12}$. Thus we conclude that the expected time to locate the target is minimized in the class of constant resetting rates r on $[-A,A]$ by setting $r=0$, and

$$\inf_{r\ge 0, r\text{ constant}}\frac{1}{2A}\int_{-A}^{A}E_0^{(r;[-A,A])}T_a\,da=\frac{5}{6}\frac{A^2}{D}=\frac{10}{3}\frac{(\text{AvgDist})^2}{D}.$$

Intuitively, it seems then that the minimum will also be obtained at $r = 0$ if the symmetric target distribution has a non-decreasing density on $[0, A]$. We now consider the linearly decreasing, symmetric density which decreases to zero. The expected distance to the target is then $\frac{2}{A^2}\int_0^A a(A-a)da = \frac{1}{3}A$. We have

$$\int_{-A}^{A} (E_0^{(r;[-A,A])}T_a)\,\frac{A-|a|}{A^2}da = \frac{2}{A^2}\int_0^A (E_0^{(r;[-A,A])}T_a)\,(A-a)da.$$

Substituting from (7.7) and performing the integration, and again making the substitution $x = \sqrt{\frac{2r}{D}}A$, we obtain

$$\int_{-A}^{A} (E_0^{(r;[-A,A])}T_a)\,\frac{A-|a|}{A^2}da = \frac{4A^2}{D}\Big(\frac{\frac{\sinh(2x)}{x} - \frac{2\sinh(x)}{x} - \cosh(x) + 1}{x^3\sinh(x)} - \frac{1}{x^2}\Big).$$

The minimum of the function in the parentheses above is obtained at $x \approx 1.3538$ and the minimum value is approximately 0.1238. Thus we conclude that the expected time to locate the target is minimized in the class of constant resetting rates r on $[-A, A]$ by setting $r \approx 0.916\frac{D}{A^2}$, and

$$\inf_{r\ge 0, r \text{ constant}} \int_{-A}^{A} (E_0^{(r;[-A,A])}T_a)\,\frac{A-|a|}{A^2}da \approx 0.495\frac{A^2}{D} = 4.455\frac{(\text{AvgDist})^2}{D}.$$

It might be interesting to pursue the above direction of calculations further. In particular, what can be said about the ratio

$$\frac{D\inf_{r\ge0,r \text{ constant}}\int_{-A}^{A}(E_0^{(r;[-A,A])}T_a)\,\mu(da)}{(\text{AvgDist}(\mu))^2} =$$

$$\frac{D\inf_{r>0}\int_{-A}^{A}\frac{1}{r}\Big[\frac{\sinh\sqrt{\frac{2r}{D}}(a+A)-\sinh\sqrt{\frac{2r}{D}}a-\sinh\sqrt{\frac{2r}{D}}A}{\sinh\sqrt{\frac{2r}{D}}A}\Big]\mu(da)}{(2\int_0^A a\mu(da))^2},$$

as one varies over all symmetric distributions μ with support $[-A, A]$?

Acknowledgment. The author thanks his colleague, Nir Gavish, for the numerical analysis on page 9.

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