Ergodic Behavior of Diffusions with Random Jumps from the Boundary

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Abstract

We consider a diffusion process on $D \subset \mathbb{R}^d$, which upon hitting ∂D, is redistributed in D according to a probability measure depending continuously on its exit point. We prove that the distribution of the process converges exponentially fast to its unique invariant distribution and characterize the exponent as the spectral gap for a differential operator that serves as the generator of the process on a suitable function space.

Key words: Diffusion processes, spectral gap, rate of convergence, invariant measure, ergodic
MSC: primary 60J60

1 Introduction and Statement of Results

Let $D \subset \mathbb{R}^d$ be a bounded domain and let

$$L = \frac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla$$

be a second order elliptic operator on D. We will assume that the domain D has a $C^{2,\alpha}$-boundary, that $a = \{a_{ij}\}_{i,j=1}^d$ is positive definite with entries in $C^{2,\alpha}(\mathbb{R}^d)$ and that $b = (b_1, \ldots, b_d)_{i=1}^d$ has entries in $C^{1,\alpha}(\mathbb{R}^d)$, for some $\alpha \in (0, 1]$. We have written the principal part of the operator L in divergence

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form for convenience and, in light of the above conditions on the coefficients, without loss of generality. The solution to the generalized martingale problem for L on D is a diffusion process in D, killed upon hitting the boundary ∂D. Let $\mathcal{P}$ denote the set of probability measures on D with the topology of weak convergence. Let $\nu_{\{\cdot\}}$ be a continuous map from ∂D to $\mathcal{P}$. Consider now a process in D obtained in the following manner. The process coincides with the diffusion in D until it hits the boundary. If it hits the boundary at $\zeta \in \partial D$, then it jumps to a point in D according to the distribution ν_ζ and starts the diffusion afresh. The same mechanism is repeated independently each time the process reaches the boundary. This process will be called a *diffusion with random jumps from the boundary*. Not surprisingly, the process is ergodic and its distribution converges in total variation exponentially fast to its invariant measure. An upper bound can be obtained by a short Doeblin-type argument, similar to the one given in [GK, Section 7]. The chief objective of this paper is to provide a characterization of the rate in terms of a certain spectral gap/eigenvalue problem. That is, we want to relate the probabilistic notion of rate of convergence to the analytic notion of spectral gap. For reversible processes, results of this form follow from the spectral theorem. However, for non-reversible processes this is much more delicate and at present there is no general theory that guarantees such a link. In this work we propose an abstract approach to this problem that can be applied to a host of processes besides diffusion with random jumps from the boundary. In a separate paper, [BAP], we study this spectral gap quantitatively, when ν_ζ is independent of the point of exit ζ. A closely related model is the Fleming-Viot-type system studied in [BHM00], [GK06] and [Löb]. The case of Brownian motion ($L = \frac{1}{2}\Delta$) with a jump measure that is independent of the point of exit and also deterministic ($\nu_\zeta \equiv \delta_{x_0}$, for some $x_0 \in D$) was studied in [GK02] for $d = 1$ and then extended to higher dimensions in [GK]. The main idea of these two papers is to study the process through its resolvent, via a Laplace transform inversion formula. This approach has some limitations, discussed below Theorem 1, which do not allow extension to the level of generality we aim to in this paper. Our approach is functional analytic and its main idea is to study the ergodic properties of the processes through its adjoint semigroup, which turns out to be easier to handle. We show that the exponential convergence to the invariant measure is equivalent to the statement that the spectral radius of a certain operator is strictly less than one.

We begin with some notation. Let $Z \equiv \{Z(t) : t \geq 0\}$ denote the diffusion process in D corresponding to L and killed at the boundary. We denote the sub-probability transition function of Z by $p^D(t, x, y)$. The law of Z with initial distribution $\rho \in \mathcal{P}$ will be denoted by P^D_ρ, and the corresponding expectation will be denoted by E^D_ρ. When $\rho = \delta_x$, for some $x \in D$, we write P^D_x and E^D_x instead. Let τ_D denote the hitting time of the boundary by the diffusion Z. It is well known that for every $x \in D$, the harmonic measure, $P^D_x(Z(\tau_D) \in \cdot)$, is absolutely continuous with respect to the Lebesgue surface measure on ∂D. Its density will be denoted by $H(x,y)$. In addition, H is L-harmonic in x and continuous in y. See [Pin95] for details. If $F : D \to \mathbb{R}$, and $\rho \in \mathcal{P}$, we will write $F(\rho)$ for $\int_D F(x)d\rho(x)$. In particular, $H(\rho,y) \equiv \int_D H(x,y)d\rho(x)$ and $p^D(t,\rho,y) \equiv \int_D p^D(t,x,y)d\rho(x)$.

We now proceed to the construction of the diffusion with random jumps process with initial distribution $\rho \in \mathcal{P}$. Let $W^{\rho,0}$ be a diffusion process on D corresponding to L, killed at the boundary and with initial distribution ρ. Let $\{W^{\nu_\zeta,n} : \zeta \in \partial D, n \in \mathbb{N}\}$ denote a family of independent diffusion processes on D which all correspond to L and are killed at ∂D, such that $W^{\nu_\zeta,n}$ has initial distribution ν_ζ. We also require that $\{W^{\nu_\zeta,n} : \zeta \in \partial D, n \in \mathbb{N}\}$ be independent of $W^{\rho,0}$. Let $\tau_0 = 0$ and $\Theta_0 = W^{\rho,0}(0)$. Let

$$\tau_1 = \sigma_1 = \inf\{t \geq 0 : W^{\rho,0}(t) \in \partial D\},\ \Theta_1 = W^{\rho,0}(\sigma_1).$$

We continue inductively:

$$\sigma_{n+1} = \inf\{t \geq 0 : W^{\nu_{\Theta_n},n} \in \partial D\},\ \Theta_{n+1} = W^{\nu_{\Theta_n},n}(\sigma_{n+1}),$$

and we let $\tau_{n+1} = \tau_n + \sigma_{n+1}$.

Lemma 1 $\lim_{n\to\infty} \tau_n = \infty$*, a.s.*

This lemma allows us to define for all $t \geq 0$:

$$X(t) = \sum_{n=0}^{\infty} \mathbf{1}_{\{t\in[\tau_n,\tau_{n+1})\}} W^{\nu_{\Theta_n},n}(t-\tau_n).$$

We call $X \equiv \{X(t) : t \geq 0\}$ a diffusion with random jumps from the boundary. The construction guarantees that X is a Markov process. It is not hard to show that because of the boundary condition, the process X cannot be reversible, even if the underlying diffusion process killed at the boundary is reversible.

In what follows, we write P_ρ (E_ρ) for the probability measure (expectation) induced by X corresponding to the initial distribution ρ. We abbreviate and write P_x (E_x) when $\rho = \delta_x$. *We also extend the latter definition to $x \in \partial D$ by letting $P_\zeta \equiv P_{\nu_\zeta}$, for $\zeta \in \partial D$.*

Note that $\{\Theta_n : n \in \mathbb{N}\}$ is a time-homogeneous Markov process, and that $\Theta \equiv \{\Theta_n : n \in \mathbb{N}_0\}$ is a non time-homogeneous Markov process. Let $\tilde{H}(x,y) = H(\nu_x,y)$, for $x,y \in \partial D$. Then, the transition functions of Θ are given by:

$$P_x(\Theta_1 \in dy) = H(x,y)dy,\quad P_x(\Theta_{n+1} \in dv|\Theta_n = u) = \tilde{H}(u,v)dv,$$
$$\text{for } n \in \mathbb{N}, x \in D \text{ and } u,v,y \in \partial D.$$

For $n \in \mathbb{N}$, we let $\tilde{H}^n$ denote the n-fold transition function for Θ, starting from time 0. In other words, $\tilde{H}^n(x,y)dy = P_x(\Theta_n \in dy)$. Hence, $\tilde{H}^1(x,y) = H(x,y)$ and for $n \geq 2$,

$$\tilde{H}^n(x,y_n) = \int\limits_D \dots \int\limits_D H(x,y_1)\tilde{H}(y_1,y_2)\dots\tilde{H}(y_{n-1},y_n)dy_1\dots dy_{n-1}.$$

(Note that $\tilde{H}$ and $\tilde{H}^1$ are different.) Since $\zeta \to \nu_\zeta$ is continuous and ∂D is compact, it follows that $\{\nu_\zeta : \zeta \in \partial D\}$ is a compact subset of $\mathcal{P}$. In particular it is tight, which guarantees that there exists $U \subset\subset D$ such that $\nu_\zeta(U) > \frac{1}{2}$ for all $\zeta \in \partial D$. As a consequence,

$$\tilde{H}(\nu_\zeta, y) \geq \int\limits_U H(z,y)d\nu_\zeta(z) \geq \frac{1}{2}\inf_{z\in U} H(z,y) > 0.$$

This shows that $\{\Theta_n : n \in \mathbb{N}\}$ satisfies the Doeblin condition. In particular, it possesses a unique invariant probability measure m; that is,

$$\tilde{H}(m,y)dy = dm(y). \tag{1.1}$$

To show that $X(t)$ possesses a density, we observe that for every $f \in C(\overline{D})$,

$$E_x f(X(t)) = E_x[f(X(t)); \tau_1 > t] + \sum_{n=1}^{\infty} E_x[f(X(t)); \tau_n \leq t < \tau_{n+1}].$$

The first term on the righthand side is equal to $\int\limits_D p^D(t,x,y)f(y)dy$. For $n \in \mathbb{N}$, we have

$$\begin{aligned} E_x[f(X(t)); \tau_n \leq t < \tau_{n+1}] &= E_x E_x[f(X(t)); \tau_n \leq t < \tau_{n+1} | \Theta_n] \\ &= \int\limits_D \int\limits_{\partial D} \int\limits_0^t p^D(t-s,\nu_\zeta,y)f(y)dP_x(\tau_n \leq s|\Theta_n = \zeta)\tilde{H}^n(x,\zeta)d\zeta dy. \end{aligned}$$

As a consequence, we observe that X possesses a transition density $p(t,x,y)$, with respect to Lebesgue measure on D given by

$$\begin{aligned} p(t,x,y) &= p^D(t,x,y) \\ &\quad + \sum_{n=1}^{\infty} \int\limits_{\partial D} \int\limits_0^t p^D(t-s,\nu_\zeta,y)dP_x(\tau_n \leq s|\Theta_n = \zeta)\tilde{H}^n(x,\zeta)d\zeta. \end{aligned} \tag{1.2}$$

Let $G(x,y)$ denote the Green's function for $p^D(t,x,y)$; that is,

$$G(x,y) = \int\limits_0^\infty p^D(t,x,y)dt.$$

The following result identifies the invariant measure of X.

Proposition 1 *Let $\nu = \int_{\partial D} \nu_\zeta dm(\zeta)$, where m is as in (1.1). That is, ν is the unique measure in $\mathcal{P}$ such that $\int_D f d\nu = \int_{\partial D} \int_D f d\nu_\zeta dm(\zeta)$, for all $f \in C(\overline{D})$. Then X has an invariant probability measure μ, which is absolutely continuous with respect to Lebesgue measure on D. Its density, also denoted by μ, is given by*

$$\mu(y) = \frac{G(\nu, y)}{\int_D G(\nu, z) dz}.$$

We recall that a family of bounded operators $\mathcal{Q} \equiv \{\mathcal{Q}_t : t \geq 0\}$ mapping some Banach space Ξ into itself is called a semigroup if $\mathcal{Q}_0$ is the identity operator on Ξ and $\mathcal{Q}_{t+s} = \mathcal{Q}_t \mathcal{Q}_s$ for all $t, s \geq 0$. A semigroup $\mathcal{Q}$ is a contraction semigroup if for all $t \geq 0$, the operator norm of $\mathcal{Q}_t$ is bounded above by 1. A semigroup $\mathcal{Q}$ on Ξ is strongly continuous if $\lim_{t \to 0} \mathcal{Q}_t x = x$ for all $x \in \Xi$.

In the sequel the L^∞ (L^1) space on D with respect to the Lebesgue measure will be denoted by L^∞ (L^1). We denote the L^∞ (L^1) norm as well as the corresponding operator norm by $\| \cdot \|_\infty$ ($\| \cdot \|_1$).

Since X has a density with respect to Lebesgue measure, it follows from the Markov property that X naturally induces a contraction semigroup $\mathcal{T} \equiv \{\mathcal{T}_t : t \geq 0\}$ on L^∞ defined by

$$(\mathcal{T}_t f)(x) = E_x f(X(t)) = \int_D p(t, x, y) f(y) dy, \tag{1.3}$$

as well as a contraction semigroup $\mathcal{S} \equiv \{\mathcal{S}_t : t \geq 0\}$ on L^1 defined by

$$\mathcal{S}_t g(y) = \int_D p(t, x, y) g(x) dx.$$

Note that $\mathcal{T}$ is the dual semigroup to $\mathcal{S}$; that is, $\mathcal{T}_t = \mathcal{S}_t^*$ for all $t \geq 0$. As we shall see in Lemma 3 below, $\mathcal{S}$ is a strongly continuous compact semigroup. By duality, this guarantees that $\mathcal{T}$ is a compact semigroup, but strong continuity is not preserved through duality. In fact,

Proposition 2 *The semigroup $\mathcal{T}$ is not strongly continuous.*

This constitutes a limitation if one wants to apply results from the rich theory of strongly continuous semigroups, including the Laplace inversion formula that was the main ingredient in [GK02] and [GK]. This problem was avoided in [GK] by considering the restriction of $\mathcal{T}$ to a suitable space of continuous functions, on which it can be shown to be strongly continuous. But then the supremum in (1.4) of Theorem 1 can be taken only over functions in that space, which does not automatically guarantee convergence in total variation as in our result. Furthermore, the inversion formula for the Laplace transform also requires some non-trivial estimates for analytical semigroups. We take a different path, showing that $\mathcal{S}$ is strongly continuous and compact. We derive the ergodic theorem for $\mathcal{S}$ through the spectral radius formula and finally obtain the ergodic theorem by duality.

Let

$$\mathcal{D} = \{f \in C^2(D) \cap L^\infty,\ \forall \zeta_0 \in \partial D \quad \lim_{D \ni x \to \zeta_0 \in \partial D} f(x) = \int_D f(y) d\nu_{\zeta_0}(y)\}.$$

We also let

$$\widetilde{\mathcal{D}} = \{f \in \mathcal{D} : Lf \in \overline{\mathcal{D}}\},$$

where $\overline{\mathcal{D}}$ is the closure of $\mathcal{D}$ in L^∞. Let $\mathcal{L}$ denote the restriction of L to $\widetilde{\mathcal{D}}$. We denote the set of eigenvalues of $\mathcal{L}$ by $\sigma_p(\mathcal{L})$. Let

$$\gamma_1 = \sup\{\text{Re } \lambda : 0 \neq \lambda \in \sigma_p(\mathcal{L})\}.$$

Before stating the main result we recall the notion of a core. Let Ξ be a Banach space. Let $\mathcal{B}$ be a linear mapping from some subspace of $\mathcal{D}(\mathcal{B})$ of Ξ into Ξ. The graph of $\mathcal{B}$ is the subspace $\{(x, \mathcal{B}x) : x \in \mathcal{D}(\mathcal{B})\} \subset \Xi \times \Xi$. We say that $\mathcal{B}_1$ is a core for $\mathcal{B}_2$ if the graph of $\mathcal{B}_1$ is a subset of the graph of $\mathcal{B}_2$ which is dense in the product topology. Finally, we also recall that a generator $\mathcal{G}$ of a semigroup $\mathcal{Q}$ on a Banach space Ξ is a linear mapping whose domain is the subspace $\mathcal{D}(\mathcal{G}) = \{x \in \Xi : \lim_{t \to 0^+} \frac{1}{t}(\mathcal{Q}_t x - x) \text{ exists }\}$. For $x \in \mathcal{D}(\mathcal{G})$, $\mathcal{G}x = \lim_{t \to 0^+} \frac{1}{t}(\mathcal{Q}_t x - x)$.
Here is our main result.

Theorem 1 *The restriction of $\mathcal{T}$ to the L^∞-closure of $\widetilde{\mathcal{D}}$ is a strongly continuous compact semigroup and $\mathcal{L}$ is a core for its generator. The spectrum of the generator consists entirely of eigenvalues and is equal to $\sigma_p(\mathcal{L})$. It has no accumulation points. Zero is an eigenvalue and all other eigenvalues have strictly negative real part. Furthermore*

$$\lim_{t \to \infty} \frac{1}{t} \ln \sup_{\|f\|_\infty \leq 1} \|E_x f(X(t)) - \int f d\mu\|_\infty = \gamma_1 \in (-\infty, 0). \tag{1.4}$$

We conclude this section with a complete calculation in the case of Brownian motion on the interval $(0, \pi)$ with deterministic jumps from each endpoint: $\nu_0 = \delta_p,\ \nu_\pi = \delta_q$, for some $p, q \in (0, \pi)$. This is a two parameter family of processes, indexed by $(p, q) \in (0, \pi) \times (0, \pi)$. To emphasize this dependence, we write $\mu_{p,q}$ and $\gamma_1(p, q)$ for μ and γ_1, respectively. For use in part (3) of the proposition below, we denote by $\lambda_n^D = -\frac{(n+1)^2}{2}$, $n \in \mathbb{N}_0$, the eigenvalues of $L = \frac{1}{2}\frac{d^2}{dx^2}$ on $(0, \pi)$ with the Dirichlet boundary condition.

Proposition 3 *Consider Brownian motion on the interval $(0,\pi)$ with deterministic jumps from 0 to $p \in (0,\pi)$ and from π to $q \in (0,\pi)$; that is, the case $L = \frac{1}{2}\frac{d^2}{dx^2}$ on $(0,\pi)$ with $\mu_0 = \delta_p$ and $\mu_1 = \delta_q$.*

(1) The set of eigenvalues of $\mathcal{L}$ is

$$\{-\frac{2\pi^2 l^2}{(\pi+q-p)^2}, -\frac{2\pi^2 l^2}{p^2}, -\frac{2\pi^2 l^2}{(\pi-q)^2},\ l \in \mathbb{N}\} \cup \{0\}.$$

In addition,

(a) When $\frac{\pi-q}{p}$ is not rational, all eigenvalues are simple.

(b) When $\frac{\pi-q}{p} = \frac{m}{n}$, for some $m,n \in \mathbb{N}$, then all eigenvalues of the form $-\frac{2\pi^2 n^2 l^2}{p^2}$, $l \in \mathbb{N}$, are of multiplicity 2. All other eigenvalues are simple.

(2) The invariant density $\mu_{p,q}$ is given by

$$\mu_{p,q}(y) = C \begin{cases} (\pi-q)y & y \in [0, p\wedge q] \\ (\pi-q)p & p \le q,\ y \in [p,q] \\ (pq + y(\pi-(p+q)) & q \le p,\ y \in [q,p] \\ (\pi-y)p & y \in [p \vee q, \pi] \end{cases}$$

where C is a normalization constant. In particular, if $(p,q) \ne (p',q')$, then $\mu_{p,q} = \mu_{p',q'}$ if and only if $p+q=\pi$, $p'=q$ and $q'=p$.

(3)

$$\gamma_1(p,q) = -\frac{2\pi^2}{\max(p^2, (\pi-q)^2, (\pi+q-p)^2)}.$$

In particular,

(a) $\gamma_1(p,q) > -2 = \lambda_1^D$ if and only if $p<q$. Thus, whenever $(p,q) \ne (p',q')$ and $\mu_{p,q} = \mu_{p',q'}$, one has $\gamma_1(p,q) \ne \gamma_1(p',q')$.

(b) $\sup_{p,q} \gamma_1(p,q) = \lim_{p\to 0, q\to\pi} \gamma_1(p,q) = -\frac{1}{2} = \lambda_0^D$,

(c) $\min_{p,q} \gamma_1(p,q) = \gamma_1(\frac{2\pi}{3}, \frac{\pi}{3}) = -\frac{9}{2} = \lambda_2^D$.

(d) $\gamma_1(p,p) = -2 = \lambda_1^D$.

(e) $\lim_{p\to\pi, q\to 0} \gamma_1(p,q) = -2 = \lambda_1^D$.

Remark. Note that for the two parameter family of processes above, the fastest exponential rate of convergence to equilibrium, which is equal to $\frac{9}{2}$, occurs when $p = \frac{2}{3}\pi$ and $q = \frac{1}{3}\pi$. There is no slowest rate, but the infimum of the rates, which is equal to $\frac{1}{2}$, is approached as $p \to 0$ and $q \to \pi$. One can show that as $p \to 0$ and $q \to \pi$, the diffusion with random jumps converges weakly to the reflected Brownian motion on $(0,\pi)$, corresponding to the operator $\frac{1}{2}\frac{d^2}{dx^2}$ with the Neumann boundary condition. The spectral gap for this operator, which gives the exponential rate of convergence to equilibrium for the reflected diffusion, is also equal to $\frac{1}{2}$, as one would expect. When $p \to \pi$ and $q \to 0$, the rate of convergence to equilibrium approaches 2. One should be

able to show that the diffusion with random jumps converges weakly to Brownian motion on the circle (of length π). The spectral gap of the corresponding generator, which gives the exponential rate of convergence to equilibrium for Brownian motion on this circle, is equal to 2, as one would expect.
Remark. Note that for all pairs (p, q), we have $\lambda_2^D \leq \gamma_1(p, q) < \lambda_0^D$. In a preprint of this article, we conjectured that these inequalities remain in effect for Brownian motion with arbitrary jump measures ν_0 from 0 and ν_π from π. In fact the right hand inequality above has now been proved for arbitrary jump measures [LLR]. Note that for all pairs of the form (p, p), we have $\gamma_1(p, p) = \lambda_1^D$. This equality remains in effect for Brownian motion with arbitrary jump measure ν from 0 and π—see [LLR] and [BAP].

2 Proofs of Lemma 1, Propositions 1, 2 and Theorem 1.

We prove Lemma 1, then Propositions 1 and 2. After that, Theorem 1 is proved through a sequence of lemmas.

Proof of Lemma 1. Let $\mathcal{F}_n$ denote the σ-algebra generated by the process up time time τ_n. Then

$$E_\rho[e^{-\lambda\sigma_{n+1}}|\mathcal{F}_n] = E^D_{\nu_{\Theta_n}}[e^{-\lambda\tau_D}].$$

By compactness, it follows that there exists some $x_0 \in \partial D$ such that

$$\max_{\zeta\in D} E^D_{\nu_\zeta}[e^{-\lambda\tau_D}] = E^D_{\nu_{x_0}}[e^{-\lambda\tau_D}] < 1.$$

Therefore, it follows that $\lim_{n\to\infty} E_\rho e^{-\lambda\tau_n} = 0$, proving the claim. $\square$

Proof of Proposition 1. By the definition of ν,

$$\begin{aligned}\int_{\partial D} H(\nu, z)\nu_z dz &= \int_{\partial D}\int_{\partial D} H(\nu_\zeta, z)dm(\zeta)\nu_z dz = \int_{\partial D}\int_{\partial D} \tilde{H}(\zeta, z)dm(\zeta)\nu_z dz \\ &= \int_{\partial D} \nu_z dm(z) = \nu,\end{aligned}$$

where the second to last equality follows from the fact that m is $\tilde{H}$-invariant. The left hand side represents the distribution of X at time τ_1, under P_ν. This shows that under P_ν, X and $X(\tau_1 + \cdot)$ are identically distributed. Since X

coincides with the killed diffusion up to time τ_1^-, we have

$$\int_D E_y f(X(t))G(\nu, y)dy = E_\nu^D \int_0^{\tau_D} E_{Z(s)} f(X(t))ds =$$
$$E_\nu \int_0^{\tau_1} E_{X(s)} f(X(t))ds = E_\nu \int_0^{\tau_1} f(X(t+s))ds =$$
$$E_\nu \int_t^{t+\tau_1} f(X(s))ds =$$
$$E_\nu^D \int_0^{\tau_D} f(Z(s))ds + E_\nu \int_{\tau_1}^{\tau_1+t} f(X(s))ds - E_\nu \int_0^t f(X(s))ds.$$

Since X and $X(\tau_1 + \cdot)$ are identically distributed under P_ν, the last two terms cancel and we obtain

$$\int_D E_y f(X(t))G(\nu, y)dy = \int_D G(\nu, y)f(y)dy.$$

□

Proof of Proposition 2. Let $\lambda_0^D < 0$ denote the principal eigenvalue for L on D with the Dirichlet boundary condition, and let ϕ^D denote a corresponding positive eigenfunction. Let $c = \inf_{\zeta \in D} \int_D \phi_0^D(x)d\nu_\zeta(x)$. From the continuity of $\zeta \to \nu_\zeta$, it follows that $c = \int_D \phi_0^D(x)d\nu_{\zeta_0}(x)$ for some $\zeta_0 \in \partial D$. Therefore $c > 0$ and we obtain

$$\begin{aligned}
\mathcal{T}_t\phi_0^D(x) &= p(t, x, \phi_0^D) \\
&\underset{(1.2)}{\geq} \int_{\partial D} \int_0^t p^D(t-s, \nu_\zeta, \phi)dP_x(\tau_1 \leq s | \Theta_1 = \zeta)H(x, \zeta)d\zeta \\
&= \int_D \int_0^t e^{\lambda_0^D(t-s)} \int_D \phi_0^D(z)d\nu_\zeta(z)dP_x(\tau_1 \leq s|\Theta_1 = \zeta)H(x,\zeta)d\zeta \\
&ce^{\lambda_0^D t} \int_0^t dP_x(\tau_1 \leq s) = ce^{\lambda_0^D t} P_x(\tau_1 \leq s).
\end{aligned}$$

In particular, $\liminf_{x \to \partial D} |\mathcal{T}_t\phi_0^D(x) - \phi_0(x)| \geq ce^{\lambda_0^D t} \underset{t \to 0}{\to} c > 0.$ □

We now state and prove a sequence of results culminating in the proof of Theorem 1. Since the domain D has a $C^{2,\alpha}$-boundary, the function $p^D(t, x, y)$ is continuous on $(0, \infty) \times \overline{D} \times \overline{D}$, [Fri64, Theorem 3.16, page 82]. This is a key fact in the following discussion.

We begin with a technical lemma.

Lemma 2 *For all $n \in \mathbb{N}$ and $t > 0$,*

$$\lim_{\epsilon \to 0} \sup_{x \in D} P_x(\tau_n \in (t-\epsilon, t]) = 0.$$

Proof. Since $P_x(\tau_1 \in (t-\epsilon, t]) = \int_D p^D(t-\epsilon, x, y) - p^D(t, x, y)dy$, the lemma in the case that $n = 1$ follows from the uniform continuity of p^D on compact subsets of $(0, \infty) \times \overline{D} \times \overline{D}$.
We now assume that $n \geq 2$. We write $\tau_n = \tau_1 + \gamma_{n-1}$, where $\gamma_{n-1} = \sum_{k=2}^n \sigma_k$. Let $\zeta \in \partial D$. We note that the random variables τ_{n-1} under P_ζ, and γ_{n-1} under $P_x(\cdot|\Theta_1 = \zeta)$ are identically distributed. Let $u, \epsilon \geq 0$. For $\zeta \in \partial D$ and $n \geq 2$,

$$\begin{aligned} &P_\zeta(\tau_n \in [u-\epsilon, u]) \\ &\quad = \int_{\partial D} \int_0^u P_{\zeta'}(\tau_{n-1} \in [u-s-\epsilon, u-s])dP_\zeta(\tau_1 \leq s|\Theta_1 = \zeta')\tilde{H}(\zeta, \zeta')d\zeta'. \end{aligned}$$

Therefore,

$$P_\zeta(\tau_n \in [u-\epsilon, u]) \leq \sup_{\zeta' \in \partial D} \sup_{v \in [0,u]} P_{\zeta'}(\tau_{n-1} \in [v-\epsilon, v]).$$

By induction,

$$\sup_{\zeta \in \partial D} P_\zeta(\tau_n \in [u-\epsilon, u]) \leq \sup_{\zeta' \in \partial D} \sup_{v \in [0,u]} P_{\zeta'}(\tau_1 \in [v-\epsilon, v]). \tag{2.1}$$

Let

$$\Upsilon_t(\epsilon) \equiv \sup_{\zeta \in \partial D} \sup_{u \leq t} P_\zeta(\tau_1 \in [u-\epsilon, u]).$$

Since

$$\begin{aligned} &P_x(\tau_n \in (t-\epsilon, t]) \\ &\quad = \int_{\partial D} \int_0^t P_\zeta(\tau_{n-1} \in (t-s-\epsilon, t-s))dP_x(\tau_1 \leq s|\Theta_1 = \zeta)H(x, \zeta)d\zeta, \end{aligned}$$

it follows from (2.1) that

$$\sup_{x \in D} P_x(\tau_n \in (t-\epsilon, t]) \leq \Upsilon_t(\epsilon).$$

To complete the proof of the lemma, we now show that

$$\Upsilon_t(\epsilon) \underset{\epsilon \to 0}{\longrightarrow} 0. \tag{2.2}$$

Since ∂D is compact and $\zeta \to \nu_\zeta$ is continuous, the family $\{\nu_\zeta : \zeta \in \partial D\}$ is a compact subset of $\mathcal{P}$. In particular it is tight. Thus,

$$\lim_{\rho\to 0} \sup_{\zeta\in\partial D} \nu_\zeta(D^\rho) = 0, \text{ where } D^\rho = \{x \in D : \ \mathrm{dist}(x, \partial D) < \rho\}. \tag{2.3}$$

As is well known, the function $u(x,t) \equiv P_x(\tau_1 > t) = P_x(\tau_D > t)$ solves $u_t = Lu$ in $D \times (0,t)$, with initial condition $u(x,0) = 1$. In particular then, it is continuous on $[0,T] \times D$ and uniformly continuous on $[0,T] \times (D - D^\delta)$, for any $T > 0$ and $\delta > 0$. Fix $\delta > 0$. By (2.3), there exists a $\rho_\delta > 0$ such that $\nu_\zeta(D^{\rho_\delta}) \le \delta$, for all $\zeta \in \partial D$. Thus, we have

$$P_\zeta(\tau_1 \in [u-\epsilon, u]) = P_{\nu_\zeta}(\tau_1 \in [u-\epsilon, u]) \le \delta + (1-\delta) \sup_{x\in D-D^{\rho_\delta}} P_x(\tau_1 \in [u-\epsilon, u]). \tag{2.4}$$

By the above-noted uniform continuity,

$$\lim_{\epsilon\to 0} \sup_{u\le t} \sup_{x\in D-D^{\rho_\delta}} P_x(\tau_1 \in [u-\epsilon, u]) = 0. \tag{2.5}$$

From (2.4), (2.5) and the definition of $\Upsilon_t(\epsilon)$ we obtain $\limsup_{\epsilon\to 0} \Upsilon_t(\epsilon) \le \delta$. Letting $\delta \to 0$ gives (2.2). □

Lemma 3 *$\mathcal{S}$ is a strongly continuous, compact semigroup.*

Proof. We have

$$\begin{aligned} E_x f(X(t)) &= E_x[f(X(t)); \tau_1 > t] + E_x[f(X(t)); \tau_1 \le t] \\ &= \int_D p^D(t,x,y) f(y) dy \\ &+ \int_D \int_{\partial D} \int_0^t p(t-s, \nu_\zeta, y) f(y) dP_x(\tau_1 \le s | \Theta_1 = \zeta) H(x,\zeta) d\zeta dy. \end{aligned}$$

We obtain

$$p(t,x,y) = p^D(t,x,y) + \int_{\partial D} \int_0^t p(t-s, \nu_\zeta, y) dP_x(\tau_1 \le s | \Theta_1 = \zeta) H(x,\zeta) d\zeta. \tag{2.6}$$

Therefore,

$$\|\mathcal{S}_t g - g\|_1 \le \| \int_D p^D(t,x,y) g(x) dx - g(y)\|_1 + \int_D |g(x)| P_x(\tau_1 \le t) dx.$$

By the bounded convergence theorem, the second term on the righthand side above converges to 0 as $t \to 0$. As for the first term, let $f \in C_c(D)$ with $\|f - g\|_1 \le \epsilon$. Then

$$\|p^D(t,g,y) - g\|_1 \le \|p^D(t,f,y) - f\|_1 + \|p^D(t, g-f, y)\|_1 + \|f - g\|_1.$$

Using [Pin95, Theorem 3.4.1], one sees that the first term goes to 0 as $t \to 0$. Each of the other two terms is bounded by $\|g - f\|_1 \leq \epsilon$. Since ϵ is arbitrary, this concludes the proof that $\mathcal{S}$ is strongly continuous.

Next, we prove that $\mathcal{S}$ is compact. Fix $t > 0$. For $n \in \mathbb{N}_0$ and $\epsilon \geq 0$, define the linear operator $\mathcal{T}_{n,\epsilon}$ by $\mathcal{T}_{n,\epsilon} f(x) = E_x[f(X(t)); \tau_n + \epsilon \leq t < \tau_{n+1}]$. Then $\mathcal{T}_{n,\epsilon} = \mathcal{S}^*_{n,\epsilon}$, where

$$\mathcal{S}_{0,0} g(y) = \int_D p^D(t, x, y) g(x) dx,$$

and for $n \in \mathbb{N}$,

$$\mathcal{S}_{n,\epsilon} g(y) = \int_D \int_{\partial D} \int_0^{t-\epsilon} p^D(t - s, \nu_\zeta, y) g(x) dP_x(\tau_n \leq s | \Theta_n = \zeta) \tilde{H}^n(x, \zeta) d\zeta dx.$$

By duality, $\|\mathcal{S}_{n,\epsilon}\|_1 = \sup_{x \in D} P_x(t \in [\tau_n + \epsilon, \tau_{n+1}])$. For every $R \in \mathbb{N}$, $\mathcal{S}_t = \sum_{n=0}^R \mathcal{S}_{n,0} + \sum_{R+1}^\infty \mathcal{S}_{n,0}$. The dual of $\sum_{R+1}^\infty \mathcal{S}_{n,0}$ is the operator that maps f to $E_x[f(X(t)); \tau_{R+1} \leq t]$. The L^∞-norm of this operator is bounded above by $\sup_{x \in D} P_x(\tau_{R+1} \leq t)$. However,

$$P_x(\tau_{R+1} \leq t) \leq \sup_{\zeta \in \partial D} P_\zeta(\tau_R \leq t) = \sup_{\zeta \in \partial D} P_\zeta(e^{-\tau_R} \geq e^{-t}).$$

Repeating the argument in the proof of Lemma 1, we can show that

$$\sup_{\zeta \in \partial D} E_\zeta[e^{-\tau_R}] \leq \alpha^R,$$

for some $\alpha \in (0, 1)$. Therefore, it follows that $\sup_{x \in D} P_x(\tau_{R+1} \leq t)$ converges to 0 as $R \to \infty$ exponentially fast. Thus by duality, $\mathcal{S}_t$ is the limit of $\sum_{n=0}^R \mathcal{S}_{n,0}$ in the operator norm. Since the subspace of compact operators is closed with respect to the operator norm, it is sufficient to prove compactness for the partial sums $\sum_{n=0}^R \mathcal{S}_{n,0}$. The latter is boils down to showing that $\mathcal{S}_{n,0}$ is compact, for all $n \geq 0$. We recall that p^D is uniformly continuous on $[\epsilon, t] \times \overline{D} \times \overline{D}$. In particular, for every $\delta > 0$, there exists $\eta > 0$ such that whenever $|y - y'| \leq \eta$, one has $|p^D(s, x, y) - p^D(s, x, y')| \leq \delta$, for all $(s, x) \in [\epsilon, t] \times \overline{D}$. Fix $n \geq 1$. Then we have

$$|S_{n,\epsilon} g(y)| \leq \max_{(s,x,y) \in [\epsilon,t] \times \overline{D} \times \overline{D}} p^D(s, x, y) \|g\|_1.$$

and

$$|S_{n,\epsilon} g(y) - S_{n,\epsilon} g(y')| \leq \delta \|g\|_1, \text{ if } |y - y'| \leq \eta.$$

These inequalities show that $\mathcal{S}_{n,\epsilon}$ maps bounded sets in L^1 to bounded, equicontinuous sets in $C(\overline{D})$. Consequently, $\mathcal{S}_{n,\epsilon}$ is compact. The same reasoning shows that $\mathcal{S}_{0,0}$ is compact. In order to complete the proof, we note that for

$n \in \mathbb{N}$,

$$\|\mathcal{S}_{n,\epsilon} - \mathcal{S}_{n,0}\|_1 = \|\mathcal{T}_{n,\epsilon} - \mathcal{T}_{n,0}\|_\infty = \sup_{x \in D} P_x(\tau_n \in (t-\epsilon, t]).$$

By Lemma 2, the righthand side goes to 0 as $\epsilon \to 0$. $\square$

We continue with some notation. Let $\mathcal{B}$ be a (possibly unbounded) linear operator on some Banach space Ξ. We let $\mathcal{D}(\mathcal{B})$ denote the domain of $\mathcal{B}$. We define $\sigma(\mathcal{B})$, the spectrum of $\mathcal{B}$ as the set of points $\lambda \in \mathbb{C}$ for which $\lambda - \mathcal{B}$ does not possess a bounded inverse on Ξ. We define $\sigma_p(\mathcal{B})$, the point spectrum of $\mathcal{B}$, as the set of eigenvalues of $\mathcal{B}$. We also recall that a linear operator on Ξ is called closed if its graph (defined above Theorem 1) is closed in the product topology. Finally, if Ξ is some vector space, we write Id_Ξ for the identity mapping on Ξ.

Since $\mathcal{S}$ is a strongly continuous semigroup, by the Hille-Yosida theorem, it possesses a densely defined closed generator $\mathcal{A}$. The next lemma is essentially [Paz83, Theorem 2.2.4]. However the statement of the theorem is only on the spectra and we also need a statement on the eigenfunctions.

Lemma 4

(1) Let $\rho \in \sigma_p(\mathcal{A})$ and let φ be a corresponding eigenfunction. Then $\mathcal{S}_t\varphi = e^{\rho t}\varphi$ for all $t \geq 0$.
(2) Let $e^{\rho t} \in \sigma_p(\mathcal{S}_t)$ for some $t > 0$ and let φ be a corresponding eigenfunction. For $k \in \mathbb{Z}$ let $x_k = \int_0^t e^{-i2\pi ks/t - \rho s}\mathcal{S}_s\varphi ds$. Then not all x_k are zero. In addition, if $x_k \neq 0$ then $\mathcal{A}x_k = (\rho + i2\pi k/t)x_k$.

Proof. For $\mu \in \mathbb{C}$ consider the family $\{e^{-\mu t}\mathcal{S}_t : t \geq 0\}$. As can be readily seen, this is a strongly continuous semigroup on L^1 and its generator is $\mathcal{A} - \mu$. Then by [Paz83, Theorem 1.2.4]

$$e^{-\mu t}\mathcal{S}_t = Id_{L^1} + (\mathcal{A} - \mu)\int_0^t e^{-\mu s}\mathcal{S}_s ds \qquad (2.7)$$

and on $\mathcal{D}(\mathcal{A})$, the second term on the righthand side is equal to $\int_0^t e^{-\mu s}\mathcal{S}_s(\mathcal{A} - \mu)ds$.

(1) Setting $\mu = 0$ in (2.7) we obtain $S_t\varphi = \varphi + \rho\int_0^t \mathcal{S}_s\varphi ds$ for all $t \geq 0$. Since $s \to \mathcal{S}_s\varphi$ is a continuous function, this implies that $\mathcal{S}_t\varphi = e^{\rho t}\varphi$ for all $t \geq 0$.
(2) Note that $\{x_k : k \in \mathbb{Z}\}$ are Fourier coefficients of the continuous, non-zero mapping $s \to e^{-\rho s}\mathcal{S}_s\varphi$. Hence for some k_0, $x_{k_0} \neq 0$. Let $\mu = i2\pi k_0/t + \rho$. Note

that $e^{-\mu t} = e^{-\rho t}$ and then $e^{-\mu t}\mathcal{S}_t\varphi = \varphi$. But by (2.7) we have

$$\varphi = \varphi + (\mathcal{A} - \mu)\int\limits_0^t e^{-i2\pi k_0 s/t - \rho s}\mathcal{S}_s\varphi ds = \varphi + (\mathcal{A} - \mu)x_{k_0},$$

completing the proof.

□

Lemma 5

(1) $\sigma(\mathcal{A})$ consists of a countable set of eigenvalues having no accumulation points, all with real part ≤ 0.
(2) For $t \geq 0$, $\sigma(\mathcal{S}_t) = e^{t\sigma(\mathcal{A})} \cup \{0\}$.
(3) The only eigenvalue of $\mathcal{A}$ with nonnegative real part is 0. It is simple and the corresponding eigenspace is spanned by μ.

Proof.

(1) Since $\mathcal{A}$ is the generator of a contraction semigroup, it follows that its spectrum consists only of complex numbers with real part ≤ 0.
For $\lambda \notin \sigma(\mathcal{A})$, let U_λ denote the resolvent of $\mathcal{A}$, that is the bounded inverse of $\lambda - \mathcal{A}$. Then

$$(\lambda - \mathcal{A})U_\lambda = Id_{L^1} \text{ and } U_\lambda(\lambda - \mathcal{A}) = Id_{\mathcal{D}(\mathcal{A})}. \tag{2.8}$$

It is known that

$$U_\lambda = \int\limits_0^\infty e^{-\lambda t}\mathcal{S}_t dt, \quad \text{if } \mathrm{Re}\ \lambda > 0,$$

where the integral converges in the operator norm. Since $\mathcal{S}$ is compact, this implies that that U_λ is compact for $\mathrm{Re}\ \lambda > 0$. Fix $\lambda > 0$. Note that (2.8) implies that $\sigma_p(\mathcal{A}) = \{(\lambda - \rho)^{-1} : \rho \in \sigma_p(U_\lambda)\}$. The operator U_λ is compact therefore $\sigma_p(U_\lambda)$ is countable and its only accumulation point is 0. However $0 \notin \sigma_p(U_\lambda)$ because by the first equality in (2.8) U_λ is one-to-one. Therefore $\sigma_p(\mathcal{A})$ is a countable set with no accumulation points.

Next we show that $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$. Since $\mathcal{A}$ is a closed operator, it follows from the closed graph theorem that whenever $\rho - \mathcal{A}$ is one-to-one and onto, then its inverse is bounded. Therefore if $\rho \in \sigma(\mathcal{A})$, then either $\rho - \mathcal{A}$ is not one-to-one, in which case $\rho \in \sigma_p(\mathcal{A})$, or else $\rho - \mathcal{A}$ is one-to-one, but is not onto. Fix $\lambda > 0$. We have

$$(\rho - \mathcal{A})U_\lambda = (\lambda - \mathcal{A})U_\lambda + (\rho - \lambda)U_\lambda = Id_{L^1} + (\rho - \lambda)U_\lambda. \tag{2.9}$$

This operator is not invertible because by assumption it is not onto. Since U_λ is compact, the Fredholm alternative guarantees that $(\lambda - \rho)^{-1} \in \sigma_p(U_\lambda)$, hence $\rho \in \sigma_p(\mathcal{A})$. This contradicts the assumption on ρ.

Finally, it follows from [Lax02, Theorem 13, p. 437] that $\sigma(\mathcal{A})$ is in fact countably infinite.

(2) An immediate consequence of Lemma 4 is that

$$e^{\sigma_p(\mathcal{A})} \subset \sigma_p(\mathcal{S}_t) \subset e^{\sigma_p(\mathcal{A})} \cup \{0\}.$$

But by part (1) $\sigma_p(\mathcal{A}) = \sigma(\mathcal{A})$ and by the compactness of $\mathcal{S}_t$, $\sigma(\mathcal{S}_t) = \sigma_p(\mathcal{S}_t) \cup \{0\}$.

(3) Suppose that the purely imaginary number $i\theta$ is an eigenvalue of $\mathcal{A}$. Fix $t > 0$ and let φ be an eigenfunction of $\mathcal{S}_t$ corresponding to the eigenvalue $e^{i\theta t}$. There is no loss of generality assuming $\|\varphi\|_1 = 1$. By definition, $\mathcal{S}_t|\varphi|$ is a probability density on $\overline{D}$, therefore $\|\mathcal{S}_t|\varphi|\|_1 = 1$. In addition, $|\mathcal{S}_t\varphi| = |e^{i\theta t}\varphi| = |\varphi|$, so $\|\mathcal{S}_t\varphi\|_1 = 1$. As a consequence, $\mathcal{S}_t|\varphi| = |\mathcal{S}_t\varphi|$, a.e. D. Fix some $y \in D$ for which the last equality holds. Explicitly,

$$\int_D |\varphi(x)| p(t,x,y)dx = |\int_D \varphi(x)p(t,x,y)dx|.$$

Since $p(t,\cdot,y) > 0$ on D, it follows immediately that $\varphi = \alpha|\varphi|$ a.e. D, for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$ (c.f. [Rud, Theorem 1.39(c)]). Hence Arg φ is constant a.e. D. There is no loss of generality assuming that φ is real-valued. This guarantees that $e^{i\theta t}$ is real-valued. Since t is arbitrary, it follows that $\theta = 0$. Next, suppose that φ_1, φ_2 are eigenfunctions of $\mathcal{A}$ corresponding to the eigenvalue 0. Since both have constant argument a.e. there is no loss of generality assuming that they are both non-negative a.e. . Furthermore, we may also assume that $\int_D \varphi_1 dx = \int_D \varphi_2 dx = 1$. Since $\mathcal{A}(\varphi_1 - \varphi_2) = 0$, it follows that $\varphi_1 - \varphi_2$ has a constant argument. Without loss of generality, $\varphi_1 - \varphi_2 \geq 0$, a.e. . The normalization assumption then implies $\varphi_1 = \varphi_2$. Hence the kernel of $\mathcal{A}$ is one dimensional. Since $\mu \in L^1$ and $\int_D f(x)d\mu(x) = \int_D \mathcal{T}_t f(x)\mu(x)dx = \int_D f(x)\mathcal{S}_t\mu(x)dx$, it follows that μ is in the domain of $\mathcal{A}$ and that $\mathcal{A}\mu = 0$.

□

Let $\mathcal{A}^*$ denote the dual of $\mathcal{A}$. More precisely,

$$\mathcal{D}(\mathcal{A}^*) = \Big\{ f \in L^\infty : \exists \varphi_f \in L^\infty \text{ such that } \int_D \mathcal{A}g(x)f(x)dx = \int_D g(x)\varphi_f(x)dx,\ \forall g \in \mathcal{D}(\mathcal{A}) \Big\},$$

and for $f \in \mathcal{D}(\mathcal{A}^*)$ we let $\mathcal{A}^* f = \varphi_f$. This mapping is well-defined because $\mathcal{D}(\mathcal{A})$ is dense in L^1. Let $f \in \mathcal{D}(\mathcal{A}^*)$. Then for all $g \in \mathcal{D}(\mathcal{A})$ we have

$$\int_D \mathcal{A}g(x)\mathcal{T}_t f(x)dx = \int_D (\mathcal{A}\mathcal{S}_t)g(x)f(x)dx = \int_D g(x)(\mathcal{T}_t\mathcal{A}^*)f(x)dx.$$

The first equality if due to the fact that a strongly continuous semigroup commutes with its generator on its domain. Therefore $\mathcal{T}_t f \in \mathcal{D}(\mathcal{A}^*)$. Let M^+ denote the closure of $\mathcal{D}(\mathcal{A}^*)$. Then $\mathcal{T}$ maps M^+ to M^+. In particular, the restriction of $\mathcal{T}$ to M^+ is a semigroup on M^+. We denote it by $\mathcal{T}^+$.

Lemma 6

(1) $\mathcal{T}^+$ is strongly continuous.
(2) Let $\mathcal{A}^+$ denote the generator of $\mathcal{T}^+$. Then $\mathcal{L}$ is a core for $\mathcal{A}^+$.
(3) $\sigma(\mathcal{A}^+) = \sigma_p(\mathcal{A}^+) = \sigma_p(\mathcal{L}) = \sigma(\mathcal{A})$.

Proof. By [Paz83, Theorem 1.10.14], $\mathcal{T}^+$ is strongly continuous and its generator, $\mathcal{A}^+$, is the restriction of $\mathcal{A}^*$ to the subspace $\mathcal{D}(\mathcal{A}^+)$, defined through

$$\mathcal{D}(\mathcal{A}^+) = \{f \in \mathcal{D}(\mathcal{A}^*) : \mathcal{A}^* f \in M^+\}.$$

This proves part (1).

We turn to the proof of part (2). Let $R_\lambda = U_\lambda^*$. If Re $\lambda > 0$ then

$$R_\lambda f = \int_0^\infty e^{-\lambda t} \mathcal{T}_t f dt.$$

Let $G_\lambda(x,y) = \int_0^\infty e^{-\lambda t} p^D(t,x,y)dt$ denote Green's function for $L-\lambda$ on D with the Dirichlet boundary condition on ∂D. Then by (2.6)

$$\begin{aligned} R_\lambda f(x) &= G_\lambda f(x) + \\ &+ \underbrace{\int_D \int_{\partial D} \int_0^\infty e^{-\lambda t} \int_0^t p(t-s, \nu_\zeta, y) f(y) dP_x(\tau_1 \le s | \Theta_1 = \zeta) H(x,\zeta) dt d\zeta dy}_{(*)}. \end{aligned}$$

Define the linear operator J_λ on L^∞ by letting

$$\begin{aligned} J_\lambda u(x) &= \int_{\partial D} \int_D u(z) d\nu_\zeta(z) E_x[e^{-\lambda \tau_1} | \Theta_1 = \zeta] H(x,\zeta) d\zeta \\ &= E_x[e^{-\lambda \tau_1} \int_D u(z) d\nu_{X(\tau_1^-)}] \end{aligned} \tag{2.10}$$

By changing the order of integration we show that $(*)$ is equal to

$$\begin{aligned} &\int_{\partial D} \int_0^\infty e^{-\lambda s} \int_D \int_s^\infty e^{-\lambda(t-s)} p(t-s, \nu_\zeta, y) f(y) dt dy dP_x(\tau_1 \le s | \Theta_1 = \zeta) H(x,\zeta) d\zeta \\ &= (J_\lambda R_\lambda) f(x) \end{aligned}$$

By [Paz83, Lemma 1.10.2], $\sigma(\mathcal{A}^*) \subset \sigma(\mathcal{A})$ and for all $\lambda \notin \sigma(\mathcal{A})$, R_λ is the inverse of $\lambda - \mathcal{A}^*$. The latter statement means

$$(\lambda - \mathcal{A}^*)R_\lambda = Id_{L^\infty} \text{ and } R_\lambda(\lambda - \mathcal{A}^*) = Id_{\mathcal{D}(\mathcal{A}^*)}. \tag{2.11}$$

We continue according to the following steps.

Step 0: Smoothness of $R_\lambda f$.

$$\{R_\lambda f : f \in L^\infty\} \subset \mathcal{D}_0. \tag{2.12}$$

Let $f \in L^\infty$. We first show that $G_\lambda f \in C_0(\overline{D})$. Fix $x_0 \in \overline{D}$. For $x \in \overline{D}$ we have

$$|G_\lambda f(x) - G_\lambda f(x_0)| \leq \|f\|_\infty \int_D |h_\epsilon(x,y) - h_\epsilon(x_0,y)|dy + 2\|f\|_\infty(\epsilon + \lambda^{-1}e^{-\lambda\epsilon^{-1}}),$$

where $h_\epsilon(x,y) = \int_\epsilon^{\epsilon^{-1}} e^{-\lambda t}p^D(t,x,y)dt$. Note that the continuity of p^D on $[\epsilon,\infty)\times\overline{D}\times\overline{D}$ implies that h_ϵ is continuous on $\overline{D}\times\overline{D}$. By letting $x \to x_0$, the first term on the righthand side converges to 0 by the bounded convergence theorem. Then by letting $\epsilon \to 0$ the second term converges to 0. This proves continuity of $G_\lambda f$ on $\overline{D}$. Since $p^D(t,x,y) = 0$ for all $x \in \partial D$ and all $t > 0$ and $y \in \overline{D}$, we also have $G_\lambda f(x) = 0$ on ∂D.

We now consider $J_\lambda R_\lambda f$. Let $u(x,\zeta) = E_x[e^{-\lambda\tau_1}|\Theta_1 = \zeta]H(x,\zeta)$. For each fixed $\zeta \in \partial D$, $u(\cdot,\zeta)$ is continuous on D because it is a harmonic function for $\lambda - L$. Furthermore, if $U \subset\subset D$ then $\sup_{(x,\zeta)\in U\times\partial D} u(x,\zeta) < \infty$. Therefore by bounded convergence $J_\lambda u(x) \in C(D)$ for every $u \in L^\infty$. In addition, it follows directly from the definition (2.10) that $\|J_\lambda u\|_\infty \leq \|u\|_\infty$. Therefore $R_\lambda f \in C(D) \cap L^\infty$. As a consequence, the mapping $r : \partial D \to \mathbb{R}$ defined by $r(\zeta) = \int_D R_\lambda f(z)d\nu_\zeta(z)$ is continuous. By (2.10), $J_\lambda R_\lambda f(x) = E_x[e^{-\lambda\tau_1}r(X(\tau_1^-))]$, and we observe that $\lim_{x\to\zeta\in\partial D} J_\lambda R_\lambda f(x) = r(\zeta)$, proving (2.12). Furthermore, note that $J_\lambda R_\lambda f \in C^2(D)$ for all $f \in L^\infty$. If we assume $f \in C(D)\cap L^\infty$, then also $G_\lambda f \in C^2(D)$ and we get

$$\{R_\lambda f : f \in C(D) \cap L^\infty\} \subset \mathcal{D}. \tag{2.13}$$

Step 1: $L = \mathcal{A}^*$ on $\{R_\lambda f : f \in C(\overline{D})\}$.

Fix $f \in C(\overline{D})$ and let $u = R_\lambda f$. Then $(\lambda - \mathcal{A}^*)u = f$. But

$$(\lambda - L)u = (\lambda - L)G_\lambda f + (\lambda - L)J_\lambda R_\lambda u = f + 0.$$

Step 2: $\{R_\lambda f : f \in C(\overline{D})\} = \mathcal{D}$.

By (2.13), $\{R_\lambda f : f \in C(\overline{D})\} \subset \mathcal{D}$. Let $u \in \mathcal{D}$ and set $f = (\lambda - L)u$. Let $v = u - R_\lambda f$. Then $v \in \mathcal{D}$ and $(\lambda - L)v = 0$. By Feynman-Kac, $v(x) = E_x[e^{-\lambda\tau_1}v(X(\tau_1^-))]$. Let $M = \sup_{x\in\overline{D}} v(x)$. The condition $v \in \mathcal{D}$ guarantees that M is attained. Clearly, $v(x) \leq ME_xe^{-\lambda\tau_1}$. Hence either $M = 0$ or that

M is only attainable on ∂D. The latter case is impossible because for every $\zeta \in \partial D$, $v(\zeta) = \int\limits_D v(z)d\nu_\zeta(z)$ and $\nu_\zeta \in \mathcal{P}(D)$. Thus, $M = 0$. Repeating the argument for $-v$, we obtain $v = 0$. Hence $u = R_\lambda f$, showing that $\mathcal{D} \subset \{R_\lambda f : f \in C(\overline{D})\}$.

Step 3: $\widetilde{\mathcal{D}}$ is a core for $\mathcal{A}^+$.

By [EK, Proposition 1.3.1], we need to show that (i) $\widetilde{\mathcal{D}}$ is dense in $\mathcal{D}(\mathcal{A}^+)$; and (ii) $\{(\lambda - \mathcal{A}^+)\varphi : \varphi \in \widetilde{\mathcal{D}}\}$ is dense in $M^+ = \overline{\mathcal{D}}(\mathcal{A}^+)$.

Since $\widetilde{\mathcal{D}} \subset \mathcal{D} \subset \mathcal{D}(\mathcal{A}^*)$ Steps 1 and 2 guarantee that $\mathcal{A}^* = L$ on $\widetilde{\mathcal{D}}$. For $f \in \widetilde{\mathcal{D}}$, $Lf \in \overline{\mathcal{D}}$ by definition. Therefore $\widetilde{\mathcal{D}} \subset \mathcal{D}(\mathcal{A}^+)$.

Since $\mathcal{T}^+$ is strongly continuous, $\cap_{n=1}^{\infty}\mathcal{D}((\mathcal{A}^+)^n)$ is dense in M^+ (c.f. [Paz83, Theorem 1.2.7]). By (2.11) $\mathcal{D}((\mathcal{A}^+)^3) = \{R_\lambda^3\varphi : \varphi \in L^\infty\}$. By (2.12) and (2.13), $R_\lambda^2\varphi \in \mathcal{D}$ for all $\varphi \in L^\infty$. Hence $R_\lambda^3\varphi \in \widetilde{\mathcal{D}}$. In particular, $\mathcal{D}(\mathcal{A}^+) \cap \widetilde{\mathcal{D}}$ is dense in M^+. Since we proved above that $\widetilde{\mathcal{D}} \subset \mathcal{D}(\mathcal{A}^+)$, we conclude that $\widetilde{\mathcal{D}}$ is a dense subset of $\mathcal{D}(\mathcal{A}^+)$, which is (i).

Let $f \in \overline{\mathcal{D}}$. By Step 1, $(\lambda - L)R_\lambda f = f$, hence $R_\lambda \in \widetilde{\mathcal{D}} \subset \mathcal{D}(\mathcal{A}^+)$. But also $(\lambda - \mathcal{A}^+)R_\lambda f = f$. Since f is arbitrary, $\{(\lambda - \mathcal{A}^+)\varphi : \varphi \in \widetilde{\mathcal{D}}\} \subseteq \overline{\mathcal{D}}$, which is (ii).

We turn to prove part (3). The semigroup $\mathcal{T}^+$ is compact because $\mathcal{T}$ is compact and $\mathcal{A}^+$ is its generator. Thus, the proof of Lemma 5-(1) applies here as well. For the second equality note that by part (2) $\sigma_p(\mathcal{L}) \subset \sigma_p(\mathcal{A}^+)$. On the other hand, if $\rho \in \sigma_p(\mathcal{A}^+)$ and φ is a corresponding eigenfunction, then $R_\lambda\varphi = (\lambda - \rho)^{-1}\varphi$ and the argument used to establish condition (ii) in Step 3 above shows that $\varphi \in \widetilde{\mathcal{D}}$. Hence $\rho \in \sigma_p(\mathcal{L})$.

To complete the proof we show $\sigma_p(\mathcal{A}) = \sigma_p(\mathcal{A}^+)$. Let $\rho \in \sigma_p(\mathcal{A})$ and let φ be a corresponding eigenfunction. Clearly, $U_\lambda\varphi = (\lambda - \rho)^{-1}\varphi$. Therefore

$$0 = \int\limits_D (Id_{L^1} - (\lambda - \rho)^{-1}U_\lambda)\varphi(x)f(x)dx = \int\limits_D \varphi(x)(Id_{L^\infty} - (\lambda - \rho)^{-1}R_\lambda)f(x)dx \tag{2.14}$$

for all $f \in L^\infty$. Since $\varphi \neq 0$, there exits $f_\varphi \in L^\infty$ such that $\int\limits_D \varphi(x)f(x)dx = 1$. Hence $(Id_{L^\infty} - (\lambda - \rho)^{-1}R_\lambda)$ is not onto. Since $R_\lambda = U_\lambda^*$ and U_λ is compact, R_λ is compact and then the Fredholm alternative implies that $(\lambda - \rho)^{-1} \in \sigma_p(R_\lambda)$. Let f be a corresponding eigenfunction. Then $\mathcal{A}^*f = \rho f$. In addition, since $f, \mathcal{A}^*f \in \mathcal{D}(\mathcal{A}^*)$, by definition of $\mathcal{A}^+$, $f \in \mathcal{D}(\mathcal{A}^+)$. Therefore $\rho \in \sigma_p(\mathcal{A}^+)$. Suppose that $\rho \in \sigma_p(\mathcal{A}^+)$ and let f be a corresponding eigenfunction. Then (2.14) holds for all $\varphi \in L^1$. This implies that $Id_{L^1} - (\lambda - \rho)^{-1}U_\lambda$ is not onto. Therefore by the Fredholm alternative $\rho \in \sigma_p(\mathcal{A})$. $\square$

Proof of Theorem 1. We need only prove (1.4) because the rest of the theorem follows from Lemma 5-(1)(3) and Lemma 6. Let $F_0 = \{u \in L^1 : \int\limits_D u(y)dy = 0\}$.

Define the operator I_0 on L^1 by letting

$$I_0 g = g - \mu \int_D g(y)dy.$$

It follows from the definition that I_0 is a projection (i.e. $I_0^2 = I_0$) onto F_0. Let $\mathcal{S}_t^0 = \mathcal{S}_t|_{F_0}$. We note that for $g \in F_0$,

$$\int_D \mathcal{S}_t^0 g(y)dy = \int_D \mathcal{S}_t g(y)dy = \int_D g(y)\mathcal{T}_t 1 dy = \int_D g(y)dy = 0.$$

Therefore $\mathcal{S}^0$ is a strongly continuous compact semigroup on F_0. Let γ be an eigenvalue of $\mathcal{S}_t$ and let φ denote its corresponding eigenfunction. We have

$$\gamma \int_D \varphi(y)dy = \int_D \mathcal{S}_t\varphi(y)dy = \int_D \varphi(y)\mathcal{T}_t 1(y)dy = \int_D \varphi(y)dy.$$

If $\gamma \neq 1$, this means that $\varphi \in F_0$, so $\gamma \in \sigma_p(\mathcal{S}_t^0)$. Suppose that $\gamma = 1$. By by Lemma 4-(2), there exists $k \in \mathbb{Z}$ such that $\mathcal{A}x_k = i2\pi k/t x_k$ and $x_k = \int_0^t e^{-i2\pi ks/t}\mathcal{S}_s\varphi ds$ is non-zero. However Lemma 5-(3) implies that $k = 0$ and $x_k = c\mu$ for some non-zero constant c. In particular, $\int_D x_k(y)dy = c \neq 0$. Since F_0 is invariant under $\mathcal{S}$, the definition of x_k implies that $\varphi \notin F_0$. Thus, we have proved that

$$\sigma(\mathcal{S}_t^0) = \sigma(\mathcal{S}_t)\backslash\{1\} = \{e^{t\rho} : \rho \in \sigma(\mathcal{A}),\ \text{Re } \rho < 0\}, \tag{2.15}$$

where the second equality follows from Lemma 5-(2).

Let R denote the spectral radius of $\mathcal{S}_1^0$. From (2.15) we have

$$\ln R = \sup\{\text{Re } \rho : \rho \in \sigma(\mathcal{A}),\ \text{Re } \rho < 0\}. \tag{2.16}$$

Since $\sup_{||f||_1 \leq 1, f \in F_0} ||\mathcal{S}_t^0 f|| = \sup_{||f||_1 \leq 1} ||\mathcal{S}_t I_0 f||$, it follows that

$$\ln R = \lim_{t\to\infty} \frac{1}{t} \ln ||\mathcal{S}_t I_0\|_1. \tag{2.17}$$

From (2.16), (2.17), Lemma 6-(3) and the definition of γ_1, we conclude that

$$\gamma_1 = \lim_{t\to\infty} \frac{1}{t} \ln ||\mathcal{S}_t I_0\|_1. \tag{2.18}$$

Next, note that

$$\mathcal{S}_t g - \mu \int_D g(y)dy = \mathcal{S}_t I_0 g.$$

This gives,

$$\sup_{\|g\|_1 \leq 1} \|\mathcal{S}_t g - \mu \int_D g(y)dy\|_1 = \|\mathcal{S}_t I_0\|_1.$$

Hence, from (2.18) we have

$$\gamma_1 = \lim_{t\to\infty} \frac{1}{t} \ln \sup_{\|g\|_1\le 1} \|\mathcal{S}_t g - \mu \int_D g(y)dy\|_1.$$

Finally, by duality

$$\sup_{\|g\|_1\le 1} \|\mathcal{S}_t g - \mu \int_D g(y)dy\|_1 = \sup_{\|f\|_\infty\le 1} \|\mathcal{T}_t f - \int_D f(x)\mu(x)dx\|_\infty.$$

□

3 Proof of Proposition 3

Proof. By shifting the interval $[0,\pi]$, we may assume that $L = \frac{1}{2}\frac{d^2}{dx^2}$ on $[-p/2, \pi - p/2]$ and that the jump measures are $\mu_{-\frac{p}{2}} = \delta_{\frac{p}{2}}$ and $\mu_{\pi-\frac{p}{2}} = \delta_{q-\frac{p}{2}}$. Any eigenfunction of $\frac{1}{2}\frac{d^2}{dx^2}$ is of the form $u(x) = Ae^{ikx} + Be^{-ikx}$ for some complex constants A, B, k. Such a function will be an eigenfunction for $\mathcal{L}$ if and only if

$$u(-p/2) = u(p/2),\ u(q-p/2) = u(\pi - p/2). \tag{3.1}$$

The corresponding eigenvalue is then $-\frac{1}{2}k^2$.
Let $z = e^{-ikp/2}$. The first boundary condition in (3.1) may be written as $Az + Bz^{-1} = Az^{-1} + Bz$. Therefore, $(A-B)z^2 + (B-A) = 0$. Thus, $z^2 = 1$ or $A = B$. We split into cases:

(1) $z^2 = 1$. Here $kp/2 = \pi l$, for some $l \in \mathbb{Z}$; therefore $k = \frac{2\pi l}{p}$. Let $a = e^{ik(q-p/2)} - e^{ik(\pi-p/2)}$. Since k is real, the second boundary condition in (3.1) then reads $Aa + B\overline{a} = 0$. Therefore,
- if $a \neq 0$, then $A = -\frac{\overline{a}}{a}B$;
- if $a = 0$, then A and B be can be arbitrarily chosen, as long as they are not both 0. Note that $a = 0$ if and only if $k(\pi - q)$ is an integer multiple of 2π. Thus, $a = 0$ if and only if $k = 0$ or $l\frac{\pi-q}{p}$ is an integer. When $k = 0$, the corresponding eigenspace is spanned by the constant function 1. When $k \neq 0$, the eigenspace is two dimensional and is spanned by e^{ikx} and e^{-ikx}.

(2) $A = B$. There is not loss of generality assuming that $A = 1$. In order to find the possible choices for k, let $z = e^{ik(q-p/2)}$ and let $z' = e^{ik(\pi-p/2)}$. The second boundary condition in (3.1) may be written as $z + \frac{1}{z} = z' + \frac{1}{z'}$. Hence, $z - z' = \frac{1}{z'} - \frac{1}{z} = \frac{z-z'}{zz'}$. As a result, either $z = z'$ or $zz' = 1$. Thus, either $e^{ik(q-p/2)} = e^{ik(\pi-p/2)}$ or $e^{ik(q-p/2)}e^{ik(\pi-p/2)} = 1$. In the former case, $k(\pi - q) = 2\pi l$ for some $l \in \mathbb{Z}$, hence $k = \frac{2\pi l}{\pi - q}$. In the latter case, $k(\pi + q - p) = 2\pi l$ for some $l \in \mathbb{Z}$, hence $k = \frac{2\pi l}{\pi+q-p}$.

Summarizing the discussion, we have shown that u is an eigenfunction for $\mathcal{L}$ if and only if

$$k \in \{\frac{2\pi l}{p}, \frac{2\pi l}{\pi - q}, \frac{2\pi l}{\pi + q - p}, l \in \mathbb{Z}\}. \tag{3.2}$$

We have also shown that all eigenvalues are simple, except for the case where $\frac{\pi-q}{p} = \frac{m}{n}$, for some $m, n \in \mathbb{N}$, where all k of the form $\frac{2\pi nl}{p}$, for some $l \in \mathbb{Z}$, correspond to eigenvalues of multiplicity 2.

We now prove the formula for the invariant density. We return to the original notation on the interval $[0, \pi]$. The boundary process $\{\Theta_n : n \in \mathbb{N}\}$ is a Markov chain on the state space $\{0, \pi\}$. Its transition function, $\tilde{H}$, satisfies $\tilde{H}(0,0) = P_p(X(\tau_1) = 0) = \frac{\pi-p}{\pi}$ and $\tilde{H}(\pi, 0) = P_q(X(\tau_1) = 0) = \frac{\pi-q}{\pi}$. Thus $\tilde{H}$ can be represented by the matrix

$$\begin{pmatrix} \frac{\pi-p}{\pi} & \frac{p}{\pi} \\ \frac{\pi-q}{\pi} & \frac{q}{\pi} \end{pmatrix}.$$

A straightforward calculation shows that m, the invariant probability for $\tilde{H}$, satisfies

$$m(0) = \frac{\pi - q}{\pi + p - q}, \; m(\pi) = \frac{p}{\pi + p - q}.$$

We recall that G, the Green's function for $p^D(t, x, y)$, is given by

$$G(x, y) = \begin{cases} \frac{2}{\pi}(\pi - x)y & y \leq x; \\ \frac{2}{\pi}(\pi - y)x & y > x. \end{cases} \tag{3.3}$$

By Proposition 1, $\mu(y) = G(\nu, y)$, where $\nu = m(0)\delta_p + m(\pi)\delta_q$. The result follows by direct substitution. $\square$

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