EXPLICIT AND ALMOST EXPLICIT SPECTRAL CALCULATIONS FOR DIFFUSION OPERATORS

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Abstract. The diffusion operator

$$H_D = -\frac{1}{2}\frac{d}{dx}a\frac{d}{dx} - b\frac{d}{dx} = -\frac{1}{2}\exp(-2B)\frac{d}{dx}a\exp(2B)\frac{d}{dx},$$

where $B(x) = \int_0^x \frac{b}{a}(y)dy$, defined either on $R^+ = (0,\infty)$ with the Dirichlet boundary condition at $x = 0$, or on R, can be realized as a self-adjoint operator with respect to the density $\exp(2Q(x))dx$. The operator is unitarily equivalent to the Schrödinger-type operator $H_S = -\frac{1}{2}\frac{d}{dx}a\frac{d}{dx} + V_{b,a}$, where $V_{b,a} = \frac{1}{2}(\frac{b^2}{a} + b')$. We obtain an explicit criterion for the existence of a compact resolvent and explicit formulas up to the multiplicative constant 4 for the infimum of the spectrum and for the infimum of the essential spectrum for these operators. We give some applications which show in particular how $\inf \sigma(H_D)$ scales when $a = \nu a_0$ and $b = \gamma b_0$, where ν and γ are parameters, and a_0 and b_0 are chosen from certain classes of functions. We also give applications to self-adjoint, multi-dimensional diffusion operators.

1. Introduction and Statement of Results

In this paper, we give an explicit formula up to the multiplicative constant 4 for the bottom of the spectrum and for the bottom of the essential spectrum for diffusion operators on the half-line $R^+ = (0,\infty)$ with the Dirichlet boundary condition at 0, and for diffusion operators on the entire line. Assuming a little more regularity, each such operator is unitarily equivalent to a certain Schrödinger-type operator, so we also obtain the same information

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for these latter operators. Recall that such an operator possesses a compact resolvent if and only if its essential spectrum is empty, or equivalently, if and only if the infimum of its essential spectrum is ∞. Thus, we obtain a completely explicit criterion for the existence of a compact resolvent. A diffusion operator with a compact resolvent is particularly nice because its transition (sub)-probability density $p(t,x,y)$ (with respect to the reversible measure) can be written in the form $p(t,x,y)=\sum_{n=0}^{\infty}\exp(-\lambda_n t)\phi_n(x)\phi_n(y)$, where $\{\phi_n\}_{n=0}^{\infty}$ is a complete, orthonormal set of eigenfunctions and $\{\lambda_n\}_{n=0}^{\infty}$, satisfying $0\le\lambda_0<\lambda_1\le\lambda_2\le\cdots$, are the corresponding eigenvalues. We give some applications of the results, which show in particular how $\inf\sigma(H_D)$ scales when $a=\nu a_0$ and $b=\gamma b_0$, where ν and γ are parameters, and a_0 and b_0 are chosen from certain classes of functions. At the end of the paper, we give applications to self-adjoint, multi-dimensional diffusion operators of the form $-\frac{1}{2}\nabla\cdot a\nabla-a\nabla Q\cdot\nabla=-\frac{1}{2}\exp(-2Q)\nabla\cdot a\exp(2Q)\nabla$ on $L^2(R^d,\exp(2Q)dx)$. The methods and the statements of the results are analytic, but many of the formulas and results have probabilistic import.

We begin with the theory on the half-line, wherein lies the crux of our method. The results for the entire line follow readily from the results for the half-line. Let $0<a\in C^1([0,\infty))$ and $b\in C([0,\infty))$. Define $B(x)=\int_0^x\frac{b}{a}(y)dy$. Consider the diffusion operator with divergence-form diffusion coefficient a and drift b

$$H_D=-\frac{1}{2}\frac{d}{dx}a\frac{d}{dx}-b\frac{d}{dx}=-\frac{1}{2}\exp(-2B)\frac{d}{dx}a\exp(2B)\frac{d}{dx}$$

on R^+ with the Dirichlet boundary condition at $x=0$. One can realize H_D as a non-negative, self-adjoint operator on $L^2(R^+,\exp(2B)dx)$ via the Friedrichs extension of the closure of the nonnegative quadratic form

$$Q_D(f,g)=\frac{1}{2}\int_0^\infty(f'ag')\exp(2B)dx,$$

defined for $f,g\in C_0^1(R^+)$, the space of continuously differentiable functions with compact support on R^+.

Let U_B denote the unitary operator from $L^2(R^+)$ to $L^2(R^+, \exp(2B)dx)$ defined by

$$U_B f = \exp(-B)f.$$

Assuming that $b \in C^1(R^+)$, define $H_S = U_B^{-1} H_D U_B$. One can check that

$$H_S = -\frac{1}{2}\frac{d}{dx}a\frac{d}{dx} + V_{b,a},$$

where

$$V_{b,a} = \frac{1}{2}(\frac{b^2}{a} + b'). \tag{1.1}$$

Assuming that $V_{b,a} = \frac{1}{2}(\frac{b^2}{a} + b')$ is bounded from below, one can realize the Schrödinger-type operator H_S as a self-adjoint operator on $L^2((0,\infty))$ via the Friedrichs extension of the closure of the semi-bounded quadratic form

$$Q_S(f,g) = \frac{1}{2}\int_0^\infty (f'ag')dx + \int_0^\infty V_{b,a} fg dx,$$

defined for $f, g \in C_0^1(R^+)$. Assuming in addition that $\int^\infty a(x)dx = \infty$, one can prove that H_S is in the limit-point case at ∞, which means in particular that H_S on $C_0^2(R^+)$ is essentially self-adjoint. (A proof in the case $a = 1$ can be found in [6, Appendix to X.1]. It can easily be extended to a satisfying the above condition.) Thus, the Friedrichs extension is in fact equal to the closure of H_S on $C_0^2(R^+)$. Note also that U_B preserves the Dirichlet boundary condition. From the above considerations, it follows that the spectra and the essential spectra of H_D and H_S coincide; in particular, H_S is also non-negative.

Conversely, given $a > 0$, every potential $V \geq 0$ can be obtained via some b as in (1.1), and modulo an additive constant, every potential V that is bounded from below can be obtained via some b as in (1.1). Indeed, let $m_V = \inf_{x \in R^+} V(x)$ and let $m_V^- = m_V \wedge 0$. Since $V - m_V^- \geq 0$, it is easy to show that the Riccati equation $\frac{1}{2}b' + \frac{1}{2}\frac{b^2}{a} = V - m_V^-$ has solutions b which exist for all $x \geq 0$.

The essential spectrum $\sigma_{\text{ess}}(-\frac{1}{2}\frac{d}{dx}a\frac{d}{dx} + V)$ of Schrödinger-type operators $-\frac{1}{2}\frac{d}{dx}a\frac{d}{dx} + V$ has been well-studied. See [7] for the results noted below. For example, if a is bounded and bounded from 0, and if the potential V satisfies $\lim_{x\to\infty} V(x) = \infty$, then the operator has a compact resolvent. Thus, the spectrum consists of an increasing sequence of eigenvalues accumulating only at infinity; in particular, $\sigma_{\text{ess}}(-\frac{1}{2}\frac{d}{dx}a\frac{d}{dx}+V)=\emptyset$. On the other hand, if V is a compact (or even relatively compact) perturbation of $-\frac{1}{2}\frac{d}{dx}a\frac{d}{dx}$, which occurs in particular if $\lim_{x\to\infty} V(x) = 0$, then the essential spectrum of $-\frac{1}{2}\frac{d}{dx}a\frac{d}{dx}+V$ coincides with that of $-\frac{1}{2}\frac{d}{dx}a\frac{d}{dx}$; thus $\sigma_{\text{ess}}(-\frac{1}{2}\frac{d}{dx}a\frac{d}{dx}+V)=[0,\infty)$. More generally, for arbitrary $a>0$, the mini-max method [6] affords an algorithm for arriving at $\inf\sigma_{\text{ess}}(-\frac{1}{2}\frac{d}{dx}a\frac{d}{dx}+V)$, although this method is mainly of theoretic import and not a practical way of calculating.

The bottom of the spectrum of $-\frac{1}{2}\frac{d}{dx}a\frac{d}{dx}+V$ is of course given by the well-known variational formula:

$$\inf\sigma(-\frac{1}{2}\frac{d}{dx}a\frac{d}{dx}+V)=\inf\frac{\int_0^\infty(\frac{1}{2}a(f')^2+Vf^2)dx}{\int_0^\infty f^2dx},$$

where the infimum is over functions $0\neq f\in C_0^1(R^+)$. The bottom of the spectrum of H_D is also given by a variational formula:

$$\inf\sigma(H_D)=\inf\frac{\frac{1}{2}\int_0^\infty a(f')^2\exp(2B)dx}{\int_0^\infty f^2\exp(2B)dx}, \tag{1.2}$$

where the infimum is over $0\neq f\in C_0^1(R^+)$.

The following theorem gives explicit formulas up to the multiplicative constant 4 for the bottom of the spectrum and for the bottom of the essential spectrum of H_D. By the spectral invariance, this then extends to the Schrödinger-type operators H_S. The formulas take on two possible forms, depending on whether $\int_0^\infty\frac{1}{a(x)}\exp(-2B(x))dx$ is finite or infinite. In Remark 2 after the theorem, it is shown how the proof for the case when the integral is finite can be reduced to the case when the integral is infinite. Remark 3 after the theorem discusses the probabilistic import of the theorem and of the above integral.

Theorem 1. *Let $0 < a \in C^1([0,\infty))$ and $b \in C([0,\infty))$. Define*

$$B(x) = \int_0^x \frac{b}{a}(y)dy.$$

Consider the self-adjoint diffusion operator

$$H_D = -\frac{1}{2}\frac{d}{dx}a\frac{d}{dx} - b\frac{d}{dx} = -\frac{1}{2}\exp(-2B)\frac{d}{dx}a\exp(2B)\frac{d}{dx}$$

on $L^2(R^+, \exp(2B)dx)$ with the Dirichlet boundary condition at 0.
If $b \in C^1(R^+)$, $\frac{b^2}{a} + b'$ is bounded from below and $\int^\infty a(x)dx = \infty$, consider also the self-adjoint Schrödinger-type operator

$$H_S = -\frac{1}{2}\frac{d}{dx}a\frac{d}{dx} + \frac{1}{2}(\frac{b^2}{a} + b')$$

on $L^2(R^+)$ with the Dirichlet boundary condition at 0.
If

$$\int^\infty \frac{1}{a(x)}\exp(-2B(x))dx = \infty, \tag{1.3}$$

define

$$\Omega^+(b,a) = \sup_{x>0}\left(\int_0^x \frac{1}{a(y)}\exp(-2B(y))dy\right)\left(\int_x^\infty \exp(2B(y))dy\right) \tag{1.4}$$

and

$$\hat{\Omega}^+(b,a) = \limsup_{x\to\infty}\left(\int_0^x \frac{1}{a(y)}\exp(-2B(y))dy\right)\left(\int_x^\infty \exp(2B(y))dy\right). \tag{1.5}$$

If

$$\int^\infty \frac{1}{a(x)}\exp(-2B(x))dx < \infty, \tag{1.6}$$

let

$$h_{b,a}(x) = \int_x^\infty \frac{1}{a(y)}\exp(-2B(y))dy \tag{1.7}$$

and define

$$\begin{aligned}\Omega^+(b,a) &= \sup_{x>0}\left(\int_0^x h_{b,a}^{-2}(y)\frac{1}{a(y)}\exp(-2B(y))dy\right)\left(\int_x^\infty h_{b,a}^2(y)\exp(2B(y))dy\right)\\ &= \sup_{x>0}\left(h_{b,a}^{-1}(x) - h_{b,a}^{-1}(0)\right)\left(\int_x^\infty h_{b,a}^2(y)\exp(2B(y))dy\right),\end{aligned} \tag{1.8}$$

and

(1.9)

$$\hat\Omega^+(b,a)=\limsup_{x\to\infty}\left(\int_0^x h_{b,a}^{-2}(y)\frac{1}{a(y)}\exp(-2B(y))dy\right)\left(\int_x^\infty h_{b,a}^2(y)\exp(2B(y))dy\right)$$

$$=\limsup_{x\to\infty}\left(h_{b,a}^{-1}(x)-h_{b,a}^{-1}(0)\right)\left(\int_x^\infty h_{b,a}^2(y)\exp(2B(y))dy\right).$$

Then

$$\frac{1}{8\Omega^+(b,a)}\le\inf\sigma(H_D)=\inf\sigma(H_S)\le\frac{1}{2\Omega^+(b,a)} \tag{1.10}$$

and

$$\frac{1}{8\hat\Omega^+(b,a)}\le\inf\sigma_{ess}(H_D)=\inf\sigma_{ess}(H_S)\le\frac{1}{2\hat\Omega^+(b,a)}. \tag{1.11}$$

In particular, H_D and H_S possess compact resolvents if and only if $\hat\Omega^+(b,a)=0$.

Remark 1. There does not exist a C for which $\inf\sigma(H_D)=\frac{C}{\Omega^+(b,a)}$, for all drifts b and all diffusion coefficients a. Indeed, on the one hand, consider the case that $b(x)=\pm\gamma$, with $\gamma\in R$, and $a=1$. Then $V_{b,a}=\frac{\gamma^2}{2}$ and thus by unitary equivalence, $\inf\sigma(H_D)=\inf\sigma(H_S)=\frac{\gamma^2}{2}$. A direct calculation in this case reveals that $\Omega^+(b,a)=\frac{1}{4\gamma^2}$; thus, $\inf\sigma(H_D)=\frac{1}{8\Omega^+(b,a)}$. On the other hand, consider the case that $b(x)=-\gamma x$, with $\gamma>0$, and $a=1$. Then $\lim_{x\to\infty}V_{b,a}(x)=\infty$, so as noted above, H_S and thus also H_D have compact resolvents. The unnormalized Hermite function $H_1(x)=x$ is an L^2-eigenfunction of H_D corresponding to the eigenvalue γ. Since it is positive, it must in fact be the principal eigenvalue. Thus, the bottom of the spectrum is equal to γ. We have

$$\Omega^+(b,a)=\sup_{x>0}(\int_0^x\exp(\gamma y^2)dy)(\int_x^\infty\exp(-\gamma y^2)dy)=$$
$$\frac{1}{\gamma}\sup_{x>0}(\int_0^x\exp(y^2)dy)(\int_x^\infty\exp(-y^2)dy)\approx\frac{.239}{\gamma}.$$

Thus, in this case, the bottom of the spectrum is approximately equal to $\frac{.239}{\Omega^+(b,a)}$. In the case $b(x)=\gamma x$, with $\gamma>0$, and $a=1$, one can check that the principal eigenfunction is $x\exp(-\gamma x^2)$, with corresponding principal eigenvalue 2. One can calculate that $\Omega^+(a,b)\approx\frac{.097}{\gamma}$, and thus the bottom of the spectrum is approximately equal to $\frac{.194}{\Omega^+(b,a)}$. Writing the bottom of the spectrum in the form $\frac{C_{b,a}}{\Omega^+(b,a)}$, we don't know whether the upper bound in the theorem is sharp; namely, $C_{b,a}\le\frac{1}{2}$.

Remark 2. In this remark, we demonstrate how formulas (1.10) and (1.11) in the case (1.6) follow from those formulas in the case (1.3), thereby reducing the proof of the theorem to the case that (1.3) holds. In the case that (1.6) holds, define the *h-transform* of H_D via the function $h_{b,a}$ in (1.7) by $H_D^{h_{b,a}}u=\frac{1}{h_{b,a}}H_D(h_{b,a}u)$. When written out, one obtains $H_D^{h_{b,a}}=-\frac{1}{2}\frac{d}{dx}a\frac{d}{dx}-(b+a\frac{h'_{b,a}}{h_{b,a}})\frac{d}{dx}$. Letting $B^{h_{b,a}}(x)=\int_0^x(\frac{b}{a}+\frac{h'_{b,a}}{h_{b,a}})(y)dy$, one has $\int^\infty\frac{1}{a(x)}\exp(-2B^{h_{b,a}}(x))dx=-\int^\infty h_{b,a}^{-2}h'_{b,a}dx=\infty$; that is, the diffusion coefficient a with the new drift $b+a\frac{h'_{b,a}}{h_{b,a}}$ satisfies (1.3). The spectrum is invariant under h-transforms [4, chapter 4—sections 3 and 10], so $\inf\sigma(H_D)=\inf\sigma(H_D^{h_{b,a}})$ and $\inf\sigma_{\text{ess}}(H_D)=\inf\sigma_{\text{ess}}(H_D^{h_{b,a}})$. These equalities along with the fact that (1.3) holds with the diffusion coefficient a and the drift $(b+a\frac{h'_{b,a}}{h_{b,a}})$ show that one obtains (1.10) and (1.11) for H_D by defining $\Omega^+(b,a)=\Omega^+(b+a\frac{h'_{b,a}}{h_{b,a}},a)$ and $\hat\Omega^+(b,a)=\hat\Omega^+(b+a\frac{h'_{b,a}}{h_{b,a}},a)$. From (1.4), one has

(1.12)
$$\Omega^+(b+a\frac{h'_{b,a}}{h_{b,a}},a)=\sup_{x>0}\left(\int_0^x\frac{1}{a(y)}\exp(-2B^{h_{b,a}}(y))dy\right)\left(\int_x^\infty\exp(2B^{h_{b,a}}(y))dy\right)$$
$$=\sup_{x>0}\left(\int_0^x h_{b,a}^{-2}(y)\frac{1}{a(y)}\exp(-2B(y))dy\right)\left(\int_x^\infty h_{b,a}^2(y)\exp(2B(y))dy\right);$$

whence the definition of $\Omega^+(b,a)$ in (1.8) in the case that (1.6) holds, and likewise for $\hat\Omega^+(b,a)$.

Remark 3. Theorem 1 and the reduction noted above in Remark 2 have some probabilistic implications, which we now describe. Let $X(t)$ be generic notation for a Markov diffusion process on the real line. Let P_x and E_x denote respectively probabilities and expectations for the process corresponding to the operator $-H_D=\frac{1}{2}\frac{d}{dx}a\frac{d}{dx}+b\frac{d}{dx}$ on $(0,\infty)$, starting at $x>0$ and killed at time

$$\tau_0 \equiv \inf\{t \geq 0 : X(t) = 0\}, \tag{1.13}$$

the first hitting time of 0. Then $P_x(\tau_0 < \infty) = 1$, for $x > 0$, if and only if (1.3) holds [4, chapter 5].

Consider first the case that (1.3) holds. At the end of section 3 we show that

$$\inf \sigma(H_D) = \sup\{\lambda \geq 0 : E_x \exp(\lambda\tau_0) < \infty\},\ x > 0. \tag{1.14}$$

Thus, in the case that (1.3) holds, (1.10) gives an explicit formula up to the multiplicative constant 4 for $\sup\{\lambda \geq 0 : E_x \exp(\lambda\tau_0) < \infty\}$.

Now consider the case that (1.6) holds. In this case, $P_x(\tau_0 < \infty) = \frac{h_{b,a}(x)}{h_{b,a}(0)}$ [4, chapter 5]. The original process, conditioned on $\{\tau_0 < \infty\}$, is itself a Markov diffusion process and it corresponds to the h-transformed operator $-H_D^{h_{b,a}}$ defined in Remark 2 [4, chapter 7]. Let $E_x^{h_{b,a}}$ denote expectations for this conditioned process starting from $x > 0$. Then it follows from Remark 2 and (1.14) that in the case that (1.6) holds, one has

$$\inf \sigma(H_D) = \sup\{\lambda \geq 0 : E_x^{h_{b,a}} \exp(\lambda\tau_0) < \infty\},\ x > 0. \tag{1.15}$$

Note from (1.4) and (1.10) that when (1.3) holds, a necessary condition for $\inf \sigma(H_D) > 0$ is that $\int^\infty \exp(2B(y))dy < \infty$. This integral condition is equivalent to $E_x\tau_0 < \infty$, for $x > 0$ [4, chapter 5—section 1]. Thus, when $P_x(\tau_0 < \infty) = 1$ holds, the finiteness of $E_x\tau_0$ is a necessary condition (but not a sufficient one) for $\inf \sigma(H_D) > 0$. Similarly, when (1.6) holds (in which case $P^{h_{b,a}}(\tau_0 < \infty) = 1$), the finiteness of $E_x^{h_{b,a}}\tau_0$ is a necessary condition (but not a sufficient one) for $\inf \sigma(H_D) > 0$. (Of course, this can also be seen from (1.14) and (1.15)—if the first moment does not exist, then a fortiori no exponential moment exists.)

An alternative probabilistic representation of $\inf \sigma(H_D)$ is this:

$$\inf \sigma(H_D) = -\lim_{n\to\infty} \lim_{t\to\infty} \frac{1}{t} \log P_x(\tau_0 \wedge \tau_n > t),\ x > 0, \tag{1.16}$$

where $\tau_n = \inf\{t \geq 0 : X(t) = n\}$. (This formula can be found essentially in [4, chapter 4].)

Formulas (1.14) and (1.15) give a probabilistic representation for the bottom of the spectrum of H_D. One can also give a similar probabilistic representation for the bottom of the essential spectrum. It follows from (1.14) and (3.9) in section 3 that if (1.3) holds, then

$$\inf \sigma_{\text{ess}}(H_D) = \lim_{l \to \infty} (\sup\{\lambda \geq 0 : E_x \exp(\lambda \tau_l) < \infty \quad x > l\}),$$

while if (1.6) holds, then

$$\inf \sigma_{\text{ess}}(H_D) = \lim_{l \to \infty} (\sup\{\lambda \geq 0 : E_x^{h_{b,a}} \exp(\lambda \tau_l) < \infty, \quad x > l\}).$$

Remark 4. After finishing this paper, the following related result due to Muckenhoupt [2], in the context of weighted Hardy inequalities, was brought to our attention. For $1 \leq p \leq \infty$, the inequality

$$\left(\int_0^\infty |U(x) \int_0^x g(t)dt|^p dx \right)^{\frac{1}{p}} \leq C \left(\int_0^\infty |V(x)g(x)|^p dx \right)^{\frac{1}{p}} \tag{1.17}$$

holds for all g and some finite C if and only if

$$B \equiv \sup_{x>0} \left(\int_x^\infty |U(y)|^p dy \right)^{\frac{1}{p}} \left(\int_0^x |V(y)|^{-p'} dy \right)^{\frac{1}{p'}} < \infty,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, and furthermore, if C_0 is the least constant C for which the above inequality holds, then $B \leq C_0 \leq p^{\frac{1}{p}} (p')^{\frac{1}{p'}} B$, for $1 < p < \infty$, and $C_0 = B$ for $p = 1, \infty$. (The integrals are interpreted according to the usual convention in the case that p or p' is ∞.) Applying this with $p = p' = 2$, $U = \exp(B)$ and $V = (\frac{a}{2})^{\frac{1}{2}} \exp(B)$, one concludes that $\inf \frac{\frac{1}{2} \int_0^\infty a(f')^2 \exp(2B)dx}{\int_0^\infty f^2 \exp(2B)dx}$ lies between $\left(8 \sup_{x>0} (\int_0^x \frac{1}{a(y)} \exp(-2B(y))dy)(\int_x^\infty \exp(2B(y))dy)\right)^{-1}$ and $\left(2 \sup_{x>0} (\int_0^x \frac{1}{a(y)} \exp(-2B(y))dy)(\int_x^\infty \exp(2B(y))dy)\right)^{-1}$, where the infimum is over $f \in C^1([0, \infty))$ which satisfy $f(0) = 0$. This is a different variational problem than the one in (1.2) for $\inf \sigma(H_D)$ because the class of admissible functions here is larger than in (1.2). In the case that (1.3) holds, Theorem 1 shows that the same bounds hold for both variational problems, since in this case, $\Omega^+(b,a) = \sup_{x>0}\left(\int_0^x \frac{1}{a(y)}\exp(-2B(y))dy\right)\left(\int_x^\infty \exp(2B(y))dy\right)$. However, when (1.6) holds, $\Omega^+(b,a)$ is defined differently, and the two variational problems yield different results. Indeed, for example, if $b = 1$ and $a = 1$, then one has $\int_x^\infty \exp(2B(y))dy) = \infty$, so the infimum in Muckenhoupt's variational problem is 0; however by (1.8), one calculates that $\Omega^+(b,a) = 1$, and it follows from Theorem 1 that the infimum in (1.2) lies between $\frac{1}{8}$ and $\frac{1}{2}$. (In fact, in this simple case it can be checked directly that $\inf \sigma(H_D) = \frac{1}{8}$.) Thus, the integral condition (1.6) turns out to be the lower threshold on the size of $a\exp(2B)$, the weight that multiplies $(f')^2$ in the variational formulas, so that the two variational formulas, one over $f \in C_0^1(R^+)$ and one over $f \in C^1([0,\infty))$ satisfying $f(0) = 0$, yield different answers. Muckenhout's proof involves a direct estimation of the integrals in (1.17). We prove Theorem 1 in a completely different way, as will be seen in sections 3 and 4.

Remark 5. It follows from the theorem that $\inf \sigma(H_D)$ and $\inf \sigma_{\text{ess}}(H_D)$ depend on a and b only through a and B.

Remark 6. For the duration of this remark, we consider a to be fixed. By a standard comparison theorem for diffusions, it follows that over the class of drifts b satisfying (1.3), the distribution of τ_0 is stochastically increasing with b. Thus, from (1.14), it follows that $\inf \sigma(H_D)$ and $\inf \sigma_{\text{ess}}(H_D)$ are nonincreasing over the class of drifts b satisfying (1.3). That is, over drifts satisfying (1.3), the more inward toward 0 the drift, the larger the bottom of the spectrum and the bottom of the essential spectrum. (It is not hard to verify that the function $\left(\int_0^x \frac{1}{a(y)}\exp(-2B(y))dy\right)\left(\int_x^\infty \exp(2B(y))dy\right)$ appearing in the definition of $\Omega^+(b,a)$ is nondecreasing in b over the class of drifts b satisfying (1.3), but this is not quite enough to arrive at the result in the above sentence.) Despite the above fact and despite Remark 5, it is *not* true that $\inf \sigma(H_D)$ and $\inf \sigma_{\text{ess}}(H_D)$ are nonincreasing as functions of

B over the class of b satisfying (1.3). An example will be given at the end of section 2.

We don't know whether $\inf \sigma(H_D)$ and $\inf \sigma_{\mathrm{ess}}(H_D)$ are nondecreasing over the entire class of drifts b satisfying (1.6), so that the more outward toward infinity the drift, the larger the bottom of the spectrum and the bottom of the essential spectrum. To prove that this is true, it would suffice to show that $b+a\frac{h'_{b,a}}{h_{b,a}}$ is nonincreasing in b over the class of drifts satisfying (1.6)—that this would suffice follows from (1.15) and the argument above for the class of drifts satisfying (1.3). What is known is this [5]:

(1.18)

For a wide class of a and b which satisfy (1.6) and for which b is on a larger order than $\frac{a(x)}{x}$, one has $b+a\frac{h'_{b,a}}{h_{b,a}} = -b + O(\frac{a(x)}{x})$ as $x \to \infty$.

This formula will be useful for one of the calculations in section 2.

We now turn to the case of the whole line. Let $0 < a \in C^1(R)$ and $b \in C(R)$, and define $B(x) = \int_0^x \frac{b}{a}(y)dy$. Let $H_D = -\frac{1}{2}\frac{d}{dx}a\frac{d}{dx} - b\frac{d}{dx}$ and consider the self-adjoint realization on $L^2(R, \exp(2B)dx)$ obtained via the Friedrichs extension of the closure of the quadratic form $Q_D(f,g) = \frac{1}{2}\int_{-\infty}^{\infty}(f'ag')\exp(2B)dx$, for $f,g \in C_0^1(R)$. In the case that $b \in C^1(R)$, $\frac{b^2}{a} + b'$ is bounded from below and $\int^{\infty} a(x)dx = \int_{-\infty} a(x)dx = \infty$, define $H_S = -\frac{1}{2}\frac{d}{dx}a\frac{d}{dx} + V_{b,a}$ to be the self-adjoint operator obtained via the Friedrichs extension of the closure of the quadratic form $Q_S(f,g) = \frac{1}{2}\int_{-\infty}^{\infty}(f'ag')dx + \int_{-\infty}^{\infty} V_{b,a}fgdx$, where $V_{b,a} = \frac{1}{2}(\frac{b^2}{a} + b')$ and $f,g \in C_0^1(R)$.

The first of the two theorems below treats $\inf \sigma_{\mathrm{ess}}(H_D)$ and the second one treats $\inf \sigma(H_D)$. The proofs of these results will be derived in just a few lines from the proof of Theorem 1.

Theorem 2. *Let $a \in C^1(R)$ and $b \in C(R)$. Define*

$$B(x) = \int_0^x \frac{b}{a}(y)dy.$$

Consider the self-adjoint diffusion operator

$$H_D = -\frac{1}{2}\frac{d}{dx}a\frac{d}{dx} - b\frac{d}{dx} = -\frac{1}{2}\exp(-2B)\frac{d}{dx}a\exp(2B)\frac{d}{dx}$$

on $L^2(R, \exp(2B)dx)$.

If $b \in C^1(R)$, $\frac{b^2}{a} + b'$ *is bounded from below and* $\int^\infty a(x)dx = \int_{-\infty} a(x)dx = \infty$, *consider also the self-adjoint Schrödinger-type operator*

$$H_S = -\frac{1}{2}\frac{d}{dx}a\frac{d}{dx} + \frac{1}{2}(\frac{b^2}{a} + b')$$

on $L^2(R)$. *Let* $\Omega^+(b,a)$ *be as in Theorem 1 and define* $\Omega^-(b,a)$ *in exactly the same way, using the half-line* $(-\infty, 0)$ *instead of* $(0,\infty)$. *Let*

$$\hat{\Omega}(b,a) = \max(\hat{\Omega}^+(b,a), \hat{\Omega}^-(b,a)).$$

Then

$$\frac{1}{8\hat{\Omega}(b,a)} \leq \inf \sigma_{ess}(H_D) = \inf \sigma_{ess}(H_S) \leq \frac{1}{2\hat{\Omega}(b,a)}.$$

In particular, H_D *and* H_S *possess compact resolvents if and only if* $\hat{\Omega}^+(b,a) = \hat{\Omega}^-(b,a) = 0$.

Remark 7. The diffusion is positive recurrent if and only if $\int_R \exp(2B(x))dx < \infty$ [4, chapter 5]. It follows from Theorem 2 that $\inf \sigma_{\text{ess}}(H_B) = 0$ if the diffusion is not positive recurrent. (See also the third to the last paragraph of Remark 3.)

Theorem 3. *Let* $a \in C^1(R)$ *and* $b \in C(R)$. *Define*

$$B(x) = \int_0^x \frac{b}{a}(y)dy.$$

Consider the self-adjoint diffusion operator

$$H_D = -\frac{1}{2}\frac{d}{dx}a\frac{d}{dx} - b\frac{d}{dx} = -\frac{1}{2}\exp(-2B)\frac{d}{dx}a\exp(2B)\frac{d}{dx}$$

on $L^2(R, \exp(2B)dx)$.

If $b \in C^1(R)$, $\frac{b^2}{a} + b'$ *is bounded from below and* $\int^\infty a(x)dx = \int_{-\infty} a(x)dx = \infty$*, consider also the self-adjoint Schrödinger-type operator*

$$H_S = -\frac{1}{2}\frac{d}{dx}a\frac{d}{dx} + \frac{1}{2}(\frac{b^2}{a} + b')$$

on $L^2(R)$*.*

If

$$\int^\infty \frac{1}{a(x)}\exp(-2B(x))dx = \int_{-\infty} \frac{1}{a(x)}\exp(-2B(x))dx = \infty, \tag{1.19}$$

define

$$\Omega(b,a) = \infty. \tag{1.20}$$

If

$$\int^\infty \frac{1}{a(x)}\exp(-2B(x))dx = \infty \quad \text{and} \quad \int_{-\infty} \frac{1}{a(x)}\exp(-2B(x))dx < \infty, \tag{1.21}$$

define

$$\Omega(b,a) = \sup_{x\in R}\left(\int_{-\infty}^x \frac{1}{a(y)}\exp(-2B(y))dy\right)\left(\int_x^\infty \exp(2B(y))dy\right).$$

If

$$\int^\infty \frac{1}{a(x)}\exp(-2B(x))dx < \infty \quad \text{and} \quad \int_{-\infty} \frac{1}{a(x)}\exp(-2B(x))dx = \infty, \tag{1.22}$$

define

$$\Omega(b,a) = \sup_{x\in R}\left(\int_x^\infty \frac{1}{a(y)}\exp(-2B(y))dy\right)\left(\int_{-\infty}^x \exp(2B(y))dy\right).$$

If

$$\int^\infty \frac{1}{a(x)}\exp(-2B(x))dx < \infty \quad \text{and} \quad \int_{-\infty} \frac{1}{a(x)}\exp(-2B(x))dx < \infty, \tag{1.23}$$

let

$$h_{b,a}(x) = \int_x^\infty \frac{1}{a(y)}\exp(-2B(y))dy$$

and define

$$\Omega(b,a)=\sup_{x\in R}\left(\int_{-\infty}^{x}h_{b,a}^{-2}(y)\frac{1}{a(y)}\exp(-2B(y))dy\right)\left(\int_{x}^{\infty}h_{b,a}^{2}(y)\exp(2B(y))dy\right)$$
$$=\sup_{x\in R}\left(h_{b,a}^{-1}(x)-h_{b,a}^{-1}(-\infty)\right)\left(\int_{x}^{\infty}h_{b,a}^{2}(y)\exp(2B(y))dy\right).$$

Then

$$\frac{1}{8\Omega(b,a)}\le\inf\sigma(H_D)=\inf\sigma(H_S)\le\frac{1}{2\Omega(b,a)}.$$

Remark 8. The diffusion process $X(t)$ corresponding to $-H_D$ is recurrent if (1.19) holds and is transient otherwise. In the transient case, if (1.21) holds, then $P_x(\lim_{t\to\infty}X(t)=-\infty)=1$; if (1.22) holds, then $P_x(\lim_{t\to\infty}X(t)=\infty)=1$; if (1.23) holds, then $P_x(\lim_{t\to\infty}X(t)=-\infty)=1-P_x(\lim_{t\to\infty}X(t)=\infty)=\frac{h_{b,a}(x)}{h_{b,a}(-\infty)}$. (For these results, see [4, chapter 5].) It follows from Theorem 3 that $\inf\sigma(H_D)=0$ if the diffusion is recurrent.

Remark 9. Similar to (1.16), one has the following probabilistic representation of $\inf\sigma(H_D)$:

$$\inf\sigma(H_D)=-\lim_{n\to\infty}\lim_{t\to\infty}\frac{1}{t}\log P_x(\tau_{-n}\wedge\tau_n>t),\ x\in R. \tag{1.24}$$

In section 2 we give some applications of Theorems 1-3. In section 3 we prove Theorem 1, postponing the proof of a key proposition to section 4. After the proof of Theorem 1 we give the quick proofs of Theorems 2 and 3. We also prove (1.14) in section 3. Finally, in section 5 we show how the one-dimensional result can be used to obtain spectral estimates for self-adjoint, multi-dimensional diffusion operators

2. Examples

The Bottom of the Spectrum. One can use Theorem 1 to study the way $\inf\sigma(H_D)$ scales in the parameters γ and ν when b is of the form $b=\gamma b_0$

and a is of the form $a = \nu a_0$. We first consider the effect of the drift alone. Consider for example the following two cases on R^+ or on R:

$$b(x) = -\gamma(1+|x|)^l \text{ and } a(x) = 1,\ \gamma > 0,\ l \in R, \tag{2.1}$$

$$b(x) = -\gamma|x|^l \text{ and } a(x) = 1,\ \gamma > 0,\ l \geq 0. \tag{2.2}$$

Proposition 1. *Consider H_D on R^+ or on R.*

1. Assume that (2.1) *holds.*

i. If $l < 0$, then $\inf \sigma(H_D) = 0$;

ii. If $l \geq 0$, then there exist constants $c_l, C_l > 0$ such that

$$c_l\gamma^2 \leq \inf \sigma(H_D) \leq C_l\gamma^2,\ \gamma > 1$$

and

$$c_l\gamma^{\frac{2}{l+1}} \leq \inf \sigma(H_D) \leq C_l\gamma^{\frac{2}{l+1}},\ 0 < \gamma \leq 1.$$

2. Assume that (2.2) *holds. Then*

$$\frac{1}{8C_l}\gamma^{\frac{2}{1+l}} \leq \inf \sigma(H_D) \leq \frac{1}{2C_l}\gamma^{\frac{2}{1+l}},\ \gamma > 0,$$

where

$$C_l = \begin{cases} \sup_{x>0} \left(\int_0^x \exp(\frac{2z^{l+1}}{l+1})dz\right)\left(\int_x^\infty \exp(-\frac{2z^{l+1}}{l+1})dz\right) & \text{on } R^+; \\ C_l = \left(\int_0^\infty \exp(-\frac{2z^{l+1}}{l+1})dz\right)^2 & \text{on } R. \end{cases}$$

Remark 10. Note that both on R^+ and on R, the rate of growth of $\inf \sigma(H_D)$ for large γ is on a slower order for the drift in (2.2) than for the drift in (2.1). The probabilistic explanation for this follows from the formulas (1.16) and (1.24) and the fact that the latter drifts are small in a (γ-dependent) neighborhood of 0, even as γ becomes large. Note also that for the drift in (2.1), the scaling power is different for $\gamma \ll 1$ than for $\gamma \gg 1$.

The bounds on the infimum of the spectrum in Proposition 1 also hold for the corresponding Schrödinger operator, $H_S = -\frac{1}{2}\frac{d^2}{dx^2} + V$, where $V = \frac{1}{2}\gamma^2(1+x)^{2l} - \frac{1}{2}\gamma l(1+x)^{l-1}$ in the case of (2.1) on R^+ and $V(x) = \frac{1}{2}\gamma^2 x^{2l} - \frac{1}{2}\gamma x^{l-1}$ in the case of (2.2) on R^+, and similarly for R.

We now consider simultaneous scaling in a and b. Consider the following case on R^+ and on R:

$$\begin{aligned} &b(x) = -\gamma(1+|x|)^l \text{ and } a(x) = \nu(1+|x|)^k, \\ &\text{where } \gamma, \nu > 0,\ l, k \in R, \text{ with } l-k > -1 \text{ and } 2l-k \geq 0. \end{aligned} \tag{2.3}$$

(Note that when $\nu = 1$ and $k = 0$, (2.3) reduces to (2.1) with $l \geq 0$.) If $2l - k < 0$ or if $l - k < -1$, then one can show that $\inf \sigma(H_D) = 0$.

Proposition 2. *Consider H_D on R^+ or on R. Assume that* (2.3) *holds. There exist constants $c_{l,k}, C_{l,k} > 0$ such that*

$$c_{l,k}\frac{\gamma^2}{\nu} \leq \inf \sigma(H_D) \leq C_{l,k}\frac{\gamma^2}{\nu},\ 0 < \nu < \gamma,$$

and

$$c_{l,k}(\frac{\gamma^{2-k}}{\nu^{1-l}})^{\frac{1}{l-k+1}} \leq \inf \sigma(H_D) \leq C_{l,k}(\frac{\gamma^{2-k}}{\nu^{1-l}})^{\frac{1}{l-k+1}},\ 0 < \gamma \leq \nu.$$

Remark 11. Note that when $\gamma \leq \nu$, the scaling dependence on the coefficient γ of the drift b has three dramatically different phases, depending on whether the exponent k of the diffusion coefficient satisfies $k < 2$, $k = 2$ or $k > 2$, while the scaling dependence of the coefficient ν of the diffusion coefficient has three dramatically different phases, depending on whether the exponent l of the drift satisfies $l < 1$, $l = 1$ or $l > 1$. However, when $\gamma > \nu$, there is only one scaling phase, and it is independent of the exponents l and k.

The bounds on the infimum of the spectrum in Proposition 2 also hold for the corresponding Schrödinger-type operator, $H_S = -\frac{1}{2}\frac{d}{dx}(\nu(1+|x|)^k)\frac{d}{dx}+V$, where $V = \frac{1}{2}\frac{\gamma^2}{\nu}(1+x)^{2l-k} - \frac{1}{2}\gamma l(1+x)^{l-1}$ in the case of R^+, and similarly for R. The parameter dependence in Proposition 2 does not seem at all apparent from looking at this operator.

We give the proof of Proposition 1; the proof of Proposition 2 is similar.

Proof of Proposition 1. We prove the proposition in the case of R^+; the case of R is handled similarly. To prove part 2, one simply makes an appropriate change of variables in the formula for $\Omega^+(-\gamma x^l,1)$ and applies Theorem 1. To get the explicit form of C_l in the case of R, one needs to do a little bit more analysis to show that the supremum over $x\in R$ occurs at $x=0$.

We now prove part 1. If $l\neq -1$, then

$$
\begin{aligned}
&\Omega^+(-\gamma(1+x)^l,1)= \\
&\sup_{x>0}\left(\int_0^x \exp(2\gamma\frac{(1+y)^{l+1}}{l+1})dy\right)\left(\int_x^\infty \exp(-2\gamma\frac{(1+y)^{l+1}}{l+1})dy\right).
\end{aligned}
\tag{2.4}
$$

For $l<-1$ the right hand integral is ∞ so $\Omega^+(-\gamma(1+x)^l,1)=\infty$. Now consider $-1<l<0$. Applying L'Hôpital's rule to the quotients

$$
\frac{\int_0^x \exp(2\gamma\frac{(1+y)^{l+1}}{l+1})dy}{(1+x)^{-l}\exp(2\gamma\frac{(1+x)^{l+1}}{l+1})},\quad \frac{\int_x^\infty \exp(-2\gamma\frac{(1+y)^{l+1}}{l+1})dy}{(1+x)^{-l}\exp(-2\gamma\frac{(1+x)^{l+1}}{l+1})},
$$

shows that $\int_0^x \exp(2\gamma\frac{(1+y)^{l+1}}{l+1})dy\sim(2\gamma)^{-1}(1+x)^{-l}\exp(2\gamma\frac{(1+x)^{l+1}}{l+1})$ and $\int_x^\infty \exp(-2\gamma\frac{(1+y)^{l+1}}{l+1})dy\sim(2\gamma)^{-1}(1+x)^{-l}\exp(-2\gamma\frac{(1+x)^{l+1}}{l+1})$, as $x\to\infty$. This shows that the supremum in (2.4) is ∞; thus $\Omega^+(-\gamma(1+x)^l,1)=\infty$. One obtains $\Omega^+(-\gamma(1+x)^l,1)=\infty$ similarly in the case $l=-1$. Applying Theorem 1 now completes the proof of part 1-i.

Consider now part 1-ii; that is, the case $l\geq 0$. Making the change of variables $z=\gamma^{\frac{1}{l+1}}(1+y)$, one obtains from (2.4),

$$
\Omega^+(-\gamma(1+x)^l,1)=\gamma^{-\frac{2}{l+1}}\sup_{x>\gamma^{\frac{1}{l+1}}}\left(\int_{\gamma^{\frac{1}{l+1}}}^x \exp(\frac{z^{l+1}}{l+1})dz\right)\left(\int_x^\infty \exp(-\frac{z^{l+1}}{l+1})dz\right).
\tag{2.5}
$$

If $l=0$, the integrals on the right hand side of (2.5) can be calculated explicitly. One finds that the supremum above is equal to 1. Part 1-ii in the case $l=0$ now follows from (2.5) and Thereom 1. From now on, we assume that $l>0$. Applying L'Hôpital's rule in the manner noted above shows that

$$
\begin{aligned}
&\int_x^\infty \exp(-\frac{z^{l+1}}{l+1})dz\sim x^{-l}\exp(-\frac{x^{l+1}}{l+1}),\quad \text{as } x\to\infty;\\
&\int_0^x \exp(-\frac{z^{l+1}}{l+1})dz\sim x^{-l}\exp(\frac{x^{l+1}}{l+1}),\quad \text{as } x\to\infty.
\end{aligned}
\tag{2.6}
$$

From (2.6) it follows that there exist constants $d_l, D_l > 0$ such that

(2.7)
$$d_l \le \sup_{x>\gamma^{\frac{1}{l+1}}} \Big(\int_{\gamma^{\frac{1}{l+1}}}^x \exp(\frac{z^{l+1}}{l+1})dz\Big)\Big(\int_x^\infty \exp(-\frac{z^{l+1}}{l+1})dz\Big) \le D_l,\ 0<\gamma\le 1.$$

Part 1-ii in the case that $0 < \gamma \le 1$ now follows from (2.5), (2.7) and Theorem 1.

Now consider part 1-ii in the case that $\gamma > 1$. Clearly,

$$\begin{aligned}&\left(\int_{\gamma^{\frac{1}{l+1}}}^{2\gamma^{\frac{1}{l+1}}} \exp(\frac{z^{l+1}}{l+1})dz\right)\Big(\int_{2\gamma^{\frac{1}{l+1}}}^\infty \exp(-\frac{z^{l+1}}{l+1})dz\Big) \le\\ &\sup_{x>\gamma^{\frac{1}{l+1}}}\Big(\int_{\gamma^{\frac{1}{l+1}}}^x \exp(\frac{z^{l+1}}{l+1})dz\Big)\Big(\int_x^\infty \exp(-\frac{z^{l+1}}{l+1})dz\Big) \le\\ &\sup_{x>\gamma^{\frac{1}{l+1}}}\Big(\int_0^x \exp(\frac{z^{l+1}}{l+1})dz\Big)\Big(\int_x^\infty \exp(-\frac{z^{l+1}}{l+1})dz\Big).\end{aligned} \tag{2.8}$$

Using (2.6) to estimate the left most and right most terms in (2.8), it follows that there exist constants $d_l, D_l > 0$ such that

(2.9)
$$d_l\gamma^{-\frac{2l}{l+1}} \le \sup_{x>\gamma^{\frac{1}{l+1}}} \Big(\int_{\gamma^{\frac{1}{l+1}}}^x \exp(\frac{z^{l+1}}{l+1})dz\Big)\Big(\int_x^\infty \exp(-\frac{z^{l+1}}{l+1})dz\Big) \le D_l\gamma^{-\frac{2l}{l+1}},$$

for $\gamma > 1$.

Part 1-ii in the case $\gamma > 1$ now follows from (2.5), (2.9) and Theorem 1. □

Theorem 1 allows one to compute the bottom of the spectrum *exactly* for an ad hoc class of Schrödinger operators, $H = -\frac{1}{2}\frac{d^2}{dx^2} + V$. Indeed, it follows from the theorem that if a and b satisfy (1.3) and $\int^\infty \exp(2B)dx = \infty$, then $\inf\sigma(H_D) = \inf\sigma(H_S) = 0$. Let $u = \exp(g)$, where g is bounded, and define $V = \frac{u''}{2u} = \frac{1}{2}((g')^2 + g'')$, $b = \frac{u'}{u} = g'$ and $a = 1$. Then b satisfies the above conditions and $H_S = -\frac{1}{2}\frac{d^2}{dx^2} + V$. Thus, $\inf\sigma(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}((g')^2 + g'')) = 0$, for all bounded g. *In particular, if g is periodic and not constant, then* $\lim_{x\to\infty}\frac{1}{x}\int_0^x V(y)dy > 0$ *but the bottom of the spectrum is 0.*

Note that either $\hat{\Omega}^+(b,a) = \Omega^+(b,a) = \infty$, or $\hat{\Omega}^+(b,a), \Omega^+(b,a) < \infty$; thus, in R^+ either both the bottom of the spectrum and bottom of the essential spectrum equal 0, or else neither of them does. It is not hard to construct examples where the bottom of the spectrum and the bottom of the essential spectrum are both positive and finite but don't coincide. For example, let $a = 1$ and let $b(x) = -1$, for $x \geq 3$. Since $\hat{\Omega}^+(b,1)$ does not depend on $\{b(x) : 0 \leq x \leq 3\}$, we have $\hat{\Omega}^+(b,1) = \frac{1}{4}$. Let $b(x) = -n$, for $1 \leq x \leq 2$, and $b(x) \leq -1$ everywhere. Then the term $\int_0^3 \exp(-2B(y))dy$ can be made arbitrarily large by choosing n arbitrarily large, and thus for sufficiently large n,

$$\Omega^+(b,1) = \sup_{x>0} \left(\int_0^x \exp(-2B(y))dy\right)\left(\int_x^\infty \exp(2B(y))dy\right) > \frac{1}{4} = \hat{\Omega}^+(b,1).$$

The Bottom of the Essential Spectrum. We consider operators on R^+. The examples can easily be extended to operators on R by making the analysis on R^+ and on R^- separately, and applying Theorem 2. Consider first the case that

(2.10)

$$b(x) = -\gamma(1+x)^l \text{ and } a(x) = \nu(1+x)^k, \quad \gamma, \nu > 0, \; l, k \in R,$$

$$\text{with } l - k > -1, \text{ or } l - k = -1 \text{ and } k \leq 1 + \frac{2\gamma}{\nu}, \text{ or } l - k < -1 \text{ and } k \leq 1.$$

The set of possible conditions on l, k above are exactly those for which (1.3) holds. One can obtain the asymptotic behavior of $\int_x^\infty \exp(2B(y))dy$ and of $\int_0^x \frac{1}{a(y)} \exp(-2B(y))dy$ by applying L'Hôpital's rule respectively to $\frac{\int_x^\infty \exp(2B(y))dy}{b^{-1}(x)\exp(2B(x))}$ and $\frac{\int_0^x \frac{1}{a(y)}\exp(-2B(y))dy}{b^{-1}(x)\frac{1}{a(x)}\exp(-2B(x))}$. Calculating and applying Theorem 1, one obtains the following result.

Proposition 3. *Consider H_D on R^+. Let a and b satisfy* (2.10).

1. Assume that $l - k < -1$ or that $l - k = -1$ and $\frac{\gamma}{\nu} \leq \frac{1}{2}$. Then $\inf \sigma_{ess}(H_D) = 0$.

2. Assume that $l - k = -1$ and $\frac{\gamma}{\nu} > \frac{1}{2}$.

i. If $k > 2$, then $\sigma_{ess}(H_D) = \emptyset$;

ii. If $k = 2$, then $0 < \inf \sigma_{ess}(H_D) < \infty$;

iii. If $k < 2$, then $\inf \sigma_{ess}(H_D) = 0$.

3. Assume that $l - k > -1$.

i. If $2l - k > 0$, then $\sigma_{ess}(H_D) = \emptyset$;

ii. If $2l - k = 0$, then $0 < \inf \sigma_{ess}(H_D) < \infty$;

iii. If $2l - k < 0$, then $\inf \sigma_{ess}(H_D) = 0$.

In particular, H_D possesses a compact resolvent if and only if 2-i or 3-i holds.

The bounds on the infimum of the essential spectrum in Proposition 3 also hold for the corresponding Schrödinger-type operator $H_S = -\frac{1}{2}\frac{d}{dx}(\nu(1+x)^k)\frac{d}{dx} + V$, where $V = \frac{1}{2}\frac{\gamma^2}{\nu}(1+x)^{2l-k} - \frac{1}{2}\gamma l x^{l-1}$. For certain values of the parameters, the results in Proposition 3 can be deduced directly from looking at H_S. For example, if $k = 0$ and $l < 1$, then $\lim_{x\to\infty} V(x)$ equals ∞ if $l > 0$ and is equal to $\frac{1}{2}\frac{\gamma^2}{\nu}$ if $l = 0$. It follows from standard perturbations results, mentioned in the first section, that in the former case $\sigma_{\text{ess}}(H_S) = \emptyset$ and in the latter case $\inf \sigma_{\text{ess}}(H_S) = \frac{1}{2}\frac{\gamma^2}{\nu}$.

However, in fact, Theorem 1 allows one to come to the same type of conclusions as in Proposition 3 in the case that a and b satisfy one of the following general conditions:

(2.11)
$$c_1(1+x)^k \le a(x) \le c_2(1+x)^k, \quad k \in R$$
$$-c_2(1+x)^m \le \int_0^x \frac{b(y)}{(1+y)^k}dy \le -c_1(1+x)^m, \text{ for large } x, \quad m > 0,\ 0 < c_1 < c_2;$$

or

$$\begin{aligned} &c_1(1+x)^k \le a(x) \le c_2(1+x)^k,\ k \le 1 \\ &\int_0^x \frac{b(y)}{(1+y)^k}dy \text{ is bounded in } x. \end{aligned} \tag{2.12}$$

It is easy to check that under (2.11) or (2.12), a and b satisfy (1.3).

Note that now b can be locally erratic, and the bottom of the essential spectrum cannot be deduced directly by looking at H_S.

Proposition 4. *Consider H_D on R^+.*

1. Assume that a and b satisfy (2.11).

i. If $2m + k - 2 > 0$, then $\sigma_{ess}(H_D) = \emptyset$;

ii. If $2m + k - 2 = 0$, then $0 < \inf \sigma_{ess}(H_D) < \infty$;

iii. If $2m + k - 2 < 0$, then $\inf \sigma_{ess}(H_D) = 0$.

2. Assume that a and b satisfy (2.12)*. Then $\inf \sigma_{ess}(H_D) = 0$.*

In particular, H_D possesses a compact resolvent if and only if 1-i holds.

To prove Proposition 4, one makes the same kind of analysis used for the proof of Proposition 3, along with the following monotonicity property which is easy to verify: for fixed a, if (1.3) holds, then for any $x_0 > 0$, $\hat{\Omega}^+(b, a)$ does not depend on $\{b(x), 0 \le x \le x_0\}$ and it is nondecreasing as a function of $\{b(x), x > x_0\}$.

In Propositions 3 and 4, the coefficients a and b are such that (1.3) holds. When (1.6) holds instead, the analysis is more complicated. We state the following analogous result for the case that (1.6) holds. Consider the following analog of (2.11):

(2.13)

$$c_1(1+x)^k \le a(x) \le c_2(1+x)^k, \quad k \in R$$

$$c_1(1+x)^m \le \int_0^x \frac{b(y)}{(1+y)^k} dy \le c_2(1+x)^m, \text{ for large } x, \quad m > 0,\ 0 < c_1 < c_2,$$

and the following analog of (2.12):

$$\begin{aligned} &c_1(1+x)^k \le a(x) \le c_2(1+x)^k,\ k > 1 \\ &\int_0^x \frac{b(y)}{(1+y)^k} dy \text{ is bounded in } x. \end{aligned} \tag{2.14}$$

It can be checked that under (2.13) or (2.14), a and b satisfy (1.6).

Proposition 5. *Consider H_D on R^+. Under some mild regularity conditions on a and b one has the following:*

1. Assume that a and b satisfy (2.13).

i. If $2m+k-2>0$, then $\sigma_{ess}(H_D)=\emptyset$;

ii. If $2m+k-2=0$, then $0<\inf\sigma_{ess}(H_D)<\infty$;

iii. If $2m+k-2<0$, then $\inf\sigma_{ess}(H_D)=0$.

2. Assume that a and b satisfy (2.14). *Then $\inf\sigma_{ess}(H_D)=0$.*

In particular, H_D possesses a compact resolvent if and only if 1-i holds.

To prove Proposition 5, one uses (1.18). This essentially reduces the problem to the one considered in Proposition 4.

We end this section with an example of the phenomenon mentioned in Remark 6. On R^+ we give an example with $a_1=a_2=1$, and with b_1 and b_2 chosen appropriately so that (1.3) holds for a_1,b_1 and a_2,b_2, and such that $B_1(x)\equiv\int_0^x b_1(y)dy\ge B_2(x)\equiv\int_0^x b_2(y)dy$, but such that

$$\inf\sigma(-\frac{1}{2}\frac{d^2}{dx^2}-b_1\frac{d}{dx})>0 \text{ and } \inf\sigma_{\rm ess}(-\frac{1}{2}\frac{d^2}{dx^2}-b_1\frac{d}{dx})=\infty, \tag{2.15}$$

while

$$\inf\sigma(-\frac{1}{2}\frac{d^2}{dx^2}-b_2\frac{d}{dx})=\inf\sigma_{\rm ess}(-\frac{1}{2}\frac{d^2}{dx^2}-b_2\frac{d}{dx})=0. \tag{2.16}$$

Let $b_1(x)=-x$ so that $B_1(x)=\int_0^x b_1(y)dy=-\frac{x^2}{2}$. Then $\Omega^+(b_1,1)<\infty$ and $\hat\Omega^+(b_1,1)=0$, so (2.15) holds. It is not hard to construct a b_2 so that $B_2(x)<B_1(x)$, but such that for each positive integer n, there exists an interval of length n over which b_2 is identically 0. We will now show that $\Omega^+(b_2,1)=\hat\Omega^+(b_2,1)=\infty$; thus, (2.16) holds. Using Theorem 1 and the probabilistic representation in (1.14), we have for the diffusion corresponding to $\frac{1}{2}\frac{d^2}{dx^2}+b_2\frac{d}{dx}$ that

$$\frac{1}{8\Omega^+(b_2,1)}\le\sup\{\lambda\ge0:E_x\exp(\lambda\tau_0)<\infty\}\le\frac{1}{2\Omega^+(b_2,1)},\ x>0. \tag{2.17}$$

Now for Brownian motion (that is, the driftless diffusion corresponding to the operator $\frac{1}{2}\frac{d^2}{dx^2}$) on the interval $(0,n)$, one has $E_x\exp(\lambda(\tau_0\wedge\tau_n))<\infty$, for $x\in(0,n)$, if and only if λ is less than the first eigenvalue for the operator $-\frac{1}{2}\frac{d^2}{dx^2}$ on $(0,n)$ with the Dirichlet boundary condition at 0 and n [4, chapter

3]; that is, if and only if $\lambda < \frac{\pi^2}{2n}$. Since the drift b_2 has intervals of length n over which it vanishes, it follows by comparison with the Brownian motion that for the diffusion corresponding to $\frac{1}{2}\frac{d^2}{dx^2} + b_2\frac{d}{dx}$, if x_n is chosen along such an interval, then $E_{x_n}\exp(\lambda\tau_0) = \infty$, if $\lambda \geq \frac{\pi^2}{2n}$. Since the finiteness or infiniteness of the expectation is independent of the starting point, it follows that in fact this holds for all $x > 0$, not just for some x_n. Since n is arbitrary, it follows that $\sup\{\lambda \geq 0 : E_x \exp(\lambda\tau_0) < \infty\} = 0$. It then follows from (2.17) that $\Omega^+(b_2, 1) = \infty$, and then by the definition of $\hat{\Omega}^+(b_2, 1)$, also $\hat{\Omega}^+(b_2, 1) = \infty$.

3. Proofs of Theorems 1-3 and of (1.14)

Proof of Theorem 1. By Remark 2, it suffices to treat the case in which (1.3) holds. Extend a and b continuously from $[0, \infty)$ to $(-1, \infty)$. For each $l \in (-1, \infty)$, let $H_D^{(l,\infty)}$ denote the corresponding self-adjoint diffusion operator on (l, ∞) with the Dirichlet boundary condition at $x = l$. Consider the problem

$$\begin{aligned} &\frac{1}{2}(au')' + bu' + \lambda u = 0,\ x \in (l, \infty); \\ &u > 0,\ x \in (l, \infty). \end{aligned} \tag{3.1}$$

Let

$$\lambda_c(l) = \sup\{\lambda :\ \text{there is a solution to (3.1)}\}. \tag{3.2}$$

By the criticality theory of second-order elliptic operators, there is a positive solution to the above equation for all $\lambda \leq \lambda_c(l)$ [4, chapter 4—section 3] and one has $\inf \sigma(H_D^{(l,\infty)}) = \lambda_c(l)$ [4, chapter 4—section 10]. It follows from the criticality theory that $\lambda_c(l)$ is right-continuous [4, chapter 4—section 4]. However, in what follows we will need left continuity. We claim that

$$\lambda_c(l) \text{ is continuous in } l. \tag{3.3}$$

We postpone the proof of (3.3) until the end of the proof of Theorem 1. Note that any positive solution as above on (l, ∞) with $l < 0$ is also a positive solution on $[0,\infty)$ and can be normalized by $u(0)=1$. Note also that (3.4) below always has a solution if $\lambda=0$. From these facts, it follows that if we consider the problem

$$\begin{aligned}&\frac{1}{2}(au')'+bu'+\lambda u=0,\ x\in(0,\infty);\\ &u>0,\ x\in(0,\infty);\\ &u(0)=1,\end{aligned} \tag{3.4}$$

then

$$\inf\sigma(H_D)=\sup\{\lambda\ge 0: \text{there is a solution to (3.4)}\}. \tag{3.5}$$

Thus, in order to prove (1.10), it suffices to prove the following proposition.

Proposition 6. *Assume that* (1.3) *holds.*
i. For $\lambda>\frac{1}{2\Omega^+(b,a)}$, *there is no solution to* (3.4)*;*
ii. For $0<\lambda<\frac{1}{8\Omega^+(b,a)}$, *there is a solution to* (3.4).

The proof of part (i) of Proposition 6 is easy, but the proof of part (ii) is nontrivial. The proof of the proposition is given in the next section.

Once (1.10) is proved, one proves (1.11) as follows. An old result of Persson [3], slightly modified to accommodate the case of a half-line, states that

$$\inf\sigma_{\text{ess}}(H_D)=\lim_{l\to\infty}(\inf\sigma(H_D^{(l,\infty)})). \tag{3.6}$$

Letting

$$\Omega_l^+(b,a)=\sup_{x>l}\left(\int_l^x\frac{1}{a(y)}\exp(-2B(y))dy\right)\left(\int_x^\infty\exp(2B(y))dy\right), \tag{3.7}$$

it follows by applying (1.10) to $H_D^{(l,\infty)}$ that

$$\frac{1}{8\Omega_l^+(b,a)}\le\inf\sigma(H_D^{(l,\infty)})\le\frac{1}{2\Omega_l^+(b,a)}. \tag{3.8}$$

We will show that

$$\hat\Omega^+(b,a)=\lim_{l\to\infty}\Omega_l^+(b,a). \tag{3.9}$$

Now (1.11) follows from (3.6), (3.8) and (3.9).

We now prove (3.9). From the definition of $\hat{\Omega}^+(b,a)$, one has for any $l>0$,

(3.10)

$$\Omega_l^+(b,a) \geq \limsup_{x\to\infty}\left(\int_l^x \frac{1}{a(y)}\exp(-2B(y))dy\right)\left(\int_x^\infty \exp(2B(y))dy\right)$$
$$= \limsup_{x\to\infty}\left(\int_0^x \frac{1}{a(y)}\exp(-2B(y))dy\right)\left(\int_x^\infty \exp(2B(y))dy\right) = \hat{\Omega}^+(b,a).$$

On the other hand, for $n=1,2,\cdots$, there exist $x_{0,n}$ and x_n with $x_{0,n}<x_n$ and $\lim_{n\to\infty}x_n=\infty$, and such that

(3.11)

$$\limsup_{l\to\infty}\Omega_l^+(b,a)-\frac{1}{n} \leq \left(\int_{x_{0,n}}^{x_n} \frac{1}{a(y)}\exp(-2B(y))dy\right)\left(\int_{x_n}^\infty \exp(2B(y))dy\right)$$
$$\leq \left(\int_0^{x_n} \frac{1}{a(y)}\exp(-2B(y))dy\right)\left(\int_{x_n}^\infty \exp(2B(y))dy\right).$$

Letting $n\to\infty$ in (3.11) and again using the definition of $\hat{\Omega}^+(b,a)$, we obtain $\limsup_{l\to\infty}\Omega_l^+(b,a)\leq\hat{\Omega}^+(b,a)$. Now (3.9) follows from this and (3.10).

We now return to prove (3.3). As noted previously, we only need prove left-continuity. Without loss of generality, we prove left-continuity at $l=0$. From its definition, λ_c is nondecreasing. Let $\lambda_1<\lambda_2<\lambda_c(0)$. It suffices to show that for $\epsilon>0$ sufficiently small, there is a solution to (3.1) with $l=-\epsilon$ and some $\lambda\geq\lambda_1$. By assumption, there is a solution to (3.1) with $l=0$ and $\lambda=\lambda_2$. Let u be such a solution. Then $u(0^+)=\lim_{x\to0^+}u(x)$ and $u'(0^+)=\lim_{x\to0^+}u'(x)$ exist and are finite. This is because any solution to (3.4) must be a linear combination of Φ_1 and Φ_2, where Φ_1 and Φ_2 are two linearly independent solutions to $\frac{1}{2}(au')'+bu'+\lambda u=0$. If $u(0^+)>0$, then solving the linear equation for $x<0$ using the boundary conditions $u(0^+)$ and $u'(0^+)$ at $x=0$, one can extend the solution u a little bit to the left so that it satisfies (3.1) with $l=-\epsilon$ and $\lambda=\lambda_2$, completing the proof.

Assume now that $u(0^+)=0$. We will show that there exists a $\hat{u}$ which is a solution to (3.1) with $l=0$ and $\lambda=\lambda_1$, and such that $\hat{u}(0)>0$.

Thus, from the previous argument, we can extend $\hat{u}$ a little bit to the left so that it satisfies (3.1) with $l = -\epsilon$ and $\lambda = \lambda_1$, completing the proof. Thus, it remains to show that such a $\hat{u}$ exists. Let ϕ be a smooth compactly supported function on R satisfying $\phi(0) = 1$ and $(\frac{1}{2}(a\phi')' + b\phi' + \lambda_1\phi)(0) = 0$. Let $v = u + \delta\phi$, where u is as above and $\delta > 0$. If δ is sufficiently small, then $v > 0$ on $(0, \infty)$ and $\frac{1}{2}(av')' + bv' + \lambda_1 v = -(\lambda_2 - \lambda_1)u + \delta(\frac{1}{2}(a\phi')' + b\phi' + \lambda_1\phi) <$ 0 on $(0, \infty)$. Thus, v is a sub-solution for (3.1) with $l = 0$ and $\lambda = \lambda_1$, and $v(0) = \delta > 0$. We claim that there is a solution $\hat{u}$ to (3.1) with $l = 0$ and $\lambda = \lambda_1$, and with $\hat{u}(0) = \delta$. Indeed, let $\hat{u}_n$ solve $\frac{1}{2}(a\hat{u}_n')' + b\hat{u}_n' + \lambda_1\hat{u}_n = 0$ in $(0, n)$, with $\hat{u}_n(0) = \delta$ and $\hat{u}_n(n) = 0$. Then by the maximum principal, $\hat{u}_n$ is increasing in n and $\hat{u}_n \le v$; thus $\hat{u} \equiv \lim_{n\to\infty} \hat{u}_n$ exists and is the desired function. □

Proof of Theorem 2. For H_D on the entire line R, the result of Persson, given in (3.6) for R^+, is

$$\inf \sigma_{\mathrm{ess}}(H_D) = \lim_{l\to\infty} \min\left(\inf \sigma(H_D^{(l,\infty)}),\ \inf \sigma(H_D^{(-\infty,-l)})\right),$$

where $H^{(-\infty,-l)}$ denotes the corresponding self-adjoint operator on $(-\infty, -l)$ with the Dirichlet boundary condition at $x = -l$. Theorem 2 follows from this and the above proof of Theorem 1. □

Proof of Theorem 3. By the criticality theory of second-order elliptic operators [4, chapter 4, sections 4 and 10],

$$\inf \sigma(H_D) = \lim_{l\to\infty} (\inf \sigma(H_D^{(-l,\infty)})) = \lim_{l\to\infty} (\inf \sigma(H_D^{(-\infty,l)})). \tag{3.12}$$

Theorem 1 can be applied to $H_D^{(-l,\infty)}$. One simply lets $-l$ play the role played by 0 in Theorem 1. (There is no need to change the definition of $B(x) = \int_0^x \frac{b}{a}(y)dy$, because the lower limit 0 can be replaced by any x_0 without affecting the formulas.) Thus, if $\int^\infty \frac{1}{a(x)} \exp(-2B(x))dx = \infty$, then defining

$$\Omega_{-l}^+(b, a) = \sup_{x>-l} \left(\int_{-l}^x \frac{1}{a(y)} \exp(-2B(y))dy\right) \left(\int_x^\infty \exp(2B(y))dy\right),$$

we have

$$\frac{1}{8\Omega^+_{-l}(b,a)} \le \inf\sigma(H_D^{(-l,\infty)}) \le \frac{1}{2\Omega^+_{-l}(b,a)}. \tag{3.13}$$

But

(3.14)

$$\lim_{l\to\infty}\Omega^+_{-l}(b,a) = \sup_{x\in R}\left(\int_{-\infty}^{x}\frac{1}{a(y)}\exp(-2B(y))dy\right)\left(\int_x^\infty \exp(2B(y))dy\right).$$

In the case that (1.19) or (1.21) holds, Theorem 3 follows from (3.12)-(3.14). The case that (1.22) holds is obtained from the case that (1.21) holds by interchanging the roles of the positive and negative half-lines. For the case that (1.23) holds, one proceeds as above in the case that (1.21) holds, but with $\Omega^+_{-l}(b,a)$ now defined by

$$\Omega^+_{-l}(b,a) =$$
$$\sup_{x>-l}\left(\int_{-l}^{x} h_{b,a}^{-2}(y)\frac{1}{a(y)}\exp(-2B(y))dy\right)\left(\int_x^\infty h_{b,a}^2(y)\exp(2B(y))dy\right).$$

□

We end this section by proving that (1.14) holds in the case that (1.3) is in effect, that is, in the case that $P_x(\tau_0<\infty)=1$. From (3.5), it is enough to show that

(3.15)

$$\sup\{\lambda\ge 0 : \text{there is a solution to (3.4)}\} = \sup\{\lambda\ge 0 : E_x\exp(\lambda\tau_0)<\infty\}.$$

Assume first that $\lambda>0$ is such that there exists a solution to (3.4) and let u be a solution. Then $u(X(t\wedge\tau_0))\exp(\lambda(t\wedge\tau_0))$ is a martingale [4, chapter 2], and thus

$$E_x u(X(t\wedge\tau_0))\exp(\lambda(t\wedge\tau_0)) = u(x). \tag{3.16}$$

Letting $t\to\infty$, it follows from Fatou's lemma that $E_x\exp(\lambda\tau_0)<\infty$.

Conversely, assume that $\lambda>0$ is such that $E_x\exp(\lambda\tau_0)<\infty$. Let $\tau_n = \inf\{t\ge 0 : X(t)=n\}$, for $n>0$. By the Feynman-Kac formula, $u_n(x)\equiv$

$E_x(\exp(\lambda\tau_0); \tau_0 < \tau_n)$ is the solution to the equation

$$\begin{aligned}&\frac{1}{2}(au')' + bu' + \lambda u = 0,\ x \in (0,n);\\ &u(0) = 1,\ u(n) = 0.\end{aligned} \tag{3.17}$$

By the maximum principal, u_n is increasing in n, and (3.4) will have a solution if and only if $\lim_{n\to\infty} u_n(x) < \infty$, in which case $u_\infty(x) \equiv \lim_{n\to\infty} u_n(x)$ is the smallest solution to (3.4). By the monotone convergence theorem and the assumption, we have $u_\infty(x) = E_x \exp(\lambda\tau_0) < \infty$. Thus λ is such that there is a solution to (3.4).

4. Proof of Proposition 6

Let $\lambda > 0$ and let f_n be the unique solution to

$$\begin{aligned}&\frac{1}{2}(af')' + bf' + \lambda f = 0 \text{ in } [0,n];\\ &f(0) = 1, f(n) = 0.\end{aligned}$$

Integrating twice and using the boundary conditions gives

$$\begin{aligned}f_n(x) = 1 &+ c_n \int_0^x \frac{1}{a(y)} \exp(-2B(y))dy\\ &- 2\lambda \int_0^x dy \frac{1}{a(y)} \exp(-2B(y)) \int_0^y dz f_n(z) \exp(2B(z)),\end{aligned} \tag{4.1}$$

where

$$c_n = \frac{-1 + 2\lambda \int_0^n dx \frac{1}{a(x)} \exp(-2B(x)) \int_0^x dy f_n(y) \exp(2B(y))}{\int_0^n \frac{1}{a(x)} \exp(-2B(x))dx}.$$

Note that, by the maximum principle, $f_n \geq 0$ and f_n is nondecreasing in n. Let $f_\infty \equiv \lim_{n\to\infty} f_n$. Recall that by assumption, $\int_0^\infty \frac{1}{a(x)} \exp(-2B(x))dx = \infty$. Thus, $c_\infty \equiv \lim_{n\to\infty} c_n = 2\lambda \int_0^\infty f_\infty(x) \exp(2B(x))dx$. Letting $n \to \infty$ in (4.1) gives

$$f_\infty(x) = 1 + 2\lambda \int_0^x dy \frac{1}{a(y)} \exp(-2B(y)) \int_y^\infty dz f_\infty(z) \exp(2B(z)). \tag{4.2}$$

By the maximum principle and the construction of f_∞, either f_∞ is the smallest solution to (3.4) or else $f_\infty = \infty$ and there are no solutions to (3.4). Using this characterization, we now proof the two parts of the proposition.

Proof of Part (i). We will show that the solution f_∞ of (4.2) is equal to ∞ if $\lambda > \frac{1}{2\Omega(b,a)}$. From (4.2) it follows that f_∞ is nondecreasing, and thus also that

(4.3)
$$f_\infty(x) \geq 1 + 2\lambda \left(\int_0^x \frac{1}{a(y)} \exp(-2B(y))dy \right) \left(\int_x^\infty \exp(2B(y))dy \right) f_\infty(x).$$

If there exists an x for which $2\lambda(\int_0^x \frac{1}{a(y)} \exp(-2B(y))dy)(\int_x^\infty \exp(2B(y))dy) \geq 1$, then (4.3) can not hold for such an x unless $f_\infty(x) = \infty$. Recalling the definition of $\Omega(b, a)$, we conclude that there is no finite solution to (4.2) if $\lambda > \frac{1}{2\Omega(b,a)}$.

Proof of Part (ii). We will show that there is a finite solution to (4.2) if $0 < \lambda < \frac{1}{8\Omega(b,a)}$. We assume that $\Omega(b, a) < \infty$ since otherwise there is nothing to prove. In particular then, we may assume that $\int^\infty \exp(2B(z))dz < \infty$. Fix $\lambda > 0$ and define the operator

$$Tf(x) \equiv 1 + 2\lambda \int_0^x dy \frac{1}{a(y)} \exp(-2B(y)) \int_y^\infty dz f(z) \exp(2B(z)), \tag{4.4}$$

operating on the domain $D_T \equiv \{f : f \geq 0 \text{ and } \int^\infty f(z) \exp(2B(z))dz < \infty\}$. Note that, by assumption, $1 \in D_T$. One can solve (4.2) by iterations. Indeed, it is clear that $T^n 1$ is increasing in n and that $f_\infty = \lim_{n\to\infty} T^n 1$, where T^n denotes the n-th iterate of T. Thus, to prove the existence of a finite solution to (4.2) it is sufficient (and necessary) to show that

$$\lim_{n\to\infty} T^n 1 < \infty. \tag{4.5}$$

Define a norm by $||f|| = \int_0^\infty f(x) \exp(2B(x))dx$. We will prove (4.5) by showing that

$$\lim_{n\to\infty} ||T^n 1|| < \infty. \tag{4.6}$$

Integrating by parts, we have

$$Tf = 1 + 2\lambda S_1 f + 2\lambda S_2 f, \tag{4.7}$$

where

$$S_1f(x) = (\int_0^x \frac{1}{a(z)} \exp(-2B(z))dz)(\int_x^\infty f(z)\exp(2B(z))dz),$$
$$S_2f(x) = \int_0^x dzf(z)\exp(2B(z))\int_0^z dt\frac{1}{a(t)}\exp(-2B(t)). \tag{4.8}$$

Thus,

$$T^n1 = 1 + \sum_{k=1}^n (2\lambda)^k (S_1+S_2)^k 1. \tag{4.9}$$

It is immediate from the definitions of S_1 and $\Omega(b,a)$ that $|S_1 1(x)| \le \Omega(b,a)$, and thus

$$||S_1 1|| \le \Omega(b,a)||1||. \tag{4.10}$$

We will prove the following inequalities:

$$||S_2 f|| \le \Omega(b,a)||f||; \tag{4.11}$$

$$||S_1^n S_2 f|| \le \Omega(b,a)(||S_1^{n-1}S_2 f|| + ||S_1^n f||),\ n \ge 1, \tag{4.12}$$

where S^0 is defined to be the identity operator. From (4.10)-(4.12), it follows that

$$||S_{\delta_1}\cdots S_{\delta_k} 1|| \le (2\Omega(b,a))^k ||1||, \tag{4.13}$$

where $\delta_j = 1$ or 2 for each $j = 1, \cdots k$. From (4.9) and (4.13) it follows that

$$||T^n 1|| \le 1 + \sum_{k=1}^n (2\lambda)^k (2^k)(2\Omega(b,a))^k = 1 + \sum_{k=1}^n (8\lambda\Omega(b,a))^k. \tag{4.14}$$

From (4.14) one concludes that (4.6) holds if $\lambda < \frac{1}{8\Omega(b,a)}$.

We now prove (4.11) and (4.12). Integrating by parts, we have

(4.15)

$$||S_2 f|| = \int_0^\infty dx \exp(2B(x)) \int_0^x dz f(z) \exp(2B(z)) \int_0^z dt \frac{1}{a(t)} \exp(-2B(t)) =$$
$$-\left(\int_x^\infty \exp(2B(z))dz\right)\left(\int_0^x dz f(z)\exp(B(z))\int_0^z dt\frac{1}{a(t)}\exp(-2B(t))\right)\Big|_0^\infty$$
$$+\int_0^\infty dx \left(\int_x^\infty \exp(2B(z))dz\right) f(x)\exp(2B(x))\int_0^x dt\frac{1}{a(t)}\exp(-2B(t))$$
$$\leq \Omega(b,a)\int_0^\infty f(x)\exp(2B(x))dx = \Omega(b,a)||f||,$$

proving (4.11).

We now turn to (4.12). We will write out the proof for $n = 2$; the very same technique holds for general n. We have

(4.16)

$$||S_1^2 S_2 f|| = \int_0^\infty dx \exp(2B(x)) \left(\int_0^x \frac{1}{a(z)}\exp(-2B(z))dz\right) \times$$
$$\left(\int_x^\infty dt \exp(2B(t)) \int_0^t ds \frac{1}{a(s)} \exp(-2B(s)) \int_t^\infty dl \exp(2B(l)) S_2 f(l)\right).$$

Integrating by parts gives

(4.17)

$$\int_t^\infty dl \exp(2B(l)) S_2 f(l) =$$
$$\int_t^\infty dl \exp(2B(l)) \int_0^l dr f(r) \exp(2B(r)) \int_1^r d\rho \frac{1}{a(\rho)} \exp(-2B(\rho)) =$$
$$-\left(\int_l^\infty \exp(2B(\nu))d\nu\right)\left(\int_0^l dr f(r)\exp(2B(r))\int_0^r d\rho\frac{1}{a(\rho)}\exp(-2B(\rho))\right)\Big|_t^\infty +$$
$$\int_t^\infty dl\left(\int_l^\infty d\nu\exp(2B(\nu))\right) f(l)\exp(2B(l))\int_0^l d\rho\frac{1}{a(\rho)}\exp(-2B(\rho)) \leq$$
$$\left(\int_t^\infty \exp(2B(\nu))d\nu\right)\left(\int_0^t dr f(r)\exp(2B(r))\int_0^r d\rho\frac{1}{a(\rho)}\exp(-2B(\rho))\right) +$$
$$\int_t^\infty dl\left(\int_l^\infty d\nu\exp(2B(\nu))\right) f(l)\exp(2B(l))\int_0^l d\rho\frac{1}{a(\rho)}\exp(-2B(\rho)).$$

Substituting (4.17) in (4.16) and using the definition of $\Omega(b,a)$, we obtain

$$||S_1^2S_2f|| \le \int_0^\infty dx \exp(2B(x)) \Big(\int_0^x \frac{1}{a(z)}\exp(-2B(z))dz\Big) \times$$
$$\Big(\int_x^\infty dt \exp(2B(t))\big(\int_0^t ds \frac{1}{a(s)}\exp(-2B(s))\big)\big(\int_t^\infty d\nu \exp(2B(\nu))\big)\times$$
$$\big(\int_0^t dr f(r)\exp(2B(r))\int_0^r d\rho \frac{1}{a(\rho)}\exp(-2B(\rho))\big)\Big)$$
$$+\int_0^\infty dx \exp(2B(x))\big(\int_0^x \frac{1}{a(z)}\exp(-2B(z))dz\big)\times$$
$$\big(\int_x^\infty dt \exp(2B(t))\big(\int_0^t ds\frac{1}{a(s)}\exp(-2B(s))\big)\times$$
$$\big(\int_t^\infty dl\big(\int_l^\infty d\nu \exp(2B(\nu))\big)f(l)\exp(2B(l))\int_0^l ds\frac{1}{a(s)}\exp(-2B(s))\big)\big)$$
$$\le \Omega(b,a)\int_0^\infty dx \exp(2B(x))\Big(\int_0^x dz\frac{1}{a(z)}\exp(-2B(z))\Big)\times$$
$$\big(\int_x^\infty dt \exp(2B(t))\big(\int_0^t dr f(r)\exp(2B(r))\int_0^r d\rho\frac{1}{a(\rho)}\exp(-2B(\rho))\big)\big)$$
$$+\Omega(b,a)\int_0^\infty dx \exp(2B(x))\big(\int_0^x \frac{1}{a(z)}\exp(-2B(z))dz\big)\times$$
$$\big(\int_x^\infty dt \exp(2B(t))\big(\int_0^t ds\frac{1}{a(s)}\exp(-2B(s))\big)\big(\int_t^\infty dl f(l)\exp(2B(l))\big)\big)$$
$$=\Omega(b,a)||S_1S_2f|| + \Omega(b,a)||S_1^2f||.$$

□

5. Application to Multi-Dimensional Diffusion Operators

Consider the multi-dimensional diffusion operator

(5.1)
$$H_D = -\frac{1}{2}\nabla\cdot a\nabla - a\nabla Q\cdot\nabla = -\frac{1}{2}\exp(-2Q)\nabla\cdot a\exp(2Q)\nabla \ \text{ on } R^d,\ d\ge 2,$$

where $a=\{a_{i,j}\}_{i,j=1}^n \in C^1(R^d)$ is positive definite and $Q\in C^1(R^d)$. One can realize H_D as a non-negative, self-adjoint operator on $L^2(R^d,\exp(2Q)dx)$ via the closure of the Friedrichs extension of the nonnegative quadratic form

$$Q_D(f,g)=\frac{1}{2}\int_{R^d}\nabla f a\nabla g\exp(2Q)dx,$$

defined for $f, g \in C_0^1(R^d)$. For $l > 0$, let $B_l(0) \subset R^d$ denote the ball of radius l centered at the origin, and let H_D^l be the self adjoint operator on $R^d - \bar{B}_l(0)$ corresponding to H_D with the Dirichlet boundary condition at $\partial B_l(0)$. More precisely, H_D^l is the Friedrichs extension of the closure of the nonnegative quadratic form

$$Q_D^l(f,g) = \frac{1}{2}\int_{R^d - \bar{B}_l(0)} \nabla f a \nabla g \exp(2Q) dx,$$

defined for $f, g \in C_0^1(R^d - \bar{B}_l(0))$. The result of Persson [3] noted in section 3 gives

$$\inf \sigma_{\text{ess}}(H_D) = \lim_{l\to\infty} \inf \sigma(H_D^l). \tag{5.2}$$

We will give upper and lower bounds on $\inf \sigma(H_D)$ and $\inf \sigma(H_D^l)$ in terms of the corresponding infima for certain one-dimensional operators. From (5.2), this will then also give upper and lower bounds on $\inf \sigma_{\text{ess}}(H_D)$. Applying Theorem 1 to the one-dimensional operators will then yield explicit bounds on $\inf \sigma(H_D)$ and $\inf \sigma_{\text{ess}}(H_D)$.

Letting $r = |x|$ and $\phi \in S^{d-1}$ denote spherical coordinates, let

$$A_{\text{rad-har}}(r,\phi) = \left(\frac{x}{|x|}\ a^{-1}(x)\frac{x}{|x|}\right)^{-1} \tag{5.3}$$

denote the representation in spherical coordinates of the reciprocal of the radially directed quadratic expression $(\frac{x}{|x|}\ a^{-1}(x)\frac{x}{|x|})$. Let $Q_r(x) = \nabla Q(x) \cdot \frac{x}{|x|}$ denote the radial derivative of Q. Write Q_r in spherical coordinates as $Q_r(r,\phi)$. For each $\phi \in S^{d-1}$, define the one-dimensional diffusion operator $H_{\text{rad-har};\phi}$ on R^+ by

(5.4)

$$H_{\text{rad-har};\phi} =$$

$$-\frac{1}{2}\frac{d}{dr}A_{\text{rad-har}}(r,\phi)\frac{d}{dr} - A_{\text{rad-har}}(r,\phi)\frac{d-1}{2r}\frac{d}{dr} - A_{\text{rad-har}}(r,\phi)Q_r(r,\phi)\frac{d}{dr} =$$

$$-\frac{1}{2}r^{1-d}\exp(-2Q(r,\phi))\frac{d}{dr}A_{\text{rad-har}}(r,\phi)\ r^{d-1}\exp(2Q(r,\phi))\frac{d}{dr}\ \text{ on } R^+.$$

Let $(\frac{x}{|x|}\ a(x)\frac{x}{|x|})(r,\phi)$ denote the representation in spherical coordinates of the radially directed quadratic expression $(\frac{x}{|x|}a(x)\frac{x}{|x|})$. Let

$$A_{\text{rad-avg}}(r)=\frac{\int_{S^{d-1}}(\frac{x}{|x|}\ a(x)\frac{x}{|x|})(r,\phi)\exp(2Q(r,\phi))d\phi}{\int_{S^{d-1}}\exp(2Q(r,\phi))d\phi} \tag{5.5}$$

and

$$Q_{r;\text{avg}}(r)=\frac{\int_{S^{d-1}}Q_r(r,\phi)\exp(2Q(r,\phi))d\phi}{\int_{S^{d-1}}\exp(2Q(r,\phi))d\phi}. \tag{5.6}$$

Define the one-dimensional diffusion operator $H_{\text{rad-avg}}$ on R^+ by

(5.7)

$$H_{\text{rad-avg}}=-\frac{1}{2}\frac{d}{dr}A_{\text{rad-avg}}(r)\frac{d}{dr}-A_{\text{rad-avg}}(r)\frac{d-1}{2r}\frac{d}{dr}-A_{\text{rad-avg}}(r)Q_{r;\text{avg}}(r)\frac{d}{dr}$$
$$=-\frac{1}{2}\exp(-2\beta(r))\frac{d}{dr}A_{\text{rad-avg}}(r)\exp(2\beta(r,\phi))\ \frac{d}{dr}\quad\text{on } R^+,$$
$$\text{where } \beta(r,\phi)=\frac{d-1}{2}\log r+\frac{1}{2}\log\int_{S^{d-1}}\exp(2Q(r,\phi))d\phi.$$

Let $H^{(l,\infty)}_{\text{rad-har},\phi}$ and $H^{(l,\infty)}_{\text{rad-avg}}$ denote the corresponding operators on (l,∞) with the Dirichlet boundary condition at $r=l$, as defined in section 3.

Remark 12. Theorem 1 can be applied to the operators $H_{\text{rad-har},\phi}$ and $H_{\text{rad-avg}}$ even though their drifts are not continuous up to 0. Indeed, the theorem applies directly to $H^{(l,\infty)}_{\text{rad-har},\phi}$ and $H^{(l,\infty)}_{\text{rad-avg}}$, for $l>0$, and one has $\inf\sigma(H_{\text{rad-har},\phi})=\lim_{l\to0}\inf\sigma(H^{(l,\infty)}_{\text{rad-har},\phi})$ and $\inf\sigma(H_{\text{rad-avg}})=\lim_{l\to0}\inf\sigma(H^{(l,\infty)}_{\text{rad-avg}})$ [4, chapter 4—sections 4 and 10]. The one change that needs to be made is that B should be defined as $B(x)=\int_{x_0}^x\frac{b}{a}(y)dy$, for some $x_0>0$. (In the proof of Corollary 2 below we use $x_0=1$.)

We will prove the following theorem.

Theorem 4.

$$\inf_{\phi\in S^{d-1}}\inf\sigma(H_{\textit{rad-har};\phi})\le\inf\sigma(H_D)\le\inf\sigma(H_{\textit{rad-avg}})$$

and

$$\inf_{\phi\in S^{d-1}}\inf\sigma(H^{(l,\infty)}_{\textit{rad-har};\phi})\le\inf\sigma(H^{(l,\infty)}_D)\le\inf\sigma(H^{(l,\infty)}_{\textit{rad-avg}}).$$

Applying Theorem 1 and (5.2) to Theorem 4, the following corollary is immediate.

Corollary 1.

$$\frac{1}{8\sup_{\phi\in S^{d-1}}\Omega^+(\beta_{\textit{rad-har};\phi}, A_{\textit{rad-har}}(\cdot,\phi))} \le \inf\sigma(H_D) \le \frac{1}{2\Omega^+(\beta_{\textit{rad-avg}}, A_{\textit{rad-avg}})};$$

$$\frac{1}{8\sup_{\phi\in S^{d-1}}\hat{\Omega}^+(\beta_{\textit{rad-har};\phi}, A_{\textit{rad-har}}(\cdot,\phi))} \le \inf\sigma_{ess}(H_D) \le \frac{1}{2\hat{\Omega}^+(\beta_{\textit{rad-avg}}, A_{\textit{rad-avg}})},$$

where

$$\beta_{\textit{rad-har};\phi}(r) = A_{\textit{rad-har}}(r,\phi)\left(Q_r(r,\phi) + \frac{d-1}{2r}\right),$$

$$\beta_{\textit{rad-avg}}(r) = A_{\textit{rad-avg}}(r)\left(Q_{r;avg}(r) + \frac{d-1}{2r}\right),$$

and Ω^+ *and* $\hat{\Omega}^+$ *are as in Theorem 1. In particular,* $\hat{\Omega}^+(\beta_{\textit{rad-avg}}, A_{\textit{rad-avg}}) = 0$ *is a necessary condition for* H_D *to possess a compact resolvent and* $\hat{\Omega}^+(\beta_{\textit{rad-har};\phi}, A_{\textit{rad-har}}(\cdot,\phi)) = 0$*, for all* $\phi\in S^{d-1}$*, is a sufficient condition.*

We give the following application of Corollary 1.

Corollary 2. *Let* $H_D = -\frac{1}{2}\nabla\cdot a\nabla$ *on* R^d*,* $d\ge 1$*.*

i. If $\lim_{r\to\infty}\frac{\inf_{\phi\in S^{d-1}} A_{\textit{rad-har}}(r,\phi)}{r^2} = \infty$*, then* $\sigma_{ess}(H_D)=\emptyset$ *and* H_D *possesses a compact resolvent;*

ii. If $\lim_{r\to\infty}\frac{A_{\textit{rad-avg}}(r)}{r^2} = 0$*, then* $\inf\sigma_{ess}(H_D) = 0$*;*

iii. If $\inf_{\phi\in S^{d-1}} A_{\textit{rad-har}}(r,\phi)\ge\lambda r^2$*, for large* r*, then* $\inf\sigma_{ess}(H_D)\ge\frac{\lambda d^2}{8}$*;*

iv. If $A_{\textit{rad-avg}}(r)\le\Lambda r^2$*, for large* r*, then* $\inf\sigma_{ess}(H_D)\le\frac{\Lambda d^2}{2}$*.*

Remark 13. Let

$$A_{\min}(r) = \inf_{|v|=1,|x|=r}(va(x)v) \quad\text{and}\quad A_{\max}(r) = \sup_{|v|=1,|x|=r}(va(x)v),$$

and note that

$$\inf_{\phi\in S^{d-1}} A_{\text{rad-har}}(r,\phi)\ge A_{\min}(r) \quad\text{and}\quad A_{\text{rad-avg}}(r)\le A_{\max}(r).$$

Parts (i)-(iii) of the above corollary, with $\inf_{\phi\in S^{d-1}} A_{\text{rad-har}}(\cdot,\phi)$ replaced by $A_{\min}$ and $A_{\text{rad-avg}}$ replaced by $A_{\max}$ are originally due to Davies [1]. The use of $\inf_{\phi\in S^{d-1}} A_{\text{rad-har}}(\cdot,\phi)$ and $A_{\text{rad-avg}}$ instead of $A_{\min}$ and $A_{\max}$ is a significant strengthening. For instance, if for $|x|\ge 1$, the radially directed vector $\frac{x}{|x|}$ is an eigenvector for $a(x)$ with eigenvalue $\gamma(|x|)>1$, and all the other eigenvalues of $a(x)$ are equal to 1, then for $r\ge 1$, one has $A_{\text{rad-har}}(r,\phi)=\gamma(r)$ while $A_{\min}(r)=1$. A two-dimensional example of such a diffusion matrix is

$$a(x)=\begin{pmatrix}\gamma(|x|)\frac{x_1^2}{|x|^2}+\frac{x_2^2}{|x|^2} & \frac{x_1x_2}{|x|^2}(\gamma(|x|)-1)\\ \frac{x_1x_2}{|x|^2}(\gamma(|x|)-1) & \gamma(|x|)\frac{x_2^2}{|x|^2}+\frac{x_1^2}{|x|^2}\end{pmatrix}.$$

Switching the roles of the eigenvalues $\gamma(|x|)$ and 1 above, one has $A_{\text{rad-avg}}(r)=1$ while $A_{\max}(r)=\gamma(r)$. A two-dimensional example of such a diffusion matrix is

$$a(x)=\begin{pmatrix}\frac{x_1^2}{|x|^2}+\gamma(|x|)\frac{x_2^2}{|x|^2} & \frac{x_1x_2}{|x|^2}(1-\gamma(|x|))\\ \frac{x_1x_2}{|x|^2}(1-\gamma(|x|)) & \frac{x_2^2}{|x|^2}+\gamma(|x|)\frac{x_1^2}{|x|^2}\end{pmatrix}.$$

Proof of Corollary 2. By the standard variational formula for $\inf\sigma(H_D^{(l,\infty)})$, it follows that $\inf\sigma(H_D^{(l,\infty)})$ is nondecreasing in a, and thus by (5.2), $\inf\sigma_{\text{ess}}(H_D)$ is also nondecreasing in a . Also from (5.2), it follows that $\inf\sigma_{\text{ess}}(H_D)$ does not depend on $\{a(x), 0<x\le l\}$, for any $l>0$. In light of these facts, (i) and (ii) follow from (iii) and (iv). Also, by the monotonicity in a, for the proof of (iii) we may assume that $A_{\text{rad-har}}(r,\phi)=\lambda r^2$, for large r, and for the proof of (iv) we may assume that $A_{\text{rad-avg}}(r)=\Lambda r^2$, for large r.

We consider (iii), the proof of (iv) following *mutatis mutandi.* From Corollary 1,

$$\inf\sigma_{\text{ess}}(H_D)\ge\frac{1}{8\sup_{\phi\in S^{d-1}}\hat{\Omega}^+(A_{\text{rad-har}}(\cdot,\phi)\frac{d-1}{2r},A_{\text{rad-har}}(\cdot,\phi))}. \tag{5.8}$$

For the pair of arguments of $\hat{\Omega}^+$ in (5.8), one has $\exp(2B(r))=\exp(\int_1^r\frac{d-1}{s}ds)=r^{d-1}$. Since we are assuming that $A_{\text{rad-har}}(r,\phi)=\lambda r^2$ for large r, we have

$\int^\infty \frac{1}{A_{\text{rad-har}}(r,\phi)} r^{1-d} dr < \infty$; thus, (1.6) holds and

(5.9)
$$h_{A_{\text{rad-har}}(\cdot,\phi)\frac{d-1}{2r}, A_{\text{rad-har}}(\cdot,\phi)}(r) = \int_r^\infty \frac{1}{A_{\text{rad-har}}(s,\phi)} s^{1-d} ds = \frac{r^{-d}}{\lambda d}, \text{ for large } r.$$

Writing $h_\phi = h_{A_{\text{rad-har}}(\cdot,\phi)\frac{d-1}{2r}, A_{\text{rad-har}}(\cdot,\phi)}$ to simplify notation, for any $l > 0$, one has from the definition of $\hat{\Omega}^+$,

$$\begin{aligned} &\hat{\Omega}^+(A_{\text{rad-har}}(\cdot,\phi)\frac{d-1}{2r}, A_{\text{rad-har}}(\cdot,\phi)) = \\ &\limsup_{r\to\infty} \left(\int_l^r h_\phi^{-2}(s) \frac{1}{A_{\text{rad-har}}(s,\phi)} s^{1-d}\right) \left(\int_r^\infty h_\phi^2(s) s^{d-1} ds\right) \\ &= \limsup_{r\to\infty} \left((h_\phi^{-1}(r) - h_\phi^{-1}(l)\right) \int_r^\infty h_\phi^2(s) s^{d-1} ds. \end{aligned} \tag{5.10}$$

(In the original definition of $\hat{\Omega}^+$, l above is replaced by 0, however using l does not change the value of the expression.) Choosing l sufficiently large and substituting $h_\phi(r) = \frac{r^{-d}}{\lambda d}$ in (5.10), one concludes that $\hat{\Omega}^+(A_{\text{rad-har}}(\cdot,\phi)\frac{d-1}{2r}, A_{\text{rad-har}}(\cdot,\phi)) = \frac{1}{\lambda d^2}$. Part (iii) now follows from this and (5.8). □

Proof of Theorem 4. We will prove the inequalities for H_D; the exact same method works for $H_D^{(l,\infty)}$. The variational formula for $\inf\sigma(H_D)$ gives

$$\inf\sigma(H_D) = \inf \frac{\frac{1}{2}\int_{R^d} \nabla f a \nabla f \exp(2Q) dx}{\int_{R^d} f^2 \exp(2Q) dx}, \tag{5.11}$$

where the infimum is over $f \in C_0^1(R^d)$.

Using spherical coordinates (r,ϕ), and letting $\inf_{\text{radial}}$ denote the infimum over radially symmetric functions $f \in C_0^1(R^d)$, we have from (5.11),

(5.12)
$$\begin{aligned} \inf\sigma(H_D) &\le \inf_{\text{radial}} \frac{\frac{1}{2}\int_{R^d} \nabla f a \nabla f \exp(2Q) dx}{\int_{R^d} f^2 \exp(2Q) dx} \\ &= \inf_{\text{radial}} \frac{\frac{1}{2}\int_0^\infty (f'(r))^2 \left(\int_{S^{d-1}} (\frac{x}{|x|}\ a(x) \frac{x}{|x|})(r,\phi) \exp(2Q(r,\phi)) d\phi\right) r^{d-1} dr}{\int_0^\infty f^2(r) \left(\int_{S^{d-1}} \exp(2Q(r,\phi)) d\phi\right) r^{d-1} dr} \\ &= \inf_{\text{radial}} \frac{\frac{1}{2}\int_0^\infty (f'(r))^2 A_{\text{rad-avg}}(r) \exp(2\beta(r)) dr}{\int_0^\infty (f^2(r) \exp(2\beta(r)) dr}, \end{aligned}$$

where $\beta(r)=\frac{d-1}{2}\log r+\frac{1}{2}\log\int_{S^{d-1}}\exp(2Q(r,\phi))d\phi$. The infimum on the right hand side of (5.12) is the bottom of the spectrum of the operator $H_{\text{rad-avg}}$ defined in (5.7). This gives the upper bound.

We now prove the lower bound. By the Schwartz inequality,

$$f_r^2(x)=(\nabla f(x)\cdot\frac{x}{|x|})^2\le(\nabla f(x)a(x)\nabla f(x))(\frac{x}{|x|}a^{-1}(x)\frac{x}{|x|}). \tag{5.13}$$

Writing (5.13) in polar coordinates and using the definition of $A_{\text{rad-har}}$, one has

$$(\nabla fa\nabla f)(r,\phi)\ge A_{\text{rad-har}}(r,\phi)f_r^2(r,\phi). \tag{5.14}$$

By the variational formula, for any $g\in C_0^1(R^+)$ and $\phi\in S^{d-1}$,

$$\begin{aligned}&\frac{1}{2}\int_0^\infty A_{\text{rad-har}}(r,\phi)(g'(r))^2r^{d-1}\exp(2Q(r,\phi))dr\\&\ge\inf\sigma(H_{\text{rad-har};\phi})\int_0^\infty g^2(r)r^{d-1}\exp(2Q(r,\phi))dr.\end{aligned} \tag{5.15}$$

From (5.14) and (5.15) one has for $f\in C_0^1(R^d)$,

$$\begin{aligned}&\frac{1}{2}\int_{R^d}\nabla fa\nabla f\exp(2Q)dx\ge\\&\frac{1}{2}\int_{S^{d-1}}\int_0^\infty A_{\text{rad-har}}(r,\phi)f_r^2(r,\phi)r^{d-1}\exp(2Q(r,\phi))drd\phi\\&\ge\int_{S^{d-1}}\inf\sigma(H_{\text{rad-har};\phi})\int_{R^+}f^2(r,\phi)r^{d-1}\exp(2Q(r,\phi))drd\phi\\&\ge\inf_{\phi\in S^{d-1}}\inf\sigma(H_{\text{rad-har};\phi})\int_{R^d}f^2\exp(2Q)dx.\end{aligned} \tag{5.16}$$

From (5.16) we conclude that

$$\inf\sigma(H_D)=\inf\frac{\frac{1}{2}\int_{R^d}\nabla fa\nabla f\exp(2Q(x))dx}{\int_{R^d}f^2\exp(2Q)dx}\ge\inf_{\phi\in S^{d-1}}\inf\sigma(H_{\text{rad-har};\phi}).$$

□

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