ON THE CONSTRUCTION AND SUPPORT PROPERTIES OF MEASURE-VALUED DIFFUSIONS ON $D \subseteq \mathbb{R}^d$ WITH SPATIALLY DEPENDENT BRANCHING.

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Abstract. In this paper, we construct a measure-valued diffusion on $D \subseteq \mathbb{R}^d$ whose underlying motion is a diffusion process with absorption at the boundary corresponding to an elliptic operator

$$L = \frac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla \text{ on } D \subseteq \mathbb{R}^d,$$

and whose spatially dependent branching term is of the form $\beta(x)z - \alpha(x)z^2$, $x \in D$, where β satisfies a very general condition and $\alpha > 0$. In the special case that α and β are bounded from above, we show that the measure-valued process can also be obtained as a limit of approximating branching particle systems.

We give criterion for extinction/survival, recurrence/transience of the support, compactness of the support, compactness of the range, and local extinction for the measure-valued diffusion. We also present a number of examples which reveal that the behavior of the measure-valued diffusion may be dramatically different from that of the approximating particle systems.

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1. Introduction. In this paper, we investigate certain properties of measure-valued diffusions $X(t) = X(t,\cdot)$ on an arbitrary domain $D \subseteq \mathbb{R}^d$, where the underlying motion is a diffusion process with absorption at the boundary corresponding to the elliptic operator

$$L = \frac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla \text{ on } D \subseteq \mathbb{R}^d, \tag{1.1}$$

and the spatially dependent branching term is of the form $\phi(z) = \beta(x)z-\alpha(x)z^2, x \in D$, with $\alpha, \beta \in C^\eta(D), \eta \in (0,1]$ and $\alpha > 0$. We will assume that the diffusion matrix $a = \{a_{i,j}\}$ satisfies $\sum_{i,j=1}^d a_{ij}(x)v_iv_j > 0$, for all $v \in \mathbb{R}^d \setminus \{0\}$ and all $x \in D$ and that $a_{i,j},\ b_i \in C^{1,\eta}(D)$. Under these conditions, there exists a unique solution to the generalized martingale problem for L on D and this solution is a diffusion process $Y(t)$ on the one-point compactification $D^* = D \cup \Delta$ of D with Δ playing the role of a cemetery state [13].

Let $\mathcal{M}_F(D)$ denote the space of finite measures on D. Under the assumption that $\alpha > 0$ and β is bounded from above, we can construct a finite measure valued process via its log-Laplace functional. We first show that for each $g \in C_b^+(D)$, the cone of bounded, continuous, nonnegative functions on D, there exists a minimal, nonnegative solution $u \in C^{2,1}(D\times(0,\infty))\cap C(D\times[0,\infty))$ to the semilinear equation

$$\begin{aligned} u_t &= Lu + \beta u - \alpha u^2,\ (x,t) \in D \times (0,\infty) \\ u(x,0) &= g(x),\ x \in D \end{aligned} \tag{1.2}$$

Furthermore, we show that $u(\cdot,t)$ is bounded for all $t \geq 0$ and that if α and β are bounded, then u is the unique solution to the mild equation

$$u(\cdot,t) = T_t g + \int_0^t ds\ T_s\phi(u(\cdot,t-s)), \tag{1.3}$$

where T_t denotes the semigroup for the diffusion process $Y(t)$. Letting $V_t(g) = u(\cdot,t)$, it then follows that $V_{t+s}(g) = V_tV_s(g)$. ¿From this we can show that

$$\mathcal{L}(t,\mu,g) \equiv \exp\left(-\int_D V_tg(x)\,\mu(dx)\right),\ g \in C_b^+(D),\ \mu \in \mathcal{M}_F(D) \tag{1.4}$$

is a Laplace transition functional. For each $\mu \in \mathcal{M}_F(D)$, this uniquely determines a probability measure P_μ on $C([0,\infty),\mathcal{M}_F(D))$ whose expectation E_μ satisfies

$$\mathcal{L}(t,\mu,g) = E_\mu \exp(- < X(t), g >)\ for\ g \in C_b^+(D), \mu \in \mathcal{M}_F(D). \tag{1.5}$$

See Theorem A1 in Appendix A for the precise statement of the result and for its proof.

Definition 1.1 The measure-valued diffusion process $(P_\mu, \mu \in \mathcal{M}_F(D))$ defined above will be called the $(L, \beta, \alpha; D)$ *-superprocess.*
Remark. We note that this definition will later be extended to a more general class of β's.

If we make the additional assumption that α is bounded from above, then we can show that the (L, β, α, D) superprocess arises as the weak limit of a rescaled, high density branching particle system. Let $Y(t)$ denote the diffusion process corresponding to the solution of the generalized martingale problem for L on D. The diffusion lives on $D^* = D \cup \{\Delta\}$, entering Δ and remaining there forever once it leaves D. For each positive integer n, consider N_n particles, each of mass $\frac{1}{n}$, starting at points $x_i^{(n)}(0) \in D, i = 1, 2, \ldots, N_n$, and performing independent branching diffusion according to the motion process $Y(t)$, with branching rate $cn, c > 0$, and branching distribution $\{p_k^{(n)}(x)\}_{k=0}^\infty$, where

$$e(x) \equiv \sum_{k=0}^{\infty} k p_k^{(n)}(x) = 1 + \frac{\gamma(x)}{n},$$

$$v^2(x) \equiv \sum_{k=0}^{\infty} (k-1)^2 p_k^{(n)}(x) = m(x) + o(1) \ as\ n \to \infty \text{ uniformly in } x,$$

$m, \gamma \in C(D)$ and $m(x) > 0$. Let $N_n(t)$ denote the number of particles alive at time t and denote their positions by $\{X_i^n(t)\}_{i=1}^{N_n(t)}$. Denote by $\mathcal{M}_F(D)$ $(\mathcal{M}_F(D^*))$ the space of finite measures on D (D^*). Define an $\mathcal{M}_F(D^*)$- valued process $X_n(t)$ by $X_n(t) = \frac{1}{n} \sum_1^{N_n(t)} \delta_{X_i^n(t)}(\cdot)$. Denote by $P^{(n)}$ the probability measure on $D([0, \infty), \mathcal{M}_F(D^*))$ corresponding to $X_n(t)$. Assume that $m(x)$ and $\gamma(x)$ are bounded from above. We show that if $w - \lim_{n \to \infty} X_n(0) = \mu \in \mathcal{M}_F(D)$, then the measure $P_\mu \in C([0, \infty), \mathcal{M}_F(D))$ defined in (1.5) satisfies

$$P_\mu = P_\mu^*|_D, \text{ where } P_\mu^* = w - lim_{n \to \infty} P^{(n)},$$

where $\beta = c\gamma(x)$ and $\alpha(x) = \frac{1}{2} c m(x)$, and where the weak limit is taken in the Skorohod space $D([0, \infty), \mathcal{M}_F(D^*))$. Furthermore, P_μ is shown to solve an appropriate martingale problem. See Theorem A2 in Appendix A for the precise statement of the result and for its proof.

We now define the properties which will be investigated in this paper

Definition 1.2 A path $X(\cdot)$ of the measure-valued diffusion *survives* if $X(t) \neq 0$ for all $t \geq 0$ and becomes *extinct* if $X(t) = 0$ for all sufficiently large t. The measure-valued process corresponding to P_μ becomes extinct (survives) if $P_\mu(X(\cdot)$ survives$) = 0$ (> 0).

Remark. Note that extinction for the process can be characterized by the existence of a P_μ-almost surely finite stopping time ζ such that $P_\mu(X(t) = 0, \text{ for } t > \zeta) = 1$.

In the sequel we will frequently use the notation $A \subset\subset B$, which means that A is bounded and $\bar{A} \subset B$.

Definition 1.3 The measure-valued process corresponding to P_μ exhibits *local extinction* if for each $D_0 \subset\subset D$, there exists a P_μ-almost surely finite stopping time ζ_{D_0} such that $P_\mu(X(t, D_0) = 0$, for $t > \zeta_{D_0}) = 1$.

Definition 1.4 Let $X(t)$ be a process that survives.

(i) The support of the measure-valued process corresponding to P_μ is *recurrent* if

$$P_\mu(X(t, B) > 0, \text{ for some } t \geq 0 \mid X(t) \text{ survives }) = 1,$$

for every open $B \subset D$.

(ii) (a) Let $d \geq 2$. The support of the measure-valued process corresponding to P_μ is *transient* if

$$P_\mu(X(t, B) > 0, \text{ for some } t \geq 0 \mid X(t) \text{ survives }) < 1,$$

for all $B \subset\subset D$, which satisfy that $D \setminus B$ is connected and *supp* $\mu \cap \bar{B} = \emptyset$.

(b) Let $d = 1$. The support of the measure-valued process corresponding to P_μ is *transient* if

$$P_\mu(X(t, B) > 0, \text{ for some } t \geq 0 \mid X(t) \text{ survives }) < 1,$$

either for all $B \subset\subset D$, satisfying $\sup B < \inf supp(\mu)$ or for all $B \subset\subset D$ satisfying $\inf B > \sup supp(\mu)$.

Definition 1.5 Let $\mu \in \mathcal{M}_F(D)$ be compactly supported. The measure-valued process corresponding to P_μ possesses the *compact support property* if

$$P_\mu\Big(\bigcup_{0 \leq s \leq t} supp\ X(s) \subset\subset D\Big) = 1, \text{ for all } t \geq 0.$$

Remark. From the results in the paper, it will follow that if any one of the properties in Definitions 1.2-1.5 holds for some P_μ with $\mu \neq 0$ of compact support, then it in fact holds for every P_μ with $\mu \neq 0$ of compact support.

One of the key tools for studying the above properties is a kind of h-transform for measure-valued processes (not in the sense of Doob) which we develop in section two. All the support properties of the superprocess will be shown to be invariant under h-transforms.

In sections three and four, we state our results concerning extinction/survival, local extinction, transience/recurrence and the compact support property. Some of our results give conditions for the above properties to hold in terms of the behavior of certain solutions to the semilinear equation while other results give conditions for the above properties to hold depending on the recurrence/transience or the conservativeness/non-conservativeness of the underlying diffusion process on D.

The results concerning transience/recurrence and local extinction extend the results in [14] where α and β are positive constants, $D = \mathbb{R}^d$, and the diffusion process corresponding to L is conservative.

In section five, we use the results of sections three and four to provide some examples which run counter to the intuition one might glean from considering the measure-valued process as a scaled high density limit of branching diffusions. In one example, we have

$$L = \exp\left(\frac{|x|^2}{d}\right)\left(\frac{1}{2}\Delta - \frac{x}{d}\cdot\nabla\right) \text{ on } D = \mathbb{R}^d, \beta = 0, \text{ and } \alpha = 1.$$

The diffusion process $Y(t)$ corresponding to L is a positive recurrent, time changed Ornstein-Uhlenbeck process. Since $\beta = 0$, the branching in the approximating particle system is critical, and since $\alpha = 1$, the offspring distribution in the particle system has bounded variance. Thus, the measure-valued process arises as the scaled, high density limit of a branching positive recurrent diffusion where the branching is critical and has finite variance. Nonetheless, it turns out that the measure-valued process does not possess the compact support property. We also provide examples to show that many of the conditions imposed in the theorems are necessary for the results to hold.

The proofs of the results of sections three and four are given in section seven. In section six, we present a decomposition theorem for surviving measure-valued diffusions in terms of a superprocess which becomes extinct but which is enriched by an immigration process. This result generalizes [5] which treated the case $\alpha = 1$ and $\beta = 0$. As an application, we point out two results from previous sections which can be proven alternatively using this decomposition.

In Appendix A, we state and prove the two existence theorems for measure-valued diffusions which were described at the beginning of this section. Appendix B gives a summary of results concerning critical and subcritical elliptic operators. These results are needed from time to time in the proofs as well as in the statements of certain theorems.

In the sequel, the notation P_μ, E_μ, and $X(t)$ will be used for the measure valued diffusion and the notation Q_x, E_x, and $Y(t)$ will be used for the diffusion on D corresponding to L. Let $\tau_D = \inf\{t \geq 0 \ : \ Y(t) \notin D\}$. $Y(t)$ is called *conservative* if $Q_x(\tau_D < \infty) = 0$ and *nonconservative* if $Q_x(\tau_D < \infty) > 0$.

2. The h-transform for measure valued processes and the $(L, \beta, \alpha; D)$-superprocess for more general $\beta's$.

In this section we define a kind of "h-transform" — but not in the sense of Doob. Using this h-transform we will extend the definition of the $(L, \beta, \alpha; D)$- superprocess to certain unbounded $\beta's$.

Let $\{X(t)\}_{t\geq 0}$ be a finite measure valued process with Laplace-transition functional

$$\mathcal{L}(s,t,\mu,\phi) \equiv E(e^{-<X(t),\phi>}|X(s)=\mu)\ ,\phi \in C_b^+(D),\ \mu \in \mathcal{M}_F(D). \tag{2.1}$$

Let $0 < h$ be a measurable function on D. Let $\phi = h\psi$ and $\mu = \frac{1}{h}\nu$ (i.e. $\frac{d\nu}{d\mu} = h$). Letting $X^h(t) \equiv hX(t)$ (i.e.$\frac{dX^h(t)}{dX(t)} = h$), we have

$$\mathcal{L}(s,t,\frac{1}{h}\nu,h\psi) = E(e^{-<X(t),h\psi>}|X(s)=\frac{1}{h}\nu) = E(e^{-<X^h(t),\psi>}|X^h(s)=\nu). \tag{2.2}$$

Define the functional

$$\mathcal{L}^h(s,t,\nu,\psi) \equiv \mathcal{L}(s,t,\frac{1}{h}\nu,h\psi),\ \text{for } \nu \in \mathcal{M}_h(D), \psi \in C_b^{+,h}(D), \tag{2.3}$$

where $\mathcal{M}_h(D) \equiv \{\nu \in \mathcal{M}(D), \frac{1}{h}\nu \in \mathcal{M}_F(D)\}$ and $C_b^{+,h}(D) \equiv \{\psi : h\psi \in C_b^+(D)\}$. By (2.2), there exists a unique measure valued process $X^h(t)$ with values in $\mathcal{M}_h(D)$, such that

$$\mathcal{L}^h(s,t,\nu,\psi) = E(e^{-<X^h(t),\psi>}|X^h(s)=\nu),$$

where

$$\nu \in \mathcal{M}_h(D), \psi \in C_b^{+,h}(D). \tag{2.4}$$

Since $X^h(\cdot) \stackrel{d}{=} hX(\cdot)$, it follows that *the support of the measure valued process is invariant under h-transforms* .

Define $\mathcal{M}_C(D) \equiv \{\mu \in \mathcal{M}_F(D) :\ \ supp(\mu) \subset\subset D\}$. The process $X^h(\cdot)$ is uniquely determined by its Laplace transition functional $\mathcal{L}^h(s,t,\nu,\phi)$ restricted to $\nu \in \mathcal{M}_C(D)$ and $\phi \in C_c^+(D)$, where $C_c^+(D)$ denotes the space of nonnegative, continuous, compactly supported functions on D. Thus, in the sequel we will work with these spaces rather than with the h-dependent spaces appearing in (2.4).

We now apply the h-transform to the $(L,\beta,\alpha;D)$-superprocess $X(t)$. Let $\mathcal{A}(u) \equiv Lu + \beta u - \alpha u^2$ on D and let $h \in C^{2,\eta}(D)$ satisfy $h > 0$ in D. Define the h-transform of the linear operator L in the usual way, namely $L^h f = \frac{1}{h}L(hf)$. Then

$$L^h f = L_0^h + \frac{Lh}{h},$$

where

$$L_0^h \equiv L + a\frac{\nabla h}{h}\cdot\nabla. \tag{2.5}$$

Define the h-transform of $\mathcal{A}$ by

$$\mathcal{A}^h(u) \equiv \frac{1}{h}\mathcal{A}(hu) = L_0^h u + \frac{(L+\beta)h}{h}u - \alpha h u^2, \tag{2.6}$$

and $V_t^h(g)$ by $V_t^h(g) \equiv \frac{1}{h}V_t(hg)$ for $g \in C_c^+(D)$. (We must restrict g to $C_c^+(D)$ in order to ensure that the new initial condition is bounded.) Then it is easy to see that $V_t^h(g)$ is the minimal nonnegative solution for the equation

$$\begin{aligned} u_t &= \mathcal{A}^h(u) \\ u(\cdot,0) &= g, \end{aligned}$$

and that

$$\mathcal{L}^h(t,\mu,g) \equiv E\left(e^{-<X^h(t),g>}|X^h(0)=\mu\right) = \exp\left(-\int_D V_t^h g(x)\,\mu(dx)\right),$$

for $g \in C_c^+(D)$, $\mu \in \mathcal{M}_C(D)$. (Note that $V_t^h(g)$ is not bounded in general for $t > 0$, but rather belongs to the space $C_b^{+,h}(D)$.) Thus the quadruple $(L,\beta,\alpha;D)$ transforms into the quadruple $(L_0^h,\beta^h,\alpha^h;D)$, where

$$L_0^h = L + a\frac{\nabla h}{h}\cdot\nabla,\ \beta^h = \frac{(L+\beta)h}{h} \text{ and } \alpha^h = \alpha h. \tag{2.7}$$

Of course, it is possible to have $\sup_D \beta^h = \infty$ even though $\sup_D \beta < \infty$. The process corresponding to $(L_0^h,\beta^h,\alpha^h;D)$ is not finite measure valued in general, but rather $\mathcal{M}_h(D)$-valued. Moreover $\frac{dX^h(t)}{dX(t)} = h$, for all $t \geq 0$.

From the point of view of partial differential equations, we have defined a transformation on semilinear elliptic operators which gives a one to one correspondence between positive solutions for the corresponding operators. For later use, we point out that in the particular case that $0 < h$ satisfies $Lh + \beta h - \alpha h^2 = 0$ on D, the h-transform leads to the quadruple $(L_0^h,\alpha h,\alpha h;D)$.

As will be clear in the sequel, the h-transform is a very useful technique for proving theorems and for coming up with interesting examples. However it should also be pointed out that the h-transform technique allows one to define a unique measure-valued (but generally not *finite* measure-valued) process for a somewhat more general class of β's — recall that in section 1, we defined the $(L,\beta,\alpha;D)$-superprocess under the assumption that β is bounded from above. Let

$$\lambda_c = \lambda_c(L+\beta) \equiv \inf\{\lambda \in R\ :\ \exists u > 0 \text{ satisfying } (L+\beta-\lambda)u = 0 \text{ in } D\}$$

denote the *generalized principal eigenvalue* for $L+\beta$ on D, and assume that

$$\lambda_c(L+\beta) < \infty, \tag{2.8}$$

(see Appendix B for more on the generalized principal eigenvalue.) The probabilistic audience likely to read this paper might prefer the following equivalent definition of λ_c
(see [13]) :

$$\lambda_c = \sup_{\substack{A \subset\subset D \\ \partial A \text{ is smooth}}} \lim_{t\to\infty} \frac{1}{t} \log E_x \left(\exp\left(\int_0^t \beta(Y(s))\, ds \right); \tau_A > t \right),$$

for any $x \in D$, where $\tau_A = \inf(t \geq 0 : Y(t) \notin A)$. ¿From the above probabilistic representation of $\lambda_c(L+\beta)$, it is clear that $\lambda_c(L+\beta) < \infty$ if β is bounded from above. For any $\lambda \geq \lambda_c$, there exist a function $0 < \phi \in C^{2,\eta}(D)$ such that $(L+\beta)\phi = \lambda\phi$ on D. Making an h-transform with $h = \phi$, one sees from (2.7) that the quadruple $(L, \beta, \alpha; D)$ transforms into $(L_0^\phi, \lambda, \alpha\phi; D)$. The latter quadruple corresponds to an $\mathcal{M}_F(D)$-valued process, $X(t)$, according to the results of section 1. Thus, by the h-transform theory above, $(L, \beta, \alpha; D)$ will correspond to the $\mathcal{M}_{1/\phi}(D)$-valued process, $X^{\frac{1}{\phi}}(t)$. Note that this construction does not depend on ϕ or λ. Indeed, let $\lambda_1, \lambda_2 \geq \lambda_c$ and let $(L+\beta-\lambda_1)\phi_1 = (L+\beta-\lambda_2)\phi_2 = 0$ on D, where $\phi_1, \phi_2 > 0$. Let $(L_0^{\phi_i}, \lambda_i, \alpha\phi_i; D)$ correspond to P_μ^i, $i = 1, 2$. Then $(P_\mu^1)^{1/\phi_1}$ coincides with $(P_\mu^2)^{1/\phi_2}$ for any $\mu \in \mathcal{M}_C(D)$, because their Laplace-functionals coincide on $C_c^+(D)$.

We summarize the above in a proposition.

Proposition 2.1. *Let $\lambda_c(L+\beta) < \infty$ and let $\alpha > 0$ on D. Then there exists a unique Borel measure-valued Markov-process $X(t)$ such that for any $\nu \in \mathcal{M}_C(D)$ and $g \in C_c^+(D)$,*

$$E_\nu \exp(- < X(t), g >) = \exp(- \int_D u(x,t)\, \nu(dx)),$$

where $u(x,t)$ is the minimal positive solution to (1.2). If β is bounded from above, then the above statement holds for $g \in C_b^+(D)$ and for $\nu \in \mathcal{M}_F(D)$.

Remark In the proofs we will use the following modification of the log-Laplace equation given in Proposition 2.1:

$$E_{X(t)} \exp(- < g, X(T) >) = \exp(- < u(\cdot, T), X(t) >), \quad P_\mu - a.s. \text{ for } T > 0.$$

This follows easily for $\sup \beta < \infty$ by the fact that $X(t) \in \mathcal{M}_F(D), P_\mu - a.s.$ and by the last sentence in Proposition 2.1. The general case can be reduced to the case when $\beta = const$ by an h-transform (see the paragraph before Proposition 2.1).

Note. The process appearing in Proposition 2.1 with this weaker condition on β will still be called an $(L, \beta, \alpha; D)$-superprocess. Survival, extinction, local extinction, recurrence/transience and h-transforms can be defined similarly for this more general setup as well.

The extension from the condition that β be bounded from above to the more general condition that $\lambda_c(L+\beta) < \infty$ is most significant in bounded domains. Indeed, if $D = \mathbb{R}^d$, L has constant coefficients, and $\lim_{\substack{|x|\to\infty \\ \frac{x}{|x|}\in U}} \beta(x) = \infty$, where $U \subset S^{d-1}$ is an open set, then it can easily be shown that $\lambda_c(L+\beta) = \infty$. On the other hand, if D is a smooth bounded domain and $L = \frac{1}{2}\Delta$, then $\lambda_c(L+\beta) < \infty$ as long as β is locally bounded and $\beta(x) \le \frac{1}{8}(\operatorname{dist}(x, \partial D))^{-2}$, for x near ∂D [9].

In the sequel we shall assume without further mention that β satisfies (2.8).

3. Extinction/survival, the range of the process and the compact support property. We begin with a theorem which characterizes extinction/survival in terms of a certain solution to the semilinear equation.

Theorem 3.1. *Let $\{D_n\}_{n=1}^{\infty}$ be a sequence of bounded domains with smooth boundaries such that $D_n \subset\subset D_{n+1} \subset\subset D$, $n = 1, 2, \ldots$, $\bigcup_{n=1}^{\infty} D_n = D$ and let $\{g_n\}_{n=1}^{\infty}$ be an increasing sequence of smooth functions on D satisfying $0 \le g_n \le n$ and*

$$g_n(x) = \begin{cases} n, & x \in D_n \\ 0, & x \in D \setminus D_{n+1}. \end{cases}$$

Let $u_n(x,t)$ be the minimal nonnegative solution to

$$\begin{aligned} &u_t = Lu + \beta u - \alpha u^2 \quad \text{in } D \times (0,\infty) \\ &u(\cdot, 0) \equiv g_n. \end{aligned} \tag{3.1}$$

Then $u(x,t) \equiv \lim_{n\to\infty} u_n(x,t)$ exists and is finite for $t > 0$. It is the minimal positive solution to

$$\begin{aligned} &u_t = Lu + \beta u - \alpha u^2 \quad \text{in } D \times (0,\infty) \\ &u(\cdot, 0) \equiv \infty, \end{aligned} \tag{3.2}$$

and it satisfies

$$\exp\left(-\int_D u(x,t)\mu(dx)\right) = P_\mu(< X(t), 1 >= 0), \mu \in \mathcal{M}_C(D). \tag{3.3}$$

Furthermore, $w(x) \equiv \lim_{t\to\infty} u(x,t)$ exists, solves the equation

$$Lu + \beta u - \alpha u^2 = 0 \quad \text{on } D, \tag{3.4}$$

and is either identically zero or positive everywhere on D. Moreover,

$$P_\mu(\textit{extinction}) = \exp\left(-\int_D w(x)\mu(dx)\right), \ \mu \in \mathcal{M}_C(D). \tag{3.5}$$

Finally, letting

$$\tilde{P}_\mu(\cdot) \equiv P_\mu(\,\cdot\,|extinction), \tag{3.6}$$

then $\tilde{P}_\mu$ *corresponds to the quadruple* $(L, \beta - 2\alpha w, \alpha; D)$.

Remark If β is bounded from above, the function g_n may be replaced by the constant n; see the last sentence in Proposition 2.1.

The next theorem characterizes the compact embeddedness of the range of the process in terms of the largest solution to the semilinear equation (3.4).

Theorem 3.2. *Let* C *denote the event that the range of the process is compactly embedded in* D*, that is*

$$C \equiv \left\{ \bigcup_{t\geq 0} supp(X(t)) \subset\subset D \right\}.$$

There exists a maximal nonnegative solution to (3.4), w_{max}*, and*

$$P_\mu(C) = \exp\left(-\int_D w_{max}\,\mu(dx)\right),\ \forall \mu \in \mathcal{M}_C(D). \tag{3.7}$$

Also

$$P_\mu(C \cap \ survival) = 0,\ \forall \mu \in \mathcal{M}_C(D). \tag{3.8}$$

The following theorem relates the compact support property to the solutions w and w_{max} appearing in Theorems 3.1 and 3.2 respectively.

Theorem 3.3. *Let* w *and* w_{max} *be as in Theorems 3.1 and 3.2. Let* C *denote the event that the range of the process is compactly embedded in* D *and let* E *denote the event of extinction. Consider the following three conditions:*

(i) *The compact support property holds*;

(ii) $P_\mu(C^c \cap E) = 0$ *for any* $\mu \in \mathcal{M}_C(D)$;

(iii) $w = w_{max}$.

Then $(i) \to (ii) \leftrightarrow (iii)$.
If $w = 0$ *then the three conditions are equivalent. Furthermore, if any of (i)-(iii) holds, then* $P_\mu(survival) > 0$ *if and only if (3.4) possesses a positive solution.*

Remark. From Theorem 3.3 and from (3.8), it follows that if the compact support property holds, then

$$P_\mu(C \triangle E) = 0 \text{ for any } \mu \in \mathcal{M}_C(D), \text{where } \triangle \text{ denotes symmetric difference.}$$

When the compact support property holds, we can use Theorem 3.3 to prove the following simple sufficient condition for survival.

Corollary 3.1. *Assume that the compact support property holds. If* $0 < \inf_{x\in D} \frac{\beta}{\alpha}(x)$*, then* $P_\mu(\mathit{survival}) > 0$*, for all* $\mu \in \mathcal{M}_C(D)$*.*

Remark. If the compact support property does not hold, then the second sentence of Corollary 3.1 is not true in general; see Example 5.3.

We now give an alternative equivalent condition for the compact support property.

Theorem 3.4. *Let* $u(x,t)$ *denote the maximal nonnegative solution to*

$$\begin{aligned} &u_t = Lu + \beta u - \alpha u^2 \ \textit{on}\ D \times (0,\infty), \\ &\lim_{t\to 0} u(x,t) = 0. \end{aligned} \tag{3.9}$$

Then

$$P_\mu\left(\bigcup_{s\le t} \mathit{supp}X(s) \subset\subset D\right) = \exp\left(-\int_D u(x,t)\,\mu(dx)\right),\ \textit{for}\ \mu \in \mathcal{M}_C(D).$$

In particular then, the $(L,\beta,\alpha;D)$*-superprocess has the compact support property if and only if* $u \equiv 0$ *is the unique nonnegative solution to (*3.9*).*

The next theorem gives a sufficient condition for $u = 0$ to be the unique nonnegative solution to (3.9) in terms of α, β and the coefficients of L, in the case $D = \mathbb{R}^d$.

Theorem 3.5. *Let* $D = \mathbb{R}^d$*. Assume that* $\inf_{x\in\mathbb{R}^d} \alpha > 0$*, that* β *is bounded from above, that*

$$\sup_{|v|=1} \sum_{i,j=1}^d a_{ij}(x) v_i v_j \le c(1 + |x|^2)$$

and that

$$|\hat{b}(x)| \le c(1 + |x|)$$

for some $c > 0$*, where* $\hat{b}_i \equiv b_i + \frac{1}{2}\sum_{j=1}^d \frac{d(a_{ij})}{dx_j}$*.*
Then $u = 0$ *is the unique nonnegative solution to (3.9) and thus the* $(L,\beta,\alpha;D)$*-superprocess has the compact support property.*

Remark. In the linear case, that is $\alpha \equiv 0$, the question of whether $u = 0$ is the unique nonnegative solution to (3.9) has been studied by numerous authors starting with Widder (see [1],[12] for example). We note that uniqueness in the linear case does not imply in general uniqueness for $\alpha > 0$. For example, consider $L = \frac{1}{2}\frac{d^2}{dx^2}$, $\beta = 0$, $,\alpha = 0$ and $D = R$. Then $u = 0$ is the unique nonnegative solution to (3.9) (see [1]). However, if $\alpha(x) = \exp\left(-(x^2+1)^2\right)$, making an h-transform with $h = \frac{1}{\alpha}$ and using Theorem 3.4 and Theorem 3.6(i), it is easy to show that there exists a nonnegative solution to (3.9). On the other hand, in Example 5.4, the operator L corresponds to a nonconservative diffusion, $\beta = 0$, but the compact support property holds. By Theorem 3.4, there is no nonnegative solution to (3.9) but it is easy to see that there exists a nonnegative solution to (3.9) if α is replaced by zero. The condition in Theorem 3.5 is weaker then the conditions appearing in the literature for the linear case. In the linear case one typically requires a two-sided bound on $\{a_{ij}\}$ of the form $c_1(1+|x|^2)^\gamma \le \sup_{|v|=1} \sum_{i,j=1}^d a_{ij}(x)v_iv_j \le c_2(1+|x|^2)^\gamma$ for some $\gamma \in [0,1]$.

We now give a connection between the compact support property and the conservativeness of the underlying diffusion process.

Theorem 3.6. *Let w be as in Theorem 3.1 and L_0^w be as in (2.5).*

(i) Assume that $\inf_{x\in D} \frac{\beta}{\alpha} > 0$. If L corresponds to a nonconservative diffusion on D, then the $(L, \beta, \alpha; D)$-superprocess does not possess the compact support property.

(ii) Assume that $w > 0$, that is, that $P_\mu(survival) > 0$, for $0 \ne \mu \in \mathcal{M}_C(D)$. If L_0^w corresponds to a nonconservative diffusion on D, then the $(L, \beta, \alpha; D)$-superprocess does not possess the compact support property.

Remark. If $\inf_{x\in D} \frac{\beta}{\alpha} \le 0$, then the second sentence in (i) is not true in general; see Example 5.4.

When L corresponds to a recurrent diffusion on D and w is bounded, then the compact support property implies the uniqueness of the nonnegative solution for the semilinear elliptic equation.

Theorem 3.7. *Assume that L corresponds to a recurrent diffusion on D. Then there is at most one positive bounded solution to*

$$Lu + \beta u - \alpha u^2 = 0 \text{ in } D. \tag{3.10}$$

In particular, it follows from Theorem 3.3 that if the $(L, \beta, \alpha; D)$-superprocess possesses the compact support property and $0 < w$ is bounded, then w is the unique positive solution to (3.10).

We conclude this section with an explicit calculation in the case that L is conservative on D and α, β are constants. This will be useful when considering the examples.

Proposition 3.1. *Assume that L is conservative on D . Let $\alpha > 0$ and β be constants. Let P_μ correspond to the quadruple $(L, \beta, \alpha; D)$. Define*

$$u_{\beta,\alpha}(t) \equiv \begin{cases} \dfrac{\beta}{\alpha}(1-e^{-\beta t})^{-1}, & \beta \ne 0 \\ \dfrac{1}{\alpha t}, & \beta = 0. \end{cases} \tag{3.11}$$

Then $u(x,t) = u_{\beta,\alpha}(t)$ is the minimal positive solution to (3.2), and thus by (3.3) and (3.5),

$$P_\mu(< X(t), 1 >= 0) = \exp(-u(\beta, \alpha, t) < \mu, 1 >), \mu \in \mathcal{M}_F(D),$$

$$w = \frac{\beta_+}{\alpha} \tag{3.12}$$

and

$$P_\mu(extinction) = \exp(-\frac{\beta_+}{\alpha} < \mu, 1 >), \mu \in \mathcal{M}_F(D), \tag{3.13}$$

where $\beta_+ \equiv \beta \vee 0$.

Proof. First let $\beta \neq 0$. Let

$$0 < u_\delta(\cdot, t) = u_\delta(t) \equiv \frac{\beta}{\alpha}(1 - e^{-\beta(t+\delta)})^{-1} \tag{3.14}$$

on $D \times [0, \infty)$. Then $u_\delta(0) = \frac{\beta}{\alpha}(1 - e^{-\beta\delta})^{-1} \equiv \lambda(\delta)$. Note that $\lambda(\delta) \to \infty$ as $\delta \to 0$. Easy computation shows that u_δ solves

$$u_\delta(t) = \lambda(\delta) + \int_0^t ds\, \phi(u_\delta(t-s)), \tag{3.15}$$

where $\phi(z) = \beta z - \alpha z^2$. Since u_δ depends only on t and (P_t) is conservative, (3.15) is equivalent to

$$u_\delta(t) = P_t\lambda(\delta) + \int_0^t ds\, P_s\phi(u_\delta(t-s)). \tag{3.16}$$

It then follows that u_δ is the minimal nonnegative solution to (3.1) with g_n replaced by the constant $n = \lambda(\delta)$. (See the remark following Theorem 3.1.) Thus, by Theorem 3.1, it follows that $u(x,t)$ (defined in Theorem 3.1) satisfies

$$u(x,t) = \lim_{\delta \to 0} u_\delta(\lambda, t) = \frac{\beta}{\alpha}(1 - e^{-\beta t})^{-1}; \tag{3.17}$$

furthermore,

$$w(x) = \lim_{t \to \infty} u(x,t) = \frac{\beta_+}{\alpha}. \tag{3.18}$$

For $\beta = 0$, define $u_\delta(\cdot, t) = \frac{1}{\alpha(t+\delta)}$ and use a similar argument. $\square$

4. Recurrence and transience of the support. Local extinction. We begin with a theorem which was proved in [14] in the case that α and β are positive constants and $D = \mathbb{R}^d$. The proof in our more general setup is almost exactly the same, except for the finiteness of the function ϕ on $D \setminus \bar{D}_0$.

Theorem 4.1. *Let $D_0 \subset\subset D$ have a smooth boundary. Let $\{D_n\}$ be a sequence of domains satisfying $D_n \subset\subset D_{n+1} \subset\subset \bar{D}_0, n = 1, 2, \cdots$ and $\bigcup_1^\infty D_n = D_0$, and let $\{\psi_n\}_{n=1}^\infty$ be a nondecreasing sequence of functions satisfying $\psi_n \in C_c^+(D), \psi_n(x) = n$ for $x \in D_n, \psi_n(x) = 0$ for $x \notin \bar{D}_0$ and $0 \le \psi_n \le n$. Let $u_n(x,t)$ denote the minimal positive solution to*

$$\begin{aligned} &u_t = Lu + \beta u - \alpha u^2 + \psi_n \text{ on } D \times (0, \infty), \\ &u(\cdot, 0) = 0. \end{aligned} \tag{4.1}$$

Then

$$\phi_n(x) \equiv \lim_{t\to\infty} u_n(x,t)$$

exists for $x \in D$, $\phi_n \in C^{2,\eta}(D)$, and ϕ_n is the minimal positive solution to

$$Lu + \beta u - \alpha u^2 + \psi_n = 0 \text{ in } D. \tag{4.2}$$

Furthermore,

$$\phi(x) = \lim_{n\to\infty} \phi_n(x)$$

exists in the extended sense, for $x \in D, \phi \in C^{2,\eta}(D \setminus \bar{D}_0)$ and $\phi(x) = \infty$ for $x \in \bar{D}_0$. ϕ is the minimal positive solution to

$$\begin{aligned} &Lu + \beta u - \alpha u^2 = 0 \text{ in } D \setminus \bar{D}_0 \\ &\lim_{x\to\partial D_0} u(x) = \infty. \end{aligned} \tag{4.3}$$

(Note that $D \setminus \bar{D}_0$ is not connected if ∂D_0 is not connected.)
Moreover, if $\mu \in \mathcal{M}_C(D)$, then

$$P_\mu(X(t, D_0) = 0, \ \forall t \ge 0) = \exp(-\int_D \phi(x)\, \mu(dx)). \tag{4.4}$$

Theorem 4.2. *Assume that $w > 0$, that is $P_\mu(survival) > 0$, for $0 \ne \mu \in \mathcal{M}_C(D)$. Let ϕ and D_0 be as in Theorem 4.1 and assume that ∂D_0 is connected. Then exactly one of the following two possibilities occurs:*

i) $\phi > w$ on $D \setminus \bar{D}_0$ for any $D_0 \subset\subset D$ and the support of the (L, β, α, D)-superprocess is recurrent.

ii) $\liminf_{x\to\partial D} \frac{\phi}{w}(x) = \inf_{x\in D\setminus\bar{D}_0} \frac{\phi}{w}(x) = 0$ for any $D_0 \subset\subset D$ and the support of the (L, β, α, D)-superprocess is transient.

Moreover, if (i) *occurs, then*

$$P_{\delta_x}(X(t, D_0) = 0 \textit{ for all } t > 0 | \textit{ extinction }) = e^{-(\phi - w)(x)}, x \in D \setminus \bar{D}_0. \tag{4.5}$$

Remark. Using the Markov property, it is easy to see that if the support of $X(t)$ is recurrent, then

$$P_\mu(X(t, D_0) > 0 \text{ for arbitrary large values of } t \mid \text{ survival }) = 1,$$

for $\mu \in \mathcal{M}_C(D)$ and $D_0 \subset D$. Consequently, recurrence excludes local extinction.

We now state a theorem which will be used in the proofs and which may be of independent interest.

Theorem 4.3. *Let* $\hat{L} = L + a\nabla Q \cdot \nabla$ *in* $D \subset \mathbb{R}^d,\ d \geq 1,$ *where* $Q \in C^{2,\eta}(D)$ *. If* L *corresponds to a recurrent (transient) diffusion process and* Q *is bounded from above (below), then* $\hat{L}$ *also corresponds to a recurrent (transient) diffusion process .*

We now give conditions for the recurrence/transience of the support in terms of w and L.

Theorem 4.4. *Assume that* $w > 0$*, that is,*
$P_\mu(survival) > 0,\ 0 \neq \mu \in \mathcal{M}_C(D)$*.*

a) If L_0^w *corresponds to a recurrent diffusion, then the support of* $X(t)$ *is recurrent.*

b) If L *corresponds to a recurrent diffusion and* w *is bounded then the support of* $X(t)$ *is recurrent.*

c) If $\frac{\beta}{\alpha}$ *is bounded from above and* $\sup_D w = \infty$*, then the support of* $X(t)$ *is transient.*

Corollary 4.1 (Comparison). *Let* P_μ *and* $\hat{P}_\mu$ *correspond to* $(L, \beta, \alpha; D)$ *and* $(L, \hat{\beta}, \hat{\alpha}; D)$ *respectively. Assume that* L *corresponds to a recurrent diffusion, that* $P_\mu(survival) > 0$ *for* $0 \neq \mu \in \mathcal{M}_C(D)$*, and that the support of the* $(L, \beta, \alpha; D)$*-superprocess is recurrent. Assume furthermore that* $\hat{\beta} \leq \beta,\ \hat{\alpha} \geq \alpha$ *and that* $\frac{\beta}{\alpha}$ *is bounded from above. Then the* $(L, \hat{\beta}, \hat{\alpha}; D)$*-superprocess either dies out or has recurrent support.*

The following theorem is an application of the above comparison result. It generalizes the result in [14], where the theorem was proved for constant $\alpha, \beta > 0$.

Theorem 4.5. *Let* L *correspond to a recurrent diffusion on* D*, and let* $\inf_{x \in D} \alpha > 0, \sup_{x \in D} \beta < \infty$*.*

a) If $w > 0$*, that is* $P_\mu(survival) > 0,$ *for* $0 \neq \mu \in \mathcal{M}_C(D)$ *, then the support of the* $(L, \beta, \alpha; D)$*-superprocess is recurrent.*

b) Assume in addition that $0 \lneqq \beta$*. Then* $w > 0$ *and the support of the* $(L, \beta, \alpha; D)$*-superprocess is recurrent.*

Conditions for having a transient support are more complicated. Before presenting the following theorem, we note that in the sequel we will frequently use concepts and results from the so-called *criticality theory* of second order elliptic operators. The definitions for *subcritical, critical* and *supercritical* operators, for the *generalized principal eigenvalue* of L on D ($\lambda_c(L,D)$), and for the *generalized principal eigenvalue at* ∞ of L on D ($\lambda_{c,\infty}(L,D)$) are presented in Appendix B. We will use these concepts without further reference.

Theorem 4.6. *Let $X(t)$ correspond to $(L,\beta,\alpha;D)$. Assume that*
(i) L corresponds to a transient diffusion on D;
(ii) $\lambda_{c,\infty}(L+\beta)<0$ if $d\geq 2$ and
either $\lambda_{c,+\infty}(L+\beta)<0$ or $\lambda_{c,-\infty}(L+\beta)<0$ if $d=1$;
(iii) $\inf_{x\in D} w>0$.
Furthermore, assume one of the following:
(iv) the compact support property holds;
(iv') $\beta\geq 0$ and $Q_x(\tau_D<\infty)\neq 1$.
Then the support of $X(t)$ is transient.

Remark. a) If (iv) holds in the above theorem, then (iii) is automatically satisfied if $\inf_{x\in D}\frac{\beta}{\alpha}>0$ or, more generally, if there exists a u such that $\inf_{x\in D} u>0$ and $Lu+\beta u-\alpha u^2\geq 0$. This follows from Theorem 3.3, from the construction of w_{max} and from Proposition 7.1.
b) Let α and β be positive constants. If (i) and (ii) hold and furthermore $Y(t)$ is conservative on D, then (iv') is trivially satisfied and (iii) holds by Proposition 3.1. Thus the support of the superprocess is transient. This particular case was proved in [14] for $D=\mathbb{R}^d$.

We now present characterizations for the recurrence of the support in terms of the behavior of the nonnegative solutions for the corresponding semilinear elliptic equation on D and on exterior domains in D.

We shall need the concept of *positive solutions of minimal growth.* If $\Omega\subset\subset D$, then $0<u\in C^2(D\setminus\bar\Omega)\cap C(D\setminus\Omega)$ is called a *positive solution of minimal growth (at ∂D)* if it satisfies (3.4) with D replaced by $D\setminus\bar\Omega$ and for any Ω', satisfying $\Omega\subsetneqq\Omega'\subset\subset D$ and any $\hat u\in C^2(D\setminus\bar\Omega')\cap C(D\setminus\Omega')$ which solves $L\hat u+\beta\hat u-\alpha\hat u^2=0$ on $D\setminus\bar\Omega'$, there exists a $c>0$ such that $u\leq c\hat u$ on $D\setminus\Omega'$. Note that if u and v are positive solutions of minimal growth, then $c<u/v<C$ for constants $c,C>0$.
The next theorem connects the concept of recurrent support to the minimality of elliptic solutions on D.

Theorem 4.7. *Assume that the $(L,\beta,\alpha;D)$-superprocess survives and satisfies the compact support property . The support of the superprocess is recurrent if and only if every positive solution of (3.4) is of minimal growth. Moreover, the above condition is sufficient for the recurrence of the support even if the compact support property fails.*

The next theorem expresses the recurrence of the support in terms of positive elliptic solutions on *exterior* domains.

Theorem 4.8. *Assume that the $(L, \beta, \alpha; D)$-superprocess survives and satisfies the compact support property . The support of the superprocess is recurrent if and only if for any $\Omega \subset\subset D$ with smooth boundary, the following exterior problem has a unique solution:*

$$\begin{aligned} Lu + \beta u - \alpha u^2 &= 0 \text{ in } D \setminus \bar{\Omega} \\ \lim_{x \to \partial\Omega} u(x) &= \infty. \end{aligned} \tag{4.6}$$

Moreover, the above condition is sufficient for the recurrence of the support even if the compact support property fails. When a unique solution to (4.6) exists, then any $u > 0$ satisfying $Lu + \beta u - \alpha u^2 = 0$ in $D \setminus \bar{\Omega}$ is a positive solution of minimal growth at ∂D.

We close this section with a characterization of local extinction.

Theorem 4.9. *The support of $X(t)$ corresponding to $(L, \beta, \alpha; D)$ exhibits local extinction if and only if there exists a positive solution u to the equation $(L+\beta)u = 0$ on D.*

The proof of Theorem 4.9 will be omitted since it is virtually identical to the proof given in [14] for the case that $\alpha, \beta > 0$ constants.

Remark. If there is no positive solution to the equation $(L + \beta)u = 0$ on D, then the operator is said to be *supercritical* ; this is equivalent to $\lambda_c(L + \beta) > 0$, where $\lambda_c(L+\beta)$ denotes the generalized principal eigenvalue of $L+\beta$ on D. (See Appendix B for more elaboration.)

Since extinction implies local extinction, we immediately get the following sufficient condition for survival:

Corollary 4.2. *If there does not exist a $u > 0$ satisfying $(L + \beta)u = 0$ on D (that is, if $\lambda_c(L + \beta) > 0$ on D) , then $P_\mu(X(t) \text{ survives}) > 0$, for $0 \not\equiv \mu \in \mathcal{M}_C(D)$.*

5. Examples. In this section, we present several examples to illustrate the results of the previous sections. In the first two examples, we show that the global behavior of the superprocess may be very different from the global behavior of the approximating particle system. In the last three examples as well as in the first example, we show that certain conditions appearing in the statements of some of the theorems are necessary for the validity of the results. Our method is based on the use of appropriate h-transforms. Recall that an h-transform leaves the support of the measure valued process invariant (see section 2). Consequently, all the concepts defined for the support of $X(t)$ in the previous sections (survival/extinction, the compact support property, local extinction, transcience/recurrence) are invariant under h-transforms. Note that w and w_{max} (defined in Theorems 3.1 and 3.2) transform into $\frac{w}{h}$ and $\frac{w_{max}}{h}$. This can be seen easily from the construction of w and w_{max}.

Example 5.1. Consider the superprocess corresponding to the quadruple

$$\left(\frac{1}{2}\Delta - \frac{x}{d}\cdot\nabla, 0, \exp(-|x|^2/d); \mathbb{R}^d\right) \tag{5.1}$$

The underlying motion corresponding to

$$L = \frac{1}{2}\Delta - \frac{x}{d}\cdot\nabla \text{ on } \mathbb{R}^d$$

is an Ornstein-Uhlenbeck process, which is positive recurrent. Since $\beta = 0$ and $\alpha(x) = \exp(-|x|^2/d)$, the superprocess may be obtained as the scaled, high density limit as $n \to \infty$ of $O(n)$ critical, binary branching Ornstein-Uhlenbeck processes, each of mass $\frac{1}{n}$ and with spatially dependent branching intensity $n\exp(-|x|^2/d)$. Note that the probability that a particle $Y(t)$ has not branched by time t is $\exp(-\int_0^t n\alpha(Y(s))ds)$; thus, since the Ornstein-Uhlenbeck process is recurrent and the branching is critical, it follows that the branching particle system will become extinct with probability one. On the other hand, we have the following result:

Claim 5.1. i) The superprocess corresponding to (5.1) survives; in fact

$$\sup_{x\in\mathbb{R}^d} P_{\delta_x}(\text{ survival }) = 1.$$

ii) The superprocess corresponding to (5.1) exhibits local extinction and thus, in particular, its support is transient.

Remark. Recall, that in Theorem 4.4 (b) we proved that if the underlying diffusion is recurrent and if in addition, the w-function for the process is positive and bounded, then the support is recurrent. Thus Example 5.1 also shows that the assumption on the boundedness of w is necessary.

Proof. In order to calculate w, we will obtain the quadruple (5.1) as the h-transform of another quadruple whose w-function is easy to calculate. Let $\hat{L} = \frac{1}{2}\Delta + \frac{x}{d}\cdot\nabla$ in $\mathbb{R}^d$ and consider the superprocess on $\mathbb{R}^d$ corresponding to $\hat{L}$, $\beta = 1$, and $\alpha = 1$; that is, corresponding to the quadruple $(\hat{L}, 1, 1, \mathbb{R}^d)$. Let $\hat{w}$ denote the w function from Theorem 3.1 for the $(\hat{L}, 1, 1, \mathbb{R}^d)$-process. ¿From Proposition 3.1, we conclude that $\hat{w} = 1$. The function $\phi(x) = \exp(-\frac{|x|^2}{d})$ satisfies $(\hat{L}+1)\phi = 0$. Applying an h-transform with $h = \phi$, if follows from (2.7) that the quadruple $(\hat{L}, 1, 1, \mathbb{R}^d)$ transforms into (5.1). Thus $w(x) = \frac{\hat{w}(x)}{\phi(x)} = \exp(\frac{|x|^2}{d})$. Since $P_{\delta_x}(survival) = 1 - \exp(-w(x))$, (i) now follows. The transience of the support follows directly from Theorem 4.4(c) (or indirectly from the local extinction property proved in the line below). Local extinction for (5.1) follows from Theorem 4.9 since $u \equiv 1$ satisfies $(L+\beta)u = Lu = 0$.

Remark. One can also get an example where L corresponds to a positive recurrent diffusion process and the support of the superprocess is transient without exhibiting local extinction. Replace $\beta \equiv 0$ by $\beta = \alpha$ in the above example; then Theorem 4.9 together with Theorem 4.6.3 in [13] guarantee that the support of the corresponding process does not exhibit local extinction. By comparison, the new w-function is even larger and thus there is a positive probability of survival. By Theorem 4.4(c), the support of the corresponding process is transient.

Example 5.2. Consider the superprocess corresponding to the quadruple

$$\left(\exp(|x|^2/d)(\frac{1}{2}\Delta - \frac{x}{d}\cdot\nabla), 0, 1; \mathbb{R}^d\right). \tag{5.2}$$

The underlying motion corresponding to $L = \exp(|x|^2/d)(\frac{1}{2}\Delta - \frac{x}{d}\cdot\nabla)$ is a time-changed Ornstein-Uhlenbeck process; since $\exp(|x|^2/d) \geq 1$, it is positive recurrent. Since $\beta = 0$ and $\alpha = 1$, the process may be obtained as the scaled, high density limit of $O(n)$ critical, binary branching time-changed Ornstein-Uhlenbeck processes, each of mass $\frac{1}{n}$ and with constant branching intensity n. By the argument given in the first example, the particle system becomes extinct with probability one. Also, since the underlying process is conservative, the particle system obviously possesses the compact support property.

Claim 5.2. The superprocess corresponding to (5.2) becomes extinct; however it does not possess the compact support property.

Proof. By Proposition 3.1, $w = 0$ for the quadruple (5.2), and thus the superprocess becomes extinct. On the other hand the function $u(x) \equiv \exp(\frac{|x|^2}{d})$, which was the w function for the quadruple (5.1), satisfies $Lu + \beta u - \alpha u^2 = Lu - u^2 = 0$. Thus, $w_{max} > 0 = w$ for the quadruple (5.2), and it follows from Theorem 3.3 that the compact support property does not hold.

Example 5.3. Consider the superprocess corresponding to the quadruple

$$\left(\exp(|x|^2/d)(\frac{1}{2}\Delta + \frac{x}{d}\cdot\nabla), \exp(|x|^2/d), \exp(|x|^2/d); \mathbb{R}^d\right) \tag{5.3}$$

In this example, $\frac{\beta}{\alpha} = 1$. According to Corollary 3.1, if $\inf_{x\in D}\frac{\beta}{\alpha}(x) > 0$, and if in addition the compact support property holds, then the superprocess survives. The claim below shows that the additional condition concerning the compact support property is necessary.

Claim 5.3. The superprocess corresponding to (5.3) becomes extinct.

Proof. The quadruple (5.3) may be obtained as the h-transform of the quadruple (5.2) via the function $h = \exp(\frac{|x|^2}{d})$. Since the superprocess corresponding to (5.2) becomes extinct, so does the superprocess corresponding to (5.3).

Example 5.4. Consider the superprocess corresponding to the quadruple

$$\left(c(x)(\frac{1}{2}\frac{d}{dx^2}+\frac{d}{dx}),0,c(x);R\right), \tag{5.4}$$

where $c(x)>0$ and $c(x)=|x|^l$ for large $|x|$, with $l>1$. By Feller's test for explosion [13, Theorem 5.1.5] and some elementary analysis, it follows that the diffusion process corresponding to $L=c(x)(\frac{1}{2}\frac{d^2}{dx^2}+\frac{d}{dx})$ is nonconservative. According to Theorem 3.6-(i), if the underlying diffusion is nonconservative and $\inf_{x\in D}\frac{\beta}{\alpha}>0$, then the compact support property does not hold. The claim below shows that the additional condition concerning $\frac{\beta}{\alpha}$ is necessary.

Claim 5.4 The range of the superprocess corresponding to (5.4) is almost surely compactly embedded and thus, in particular, the superprocess possesses the compact support property.

Proof. Consider the quadruple $(\frac{1}{c(x)}L,0,1,;\mathbb{R}^d)$. The diffusion corresponding to the operator $\frac{1}{c(x)}L$ is conservative, and by Theorem 3.5, the compact support property holds for this quadruple. Thus, it follows from Proposition 3.1 and Theorem 3.3 that $w_{max}=w=0$ for this quadruple. In particular then, there are no positive solutions u to $\frac{1}{c(x)}Lu-u^2=0$. Thus, there are also no positive solutions u to $Lu-c(x)u^2=0$; hence $w_{max}=w=0$ for the original quadruple (5.4), and it follows by Theorem 3.2 that the range of the process corresponding to the quadruple (5.4) is almost surely compactly embedded.

Example 5.5. Consider the superprocess corresponding to the quadruple

$$\left(\frac{1}{2}\Delta+kx\cdot\nabla,kd+\epsilon,e^{k|x|^2};\mathbb{R}^d\right), \tag{5.5}$$

where $\epsilon>0$. The diffusion corresponding to $L=\frac{1}{2}\Delta+kx\cdot\nabla$ is conservative. Also, we have $\lambda_c^\infty=-\infty$, if $d\geq 2$, and $\lambda_c^{\pm\infty}=-\infty$, if $d=1$ [14, example 2]. Thus conditions (i), (ii), and (iv$'$) of Theorem 4.6 hold. According to that theorem, if in addition to the above conditions, $\inf_{x\in\mathbb{R}^d}w(x)>0$ (condition (iii)), then the support of the superprocess would be transient. The claim below shows the necessity of condition (iii) concerning w. (For examples where (i), (iii) and $(iv)'$ are satisfied in Theorem 4.6, but (ii) is not and the superprocess has a recurrent support, see [14, Theorem 4].)

Claim 5.5. The superprocess corresponding to (5.5) survives and has a recurrent support.

Proof. We have $\lambda_c(L) = -kd$ [14, example 2]. Since $\lambda_c(L) \neq \lambda_c^{\infty}$, if $d \geq 2$, and $\lambda_c(L) \neq \lambda_c^{\pm\infty}$, if $d = 1$, it follows from Theorem 4.7.2 in [13] that $L - \lambda_c(L)$ on $\mathbb{R}^d$ is a critical operator. The ground state for $L - \lambda_c(L)$ is $\phi_c(x) = \exp(-k|x|^2)$. Let $\beta = kd + \epsilon$ and $\alpha = e^{k|x|^2}$. Note that

$$\lambda_c^{\infty}(L + \beta) = -\infty.$$

Make an h-transform on $(L, \beta, \alpha; \mathbb{R}^d)$ with $h = \phi_c$. Then

$$(L + \beta)^{\phi_c} = (L + kd)^{\phi_c} + \epsilon$$

where $(L + kd)^{\phi_c}$ corresponds to a recurrent diffusion process. The h-transformed superprocess corresponds to $((L + kd)^{\phi_c}, \epsilon, \epsilon; D)$. Since $(L + kd)^{\phi_c}$ corresponds to a recurrent diffusion process, by Theorem 4.5(b), the process survives with positive probability and the support is recurrent. The same holds then for the original process as well.

Remark. Claim 5.5 also shows that the result in Corollary 4.1 does not hold in general for transient diffusions. Indeed, $1 \leq \alpha$ but, using Theorem 4.6, it is easy to show that the superprocess corresponding to $(L, \beta, 1; \mathbb{R}^d)$ has a transient support.

6. Decomposition with immigration. In this section we present a generalization of the result in [5] on the decomposition of superprocesses with immigration. Let $\mathcal{N}(D)$ denote the class of discrete measures on D and let $X(t)$ correspond to the quadruple $(L, \beta, \alpha; D)$. Denote the corresponding probabilities by $\{P_\mu\}$. Let $\tilde{X}(t)$ be the superprocess corresponding to the quadruple $(L, \beta - 2\alpha w, \alpha; D)$, where w is as in Theorem 3.1. By Theorem 3.1, $\tilde{X}(t)$ corresponds to $P_\mu(\cdot \mid \textit{extinction})$. Let $B^\nu(s)$ denote the branching diffusion with motion process corresponding to L_0^w on D and branching term $\alpha w(x)(z^2 - z)$ (that is, binary branching at the spatially dependent rate $\alpha w(x)$), with initial measure $\nu \in \mathcal{N}(D)$. Denote by $N(s)$ the number of particles at time s for $s \geq 0$, and let

$$Z_\nu(s) = 2 \sum_{1}^{N(s)} \alpha(B_i^\nu(s)) \delta_{B_i^\nu(s)} \text{ for } B^\nu(s) = \{B_i^\nu(s)\}_{i=1}^{N(s)}.$$

Conditional on $\{Z_\nu(s)\}_{s=0}^{\infty}$, let $(R(t), \mathbb{P}^{\mu,\nu})$ be the superprocess obtained by taking the process $\tilde{X}(t)$ with starting measure μ, and adding immigration, where the immigration at time t is according to the measure $Z_\nu(t)$. This is described mathematically by the conditional Laplace functional

$$\begin{aligned} &\mathbb{E}^{\mu,\nu}(\exp(- < R(t), f > - < Z(t), k >) \mid Z_\nu(s), s \geq 0) \\ (6.1) \quad & \\ &= \exp(- < \mu, \tilde{u}_f(\cdot, t) > - \int_0^t ds < Z_\nu(s), \tilde{u}_f(\cdot, t - s) > - < Z_\nu(t), k >), \end{aligned}$$

for $f,k \in C_c^+$, where $\tilde{u}_f$ denotes the minimal nonnegative solution to (1.2) with g replaced by f and β and α replaced by $(\beta - 2\alpha w)$ and αw respectively (see also [5] and references therein). Denote by N_μ the law of the Poisson random measure on D with intensity $w\mu$ and define the random initial measure η by

$$\mathcal{L}(\eta) \equiv \delta_\mu \times N_\mu.$$

Theorem 6.1. *The law of $R(t)$ under $\mathbb{P}^\eta$ is P_μ.*

In [5], this result was proved when α and β are constant and the underlying motion is a conservative Markov process (note that if L corresponds to a conservative diffusion and α and β are positive constants, then $\alpha w = \beta$ and $L^w = L$). The proof of Theorem 6.1 is a modification of the proof of Theorem 3.2. in [5], using Theorem 3.1 of this paper.

Remark Letting $k = 0$ and choosing appropriate functions f in (6.1), it is not hard to give alternative proofs for Theorem 4.4 (a) and Theorem 3.6(ii).

Sketch of the proof. First, note that it is enough to prove the statement for $\beta = const$. This follows by applying an h-transform with an h for which $\beta^h =$ constant. (The existence of such an h follows by the paragraph before Proposition 2.1.) Recall that $X^h(t) = hX(t)$. On the other hand, since $\alpha^h = \alpha h$, $w^h = \frac{w}{h}$, and the new starting measure is $h\mu$, the branching rate αw and the intensity $w\mu$ for the Poisson random measure are invariant under h-transforms, while the immigration term $Z_\nu(t)$ transforms into $h \cdot Z_\nu(t)$.

Assume now that $\beta = const$. Using the method of the proof of Theorem 3.2 in [5], one has to show two things — that the one-dimensional distributions of $R(t)$ under $\delta_\mu \times N_\mu$ and the one-dimensional distributions of $X(t)$ under P_μ coincide, and that $R(t)$ under $\delta_\mu \times N_\mu$ is a Markov process. For the latter, one can use Lemma 3.3 in [5] the same way as it is used in the proof of Theorem 3.2 in [5]. Concerning the one-dimensional distributions, the statement can be formulated as follows. Let $f \in C_b^+(D)$. Let u be the minimal nonnegative solution to

$$\begin{aligned} &u_t = Lu + \beta u - \alpha u^2 \quad \text{on } D \times (0,\infty) \\ &u(\cdot,0) = f, \end{aligned} \tag{6.2}$$

and let $\tilde{u}$ be the minimal nonnegative solution to

$$\begin{aligned} &u_t = Lu + (\beta - 2\alpha w)u - \alpha u^2 \quad \text{on } D \times (0,\infty) \\ &u(\cdot,0) = f. \end{aligned} \tag{6.3}$$

Then $u - \tilde{u} = wG$, where G is the minimal nonnegative solution to

$$\begin{aligned} &G_t = L_0^w G - \alpha w(G^2 - G) + 2\alpha\tilde{u}(1-G) \quad \text{on } D \times (0,\infty) \\ &G(\cdot,0) = 0 \end{aligned} \tag{6.4}$$

(c.f. [5], p.194.). (Note that $u-\tilde{u}$ is nonnegative by equation (22)in [5], or alternatively by the parabolic maximum principle .) Using the fact that $\frac{Lw}{w} = \alpha w - \beta$ and applying an h-transform on (6.4) with $h = \frac{1}{w}$, one obtains that wG is the minimal nonnegative solution to

$$
\begin{aligned}
&v_t = Lv + (\beta - 2\alpha\tilde{u})v - \alpha v^2 + 2\alpha w\tilde{u} \quad \text{on } D \times (0,\infty) \\
&v(\cdot,0) = 0. \qquad (6.5)
\end{aligned}
$$

A simple computation shows that $u-\tilde{u}$ also satisfies (6.5). In order to complete the proof, we have to show the minimality of $u-\tilde{u}$. Recall that by the construction of u in Lemma A1, u ($\tilde{u}$) is given as the monotone increasing limit of certain solutions u_n ($\tilde{u}_n$), where $u_n(\cdot,t)$ ($\tilde{u}_n(\cdot,t)$) vanishes outside D_n, $D_n \subset\subset D$ for all $t \geq 0$. Since $u - \tilde{u} = \lim_{n\to\infty}(u_n - \tilde{u}_n)$, the minimality of $u-\tilde{u}$ follows by the parabolic maximum principle . $\square$

7. Proofs for Sections 3 and 4.

In the sequel we will frequently use the following two maximum principles. In [14], these maximum principles were proved for $\beta, \alpha > 0$ constants; the proofs go through for our case without difficulty.

Proposition 7.1 (Elliptic maximum principle).
Let $D \subset \mathbb{R}^d$ be a bounded $C^{2,\eta}$-domain, $\eta \in (0,1]$ and let $\beta, \alpha \in C^\eta(\bar{D})$, $\alpha > 0$. Let $v_1, v_2 \in C^{2,\eta}(D) \cap C(\bar{D})$ satisfy $v_1, v_2 > 0$ in D, $Lv_1 + \beta v_1 - \alpha v_1^2 \leq \min(0, Lv_2 + \beta v_2 - \alpha v_2^2)$ in D and $v_1 \geq v_2$ on ∂D. Then $v_1 \geq v_2$ in $\bar{D}$.

Proposition 7.2 (Parabolic maximum principle).
*Let $D \subset \mathbb{R}^d$ be a bounded region and let $\beta, \alpha \in C^\eta(\bar{D}), \eta \in (0,1]$, with $\alpha > 0$. Let $0 \leq v_1, v_2 \in C^{2,1}(D \times (0,\infty)) \cap C(\bar{D} \times (0,\infty))$ satisfy $Lv_1 + \beta v_1 - \alpha v_1^2 - (v_1)_t \leq Lv_2 + \beta v_2 - \alpha v_2^2 - (v_2)_t$
in $D \times (0,\infty)$,$v_1(x,0) \geq v_2(x,0)$ for $x \in D$, and $v_1(x,t) \geq v_2(x,t)$ for $x \in \partial D$ and $t > 0$. Then $v_1 \geq v_2$ in $\bar{D} \times [0,\infty)$.*

In the sequel we will use the following more general form of the log-Laplace equation as well:

$$
E_\mu \exp(- < g, X(t) > - \int_0^t < \psi, X(s) > ds) = \exp(-\int_D u(x,t)\mu(dx)),
$$

for all

$$
0 \leq g, \psi \in C_c(D), \mu \in \mathcal{M}_C(D),
$$

where $u \in C^{2,1}(D \times [0,\infty))$ is the minimal nonnegative solution of the equation

$$
\begin{aligned}
&u_t = Lu + \beta u - \alpha u^2 + \psi \quad \text{on } D \times (0,\infty) \\
&u(\cdot,0) = g(\cdot).
\end{aligned}
$$

(See [8], p.96).

We will also need the following lemma.

Lemma 7.1. *Let $x_0 \in D$ and $R > 0$ be such that $B_R(x_0) \subset\subset D$. Then for any $t > 0$, and any finite measure μ compactly supported in $B_R(x_0)$,*

$$P_\mu(<1, X(t)>=0, X(s, D \setminus \bar{B}_R(x_0)) = 0, \ \textit{for } 0 \le s \le t) > 0,$$

Proof of Lemma 7.1. Fix $x_0 \in D$ and $R > 0$ such that $B_R(x_0) \subset\subset D$. Let $\alpha \ge A$ and $\beta \le B$ on $B_R(x_0)$. Define the events

$$E_t \equiv \{<1, X(t)>=0\}, \ A_t \equiv \{X(s, D \setminus B_R(x_0)) = 0, \ 0 \le s \le t\}.$$

Let $\{\psi_n\}$ be a nondecreasing sequence of smooth functions, satisfying $0 \le \psi_n \le n$ and

$$\psi_n(y) = \begin{cases} 0, \ if \ |y - x_0| \le R \\ n, \ if \ y \in D \setminus B_{R+\frac{1}{n}}(x_0). \end{cases}$$

Let u_n be the minimal nonnegative solution to

$$\begin{aligned} &u_t = Lu + \beta u - \alpha u^2 + \psi_n \text{ on } D \times (0, \infty) \\ &u(\cdot, 0) = g_n, \end{aligned}$$

where $\{g_n\}$ is as in the statement of Theorem 3.1. Then using the log-Laplace equation, it is easy to see that

$$\begin{aligned} &P_\mu(E_t \cap A_t) = \\ &= \lim_{n\to\infty} E_\mu \exp(-<g_n, X(t)> - \int_0^t <\psi_n, X(s)> ds) = \lim_{n\to\infty} e^{-\int u_n(x,t)\, d\mu}. \end{aligned} \tag{7.1}$$

Without loss of generality, we may take $x_0 = 0$. Define

$$f_\delta(t) = 1 + \frac{1}{t+\delta}, \ \delta > 0.$$

Let $\eta(x)$ be a smooth function defined on $\mathbb{R}^d$ and satisfying $0 \le \eta(x) \le |x|$ and $\eta(x) = |x|$, for $|x| \ge R/2$. Define $\rho(x) = (R - \eta(x))^{-2}$ and $G_{c,\lambda} = C + \lambda\rho(x)$. Let $v_\delta(x,t) = f_\delta(t) \cdot G_{c,\lambda}(x)$. Clearly,

$$\begin{aligned} &\inf_{|x|<R} v_\delta(x, 0) \to \infty \text{ as } \delta \to 0, \text{ and} \\ &\lim_{|x|\to R} v_\delta(x,t) = \infty, \ \forall t > 0. \end{aligned} \tag{7.2}$$

Furthermore, for C and λ are sufficiently large, in $B_R(x_0)$,

$$Lv_\delta + \beta v_\delta - \alpha v_\delta^2 - (v_\delta)_t \le Lv_\delta + Bv_\delta - Av_\delta^2 - (v_\delta)_t \le 0, \text{ in } B_R(x_0). \tag{7.3}$$

To see this, write

$$Lv_\delta + Bv_\delta - Av_\delta^2 - (v_\delta)_t = f_\delta \cdot (LG_{c,\lambda} + BG_{c,\lambda} - AG^2_{c,\lambda}) + A(f_\delta - f_\delta^2)G^2_{c,\lambda} - G_{c,\lambda} f'_\delta.$$

One can check that $LG_{c,\lambda} + BG_{c,\lambda} - AG^2_{c,\lambda} \le 0$ for C, λ large. Moreover, if $C \ge \frac{1}{A}$, then $AG_{c,\lambda} \ge 1$ and thus

$$A(f_\delta - f_\delta^2)G^2_{c,\lambda} - G_{c,\lambda} f'_\delta = G_{c,\lambda}(-\frac{AG_{c,\lambda}}{t+\delta} + \frac{1}{(t+\delta)^2}(1 - AG_{c,\lambda})) < 0.$$

Using (7.2), (7.3) and the parabolic maximum principle, it follows that for $n \ge 1$,

$$u_n(x,t) \le v_0(x,t) \equiv (1 + \frac{1}{t})(C + \lambda(R - \eta(x))^{-2}), \text{ for } |x| < R,$$

if C and λ are large enough. Therefore, by (7.1),

$$P_\mu(E_t \cap A_t) \ge e^{-\int v_0(x,t)\, d\mu} > 0. \quad \square$$

The following proposition will be used several times.

Proposition 7.3. *The function ϕ from Theorem 4.1 is a positive solution of minimal growth at ∂D.*

Proof. For the case that $\alpha, \beta = const$, $D = \mathbb{R}^d$ and $Y(t)$ is conservative, this follows by the construction of ϕ in [14, p.250], or by [14, equation 5.7 and the paragraph that follows]. The same proof goes through in general. $\square$

Proof of Theorem 3.1. (i) As a first step, we construct u. It is easy to see that

$$\lim_{n\to\infty} \exp(- < g_n, X(t) >) = 1_{\{<1,X(t)>=0\}}. \tag{7.4}$$

Let $\mu \in \mathcal{M}_C(D)$. Using the log-Laplace equation, we have

$$E_\mu \exp(- < g_n, X(t) >) = \exp(- \int_D u_n(x,t)\, \mu(dx)), \tag{7.5}$$

where u_n is the minimal nonnegative solution to

$$\begin{aligned} &u_t = Lu + \beta u - \alpha u^2 \quad \text{on } D \times (0,\infty) \\ &u(\cdot, 0) \equiv g_n. \end{aligned}$$

By (7.5), it follows that u_n is monotone nondecreasing in n. Thus, from (7.4) and (7.5), we have

(7.6)

$$P_\mu(< 1, X(t) >= 0) = \lim_{n\to\infty} \exp\left(- \int_D u_n(x,t)\mu(dx)\right) = e^{-\int_D u(x,t)\mu(dx)},$$

for $\mu \in \mathcal{M}_C(D)$, where $u(x,t) \equiv \lim_{n\to\infty} u_n(x,t)$. For $t > 0$, the boundedness of $u(\cdot,t)$ on compact subsets of D follows from Lemma 7.1. Using an argument similar to the one appearing in [14, p.260], it follows that $u > 0$ solves $u_t = Lu + \beta u - \alpha u^2$ on $D \times (0,\infty)$. ¿From (7.6) it follows that $u(x,t)$ is monotone nonincreasing in t. Letting $w(x) \equiv \lim_{t\to\infty} u(x,t)$, and using the fact that $\int_D u(x,t)\mu(dx) < \infty, t > 0$, we conclude that

$$P_\mu(\text{ extinction}) = \exp\,(-\int_D w(x)\mu(dx)),\ \ \mu \in \mathcal{M}_C(D). \tag{7.7}$$

We now show that $w \in C^{2,\eta}(D)$ and that w satisfies $Lw + \beta w - \alpha w^2 = 0$ in D. Let $\{D_m\}_{m=1}^\infty$ be a sequence of domains with smooth boundaries such that $D_m \subset\subset D_{m+1} \subset\subset D,\ m = 1,2,\ldots,\ \bigcup_1^\infty D_m = D$. Let $Y(t)$ be the diffusion corresponding to L on D with the expectations $\{E_x\}$ and let

$$\tau_m \equiv \inf\{t \geq 0 : Y(t) \notin D_m\}.$$

By Itô's formula, we have

$$\begin{aligned} u_n(x,t) &= E_x u_n(Y((t-1)\wedge\tau_m),\, t-(t-1)\wedge\tau_m)) \\ &\quad + E_x \int_0^{(t-1)\wedge\tau_m} (\beta u_n - \alpha u_n^2)(Y(s),t-s)\,ds \end{aligned}$$

It then follows by dominated convergence that

$$\begin{aligned} u(x,t) &= E_x u(Y((t-1)\wedge\tau_m),\, t-(t-1)\wedge\tau_m)) \\ &\quad + E_x \int_0^{(t-1)\wedge\tau_m} (\beta u - \alpha u^2)(Y(s),t-s)\,ds \end{aligned}$$

Letting $t \to \infty$, it follows again by the dominated convergence theorem that

$$w(x) = E_x w(Y(\tau_m)) + E_x \int_0^{\tau_m} (\beta w - \alpha w^2)(Y(s),t-s)\,ds.$$

Let $f_1(x) = E_x w(Y(\tau_m))$ and let $f_2(x) = E_x \int_0^{\tau_m} (\beta w - \alpha w^2)(Y(s),t-s)\,ds$. Let G_m denote the Green's function for L in D_m. Then as is well known,

$$f_1(x) = \int_{\partial D_m} \frac{\partial G_m}{\partial n_y}(x,y) w(y)\,dy,$$

where $\dfrac{\partial}{\partial n_y}$ denotes the inward normal derivative with respect to y, and

$$f_2(x) = \int_{D_m} G_m(x,y)(\beta w - \alpha w^2)(y)\,dy.$$

Since L has nice coefficients and ∂D_m is a smooth boundary, $\frac{\partial G_m}{\partial n_y}(x,y)$ and $G_m(x,y)$ are continuous in x for $x \neq y$. Using this along with the dominated convergence theorem, the fact that w is bounded on compacts and the fact that $\int_{D_m} G_m(x,y)\,dy < \infty$, it follows that f_1 and f_2 are continuous; thus w is continuous. Once we know that w is continuous, it follows from standard theory that f_1 is in fact $C^{2,\eta}$ and satisfies $Lf_1 = 0$ in D_m. Similarly, it now follows by the bootstrap method that f_2 is $C^{2,\eta}$ and satisfies $Lf_2 = \beta w - \alpha w^2$ in D_m (see [7, p.21]). Since m is arbitrary, and since $w = f_1 + f_2$, we conclude that $w \in C^{2,\eta}$ and $Lw + \beta w - \alpha w^2 = 0$ in D.

Using the strong maximum principle (Theorem 3.2.6 in [13]), it follows that either $w \equiv 0$ on D or $w > 0$ on D.

It remains to prove the final statement in the theorem. First, note that although the support of $X(t)$ is not necessarily compactly embedded in D, by the remark following Proposition 2.1 and by monotone convergence, (7.5) and (7.6) remain valid for μ replaced by $X(t)$, that is, for $t, T > 0$

$$E_{X(t)} \exp(- < g_n, X(T) >) = \exp(- < u_n(\cdot, T), X(t) >), \quad P_\mu - a.s.$$

and

$$P_{X(t)}(< 1, X(T) >= 0) = e^{-<u(\cdot,T),X(t)>}, \quad P_\mu - a.s.$$

Using this and the Markov property, we have for $f \in C_c^+$

$$E_\mu \left(e^{-<f,X(t)>}; \text{extinction}\right) = \lim_{T\to\infty} E_\mu \left(e^{-<f,X(t)>}; < 1, X(t+T) >= 0\right)$$

$$= \lim_{T\to\infty} E_\mu e^{-<f,X(t)>} P_{X(t)}(< 1, X(T) >= 0) = \lim_{T\to\infty} E_\mu e^{-<f,X(t)>} e^{-<u(\cdot,T),X(t)>}$$

$$= \lim_{T\to\infty} E_\mu e^{-<f(\cdot)+u(\cdot,T),X(t)>}.$$

We claim that

$$\lim_{T\to\infty} E_\mu e^{-<f(\cdot)+u(\cdot,T),X(t)>} = \lim_{T\to\infty} e^{-<\hat{u}^{f,T}(\cdot,t),\mu>},$$

where $\hat{u}^{f,T}$ is the minimal nonnegative solution to (1.2) with $g(\cdot)$ replaced by $f(\cdot) + u(\cdot, T)$. To see this, first assume that $\sup_D \beta < \infty$. Let $u_f(\cdot, t)$ be the minimal nonnegative solution to (1.2) with $g(\cdot)$ replaced by $f(\cdot)$. Then the finiteness of $\hat{u}^{f,T}$ follows from the fact that $v(\cdot,t) = u_f(\cdot,t) + u(\cdot, T+t)$ satisfies $Lv + \beta v - \alpha v^2 \leq v_t$, and is thus a supersolution (see the construction of the minimal nonnegative solution to (1.2) given in the proof of Lemma A1). Also, by the construction given in the proof of Lemma A1 and from (A.4),

$$E_\mu e^{-<f(\cdot)+u(\cdot,T),X(t)>} = e^{-<\hat{u}^{f,T}(\cdot,t),\mu>}.$$

This equation holds then for the general case as well, since by the paragraph before Proposition 2.1, there exist an h-transform for which the new β is constant. (Note that f, u and $\hat{u}^{f,T}$ transform into $\frac{f}{h}, \frac{u}{h}$ and $\frac{\hat{u}^{f,T}}{h}$ respectively, and μ and $X(t)$ transform into $h\mu$ and $hX(t)$ respectively.) Therefore by (7.7),

$$E_\mu(\exp(-<f, X(t)>) \mid \text{ extinction }) = \lim_{T\to\infty} e^{-<\hat{u}^{f,T}(\cdot,t)-u(\cdot,T+t),\mu>}.$$

It is easy to check that $v^{(T)}(x,t) = \hat{u}^{f,T}(x,t) - u(x, T+t)$ satisfies $v^{(T)} \geq 0$ and

$$\begin{aligned} &v_t^{(T)} = Lv^{(T)} + (\beta - 2\alpha\hat{u}^T)v^{(T)} - \alpha(v^{(T)})^2 \quad \text{on } D \times (0,\infty) \\ &v^{(T)}(\cdot, 0) = f(\cdot). \end{aligned}$$

For the minimality of $v^{(T)}$ one can use the argument in the paragraph following (6.5). Since $(\beta - 2\alpha\hat{u}^T) \uparrow (\beta - 2\alpha w)$, using an argument similar to the one appearing in [14, p.260], it follows that $v \equiv \lim_{T\to\infty} v^{(T)}$ is the minimal nonnegative solution to

$$\begin{aligned} &v_t = Lv + (\beta - 2\alpha w)v - \alpha v^2 \quad \text{on } D \times (0,\infty) \\ &v(\cdot, 0) = f(\cdot). \end{aligned}$$

This completes the proof. $\square$

Proof of Theorem 3.2. Let $\{D_n\}_{n=1}^\infty$ be a sequence of domains with smooth boundaries such that $D_n \subset\subset D_{n+1} \subset\subset D$, $n = 1, 2, \ldots$, $\bigcup_1^\infty D_n = D$. Let w_n be the minimal positive solution to

$$\begin{aligned} &Lu + \beta u - \alpha u^2 = 0 \quad in\ D_n \\ &\lim_{x\to\partial D_n} u(x) = \infty. \\ &u(x) = \infty\ if\ x \in D \setminus D_n. \end{aligned}$$

Similarly as in the proof of Theorem 1 in [14], one can show that $w_n > 0$ exists and that

$$P_\mu(X(t, D \setminus D_n) = 0,\ \forall t > 0) = \exp(-\int_D w_n(x)\,\mu(dx)).$$

By trivial probabilistic considerations, w_n is monotone nonincreasing in D. Let $\hat{w} \equiv \lim_{n\to\infty} w_n$ in D. Then

$$P_\mu(C) = \exp(-\int_D \hat{w}(x)\,\mu(dx)). \tag{7.8}$$

Using standard arguments , one can show that $\hat{w}$ satisfies

$$Lu + \beta u - \alpha u^2 = 0 \text{ in } D \tag{7.9}$$

(see the proof of Theorem 1 in [14]). Using the elliptic maximum principle , it follows from the construction, that $\hat{w} = w_{max}$ is the maximal solution to (7.9).

For the last statement of the theorem, let $\hat{u}_n(x,t)$ be the function satisfying

$$P_{\delta_x}(< 1, X(t) >= 0\,;\, X(s, D \setminus D_n) = 0,\ s \le t) = e^{-\hat{u}_n(x,t)}.$$

In the case of a ball, $\hat{u}_n(x,t) = \lim_{m\to\infty} u_m(x,t)$, where u_m is as in the proof of Lemma 7.1; the general construction is similar. ¿From the obvious monotonicity, we can define

$$f(x) \equiv \lim_{n\to\infty} \lim_{t\to\infty} u_n(x,t).$$

Then

$$P_{\delta_x}(X(t) \text{ dies out} \cap C) = e^{-f(x)},$$

where C is as in the statement of the theorem. Using standard compactness arguments again, one can show that f satisfies (7.9). Therefore $f \le w_{max}$; thus from (7.8) it follows that

$$P_{\delta_x}(X(t) \text{ dies out } \cap C) = P_{\delta_x}(C). \quad \square$$

Proof of Theorem 3.3. The equivalence of (ii) and (iii) follows from (3.5), (3.7) and (3.8). We now show that $(i) \to (ii)$. If $< 1, X(t) >= 0$, then by the compact support property and the fact that $X(0) \in \mathcal{M}_C(D)$, it follows that

$$\bigcup_{s=0}^{\infty} suppX(s) = \bigcup_{s=0}^{t} suppX(s) \subset\subset D.$$

Therefore, $P_\mu(C^c \mid \text{extinction }) = 0$, for $\mu \in \mathcal{M}_C(D)$.

Assume now that $w = w_{max} = 0$. Then by (7.8), $P_\mu(C) = 1$ for $\mu \in \mathcal{M}_C(D)$, and therefore $X(t)$ obviously possesses the compact support property. The last statement of the theorem now follows trivially. $\square$

Proof of Corollary 3.1. For K sufficiently large, $u \equiv K$ satisfies $Lu + \beta u - \alpha u^2 = \beta K - \alpha K^2 \ge 0$, thus by the elliptic maximum principle and the construction of w_{max} (see the proof of Theorem 3.1) it follows that $w_{max} \ge K$. Therefore, by (ii) of Theorem 3.3, $w \ge K$, and then by Theorem 3.1, $P_\mu(survival) > 0$. $\square$

Proof of Theorem 3.4. Let $\{D_m\}_{m=1}^\infty$ be a sequence of domains with smooth boundaries such that $D_m \subset\subset D_{m+1} \subset\subset D,\ m = 1, 2, \ldots,\ \bigcup_1^\infty D_m = D$. For each m, define a nondecreasing sequence of smooth functions $\{\psi_n^{(m)}\}_{n=1}^\infty$ satisfying $0 \le \psi_n^{(m)} \le n$ and

$$\psi_n^{(m)}(x) = \begin{cases} 0,\ if\ x \in D_m \\ n,\ if\ x \in D \setminus D_m^{1/n}, \end{cases}$$

where $D_m^{1/n} \equiv \{x \in D : dist(x, D_m) < \frac{1}{n}\}$. Also, for each m, define the event $A_t^m \equiv \{X(s, D \setminus \bar{D}_m) = 0,\ s \le t\}$. Let $\mu \in \mathcal{M}_C(D)$, $supp(\mu) \subset D_1$. Using the log-Laplace equation it follows that

$$\begin{aligned} P_\mu(A_t^m) &= \lim_{n\to\infty} E_\mu \exp(-\int_0^t < \psi_n^{(m)}, X(s) > ds) = \\ &= \lim_{n\to\infty} \exp(-\int_D u_{n,m}(x,t)\, d\mu(x)), \end{aligned} \tag{7.10}$$

where $u_{n,m}$ is the minimal nonnegative solution to

$$\begin{aligned} &u_t = Lu + \beta u - \alpha u^2 + \psi_n^{(m)} \quad on\ D \times (0,\infty) \\ &u(\cdot, 0) \equiv 0. \end{aligned}$$

It follows from the log-Laplace equation, or alternatively from the parabolic maximum principle , that $u_{n,m}$ is monotone nondecreasing in n. Using standard bootstrap arguments together with an appropriate upper solution, one can show that $u_m(x,t) \equiv \lim_{n\to\infty} u_{n,m}(x,t)$ is finite and satisfies

$$\begin{aligned} &u_t = Lu{+}\beta u - \alpha u^2 \quad \text{on } D \times (0,\infty) \\ &\lim_{t\to 0} u(x,t) = 0,\ x \in D_m \\ &\lim_{x\to\partial D_m} u(x,t) = \infty,\ t > 0. \end{aligned} \tag{7.11}$$

(An upper solution for (7.11) was constructed for $\beta = \alpha = 1; L = \frac{D}{2}\Delta,\ D > 0$ in ([15], p.1380). A modification of that function works for the general case, as well.) In fact, by the construction of u_m, it is easy to see that u_m is the *minimal* nonnegative solution to (7.11). Since $u_{n,m}$ is nonincreasing in m, the same holds for u_m. Let $u(x,t) \equiv \lim_{m\to\infty} u_m(x,t)$. By standard bootstrap arguments, $0 \le u(x,t)$ and

$$\begin{aligned} &u_t = Lu + \beta u - \alpha u^2 \quad \text{on } D \times (0,\infty) \\ &u(\cdot, 0) \equiv 0. \end{aligned} \tag{7.12}$$

(Note that $\lim_{t\to 0} u(x,t) = 0$ follows from the fact that u_m is nonincreasing in m.) Using the parabolic maximum principle together with the above construction of u, it follows that u is the maximal nonnegative solution to (7.12). By (7.10) and monotonicity, we have

$$P_\mu(\bigcup_{s\le t} suppX(s) \subset\subset D) = \exp(-\int_D u(x,t)\, \mu(dx)). \tag{7.13}$$

The theorem now follows from (7.12), (7.13) and the maximality of u. $\square$

Proof of Theorem 3.5. Let $u \geq 0$ be a solution to (3.9). Resolve L into spherical coordinates as follows:

$$L = p(x)\frac{d^2}{dr^2} + q(x)\frac{d}{dr} + \text{terms involving differentiation not only in r},$$

where $r = |x|$. Then, by the assumptions of the theorem, $\frac{p(x)}{1+|x|^2}$ and $\frac{q(x)}{1+|x|}$ are bounded from above. Let $\beta \leq B$ and $0 < A \leq \alpha$ on $\mathbb{R}^d$. Define

$$u_R(x,t) = (\lambda + \gamma r^2) \cdot (R-r)^{-2} \cdot e^{Kt}, \quad |x| < R,\ t \geq 0,$$

where $r = |x|$ and $\gamma, K$and λ are to be fixed later. We will show that

$$Lu_R + \beta u_R - \alpha u_R^2 - (u_R)_t \leq 0, \text{for all } R > 0, \tag{7.14}$$

if γ, K, λ are sufficiently large. Since $\lim_{r \to R} u_R(x,t) = \infty$, for x, t fixed, and since $u_R(x,0) > 0$, it follows from the parabolic maximum principle that $u(x,t) \leq u_R(x,t)$ for $|x| < R, t > 0$. Since $\lim_{R \to \infty} u_R(x,t) = 0$, we conclude that $u \equiv 0$.

It remains to prove (7.14). Note that

$$e^{-Kt}(Lu_R + \beta u_R - \alpha u_R^2 - (u_R)_t) \leq e^{-Kt}(Lu_R + Bu_R - Au_R^2 - (u_R)_t) =$$

$$= 6(R-r)^{-4} \cdot p(\lambda + \gamma r^2) + [2p \cdot 2\gamma r + q(\lambda + \gamma r^2)] \cdot 2(R-r)^{-3} +$$
$$+ [p \cdot 2\gamma + q \cdot 2\gamma r + (B-K)(\lambda + \gamma r^2)](R-r)^{-2} - [\alpha(\lambda + \gamma r^2)^2 e^{Kt}](R-r)^{-4} \equiv \Sigma.$$

First, consider $r > 1$. Choose an M such that $p(x) \leq Mr^2, q(x) \leq Mr$, for $|x| = r > 1$. Then, to show that $\Sigma \leq 0$ for $r > 1$, it is enough to prove that

$$\begin{aligned} &6Mr^2(\lambda + \gamma r^2) + \\ &\qquad + 2(4\gamma M r^3 + Mr(\lambda + \gamma r^2))(R-r) + \\ &\qquad + [4Mr^2\gamma - C(\lambda + \gamma r^2)](R-r)^2 - \\ &\qquad - \alpha(\lambda + \gamma r^2)^2 \equiv I + II + III - IV \leq 0, \end{aligned}$$

where $C \equiv K - B$.

We consider separately the cases $R \leq 2r$ and $R > 2r$. When $R \leq 2r$, an easy computation shows that

$$I + II \leq 17\, Mr^2(\lambda + \gamma r^2).$$

Let $4M < C$, that is, $4M + B < K$. Then $III < -C\lambda(R-r)^2 < 0$, and thus, it is enough to have

$$17\, Mr^2(\lambda + \gamma r^2) \leq \alpha\, (\lambda + \gamma r^2)^2,$$

which is satisfied if $\frac{17\,M}{A} \leq \gamma$.

Consider now the case $R \geq 2r$. Then

$$\begin{aligned} II &\leq 8\gamma M r^2(R-r)^2 + 2M\lambda(R-r)^2 + 2M\gamma r^2(R-r)^2 = \\ &= (10\,M\gamma r^2 + 2M\lambda)(R-r)^2 \leq \\ &\leq (10\,\gamma + 2\lambda)\cdot Mr^2(R-r)^2. \end{aligned}$$

(In the last inequality we used the fact that $r > 1$.) Let $-4M + C \equiv C_0 > 0$. Then

$$III = (-C\lambda - C_0 r^2\gamma)(R-r)^2,$$

and thus

$$II + III \leq -C\lambda(R-r)^2 + [(10\,\gamma + 2\lambda)M - C_0\gamma]\cdot r^2(R-r)^2.$$

Also, $I - IV \leq 0$ if $\frac{6M}{A} \leq \gamma$.

In light of the above calculations, the inequality $\Sigma \leq 0$ for all $0 \leq r < R$ will be satisfied if γ, K and λ are chosen as follows. First, choose $\gamma = \frac{17\,M}{A}$. Then choose λ so large that $\Sigma \leq 0$ for $r \leq 1$. This is possible because of the λ^2 term in IV. Finally, let K be so large that

$$(10 + \frac{2\lambda}{\gamma})\cdot M \leq C_0 = -4M + C = -4M - B + K,$$

that is

$$(14 + \frac{2\lambda}{\gamma})M + B \leq K. \quad \square$$

Proof of Theorem 3.6. (*i*) Let $\{D_m\}_{m=1}^{\infty}$ be a sequence of domains with smooth boundaries such that $D_m \subset\subset D_{m+1} \subset\subset D,\ m = 1, 2, \ldots,\ \bigcup_1^{\infty} D_m = D$. Let $Y(t)$ be the diffusion corresponding to L on D with the probabilities $\{Q_x\}$ and let

$$\tau_m \equiv \inf\{t \leq 0 : Y(t) \notin D_m\}.$$

Assuming to the contrary that the compact support property holds, and using the notation and results of Theorem 3.4 and its proof, it follows that $\lim_{m\to\infty} u_m(x,t) = 0$. On the other hand, $v_m(x,t) \equiv Q_x(\tau_m > t)$ satisfies

$$\begin{aligned} Lv - v_t &= 0 \text{ on } D_m \times (0,t) \\ v(\cdot, 0) &= 1 \\ \lim_{x\to\partial D_m} v(x,t) &= 0, t > 0 \\ 0 \leq &v \leq 1. \end{aligned}$$

Let $\hat{v}_m = K(1 - v_m)$. Then for sufficently small K, $\hat{v}_m$ satisfies

$$\begin{aligned} Lu + \beta u - \alpha u^2 - u_t = \beta u - \alpha u^2 &\geq 0 \text{ on } D_m \times (0,t) \\ u(\cdot, 0) &= 0 \\ \lim_{x \to \partial D_m} u(x,t) &= K, t > 0 \\ 0 \leq u &\leq K. \end{aligned}$$

By the parabolic maximum principle it follows $0 \leq \hat{v}_m \leq u_m$ and thus $\hat{v}_m \to 0$ as $m \to \infty$. Therefore $\lim_{m\to\infty} v_m(x,t) = 1$ and thus $\lim_{m\to\infty} Q_x(\tau_m > t) = 1, \forall x \in D, t > 0$. This shows that the diffusion is conservative and completes the proof of (i).

(ii) Since the compact support property is invariant under h-transforms, the superprocess $X(t)$ corresponding to $(L, \beta, \alpha; D)$ has the compact support property if and only if the same holds for $X^w(t)$ corresponding to $(L_0^w, \alpha w, \alpha w; D)$. The argument of (i) can now be applied to the latter process. □

Proof of Theorem 3.7.
We will show that every bounded positive solution is minimal. Let h be a bounded positive solution to (3.10) and let $0 < \hat{h}$ be any positive solution. Then $f \equiv \frac{\hat{h}}{h}$ solves the h-transformed equation

$$L_0^h f + \alpha h f - \alpha h f^2 = 0 \text{ in } \ D.$$

To show that h is minimal, we will show that $f \geq 1$. By Theorem 4.3, L_0^h corresponds to a recurrent diffusion process. We consider two cases separately. Assume first that $f < 1$ on D. Then $L_0^h f < 0$ follows on D, which contradicts the recurrent property of the diffusion corresponding to L_0^h (see Theorem 4.3.9. in [13]). Assume now that $f(x_0) < 1$ for some $x_0 \in D$ and that $\hat{\Omega} \equiv \{x \in D : f(x) < 1\} \neq D$. Let Ω be the connected component of $\hat{\Omega}$ containing x_0. Clearly, $L_0^h f < 0$ on Ω . Since $f(x) = 1$ for $x \in \partial\Omega$, applying Itô's formula and using recurrence gives

$$1 = E_{x_0} f(Y(\sigma_\Omega)) \leq f(x_0) < 1,$$

which is a contradiction. □

Proof of Theorem 4.1. The finiteness of ϕ on $D \setminus \bar{D}_0$ follows from (4.4) and Lemma 7.1. The rest of the proof is identical with the proof of Theorem 1 in [14] which treated the case in which $D = \mathbb{R}^d$ and α, β are positive constants. □

Proof of Theorem 4.2. First, we show that there are two possibilities :

$$1)\ \phi > w \text{ on } D \setminus \bar{D}_0$$

or

$$2)\ \liminf_{x \to \partial D} \frac{\phi}{w}(x) = \inf_{x \in D \setminus \bar{D}_0} \frac{\phi}{w}(x) = 0.$$

Making an h transform with $h = w$, ϕ transforms into $\phi^w = \frac{\phi}{w}$, and ϕ^w satisfies

$$L_0^w u + \alpha w u - \alpha w u^2 = 0 \text{ in } D \setminus \bar{D}_0.$$

For the transformed equation, we have $\dfrac{\beta^w}{\alpha^w} = \dfrac{\alpha w}{\alpha w} = 1$, and thus the same proof as in [14, proof of Theorem 2] shows that either

$$\phi^w > 1 \text{ on } D \setminus \bar{D}_0$$

or

$$\liminf_{x \to \partial D} \phi^w(x) = \inf_{x \in D \setminus \bar{D}_0} \phi^w(x) = 0.$$

Let $A \equiv \{X(t, D_0) = 0,\ \forall t > 0\}$. Replacing w by ϕ in the proof of Theorem 3.1, one can see that $P(\cdot|A)$ corresponds to $(L, \beta - 2\alpha\phi, \alpha; D \setminus \bar{D}_0)$. Applying (3.5) to this new quadruple, we obtain

$$P_\mu(\text{ extinction } |A) = \exp(-\int_{D \setminus \bar{D}_0} w^*(x)\, \mu(dx)),$$

where w^* solves the equation $Lu + (\beta - 2\alpha\phi)u - \alpha u^2 = 0$ in $D \setminus \bar{D}_0$. Since either $w^* \equiv 0$ in $D \setminus \bar{D}_0$ or w^* is positive in $D \setminus \bar{D}_0$, there are two possibilities :

(7.15)
$$1)\ P_\mu(\textit{survival } \cap\ A) = 0, \text{ for all } \mu \text{ satisfying } \textit{supp}\, \mu \subset\subset D \setminus \bar{D}_0,$$
(7.16)
$$2)\ P_\mu(\textit{survival } \cap\ A) > 0, \text{ for all } \mu \text{ satisfying } \textit{supp}\, \mu \subset\subset D \setminus \bar{D}_0.$$

Consider (7.15). We will show that $\phi > w$. We have $P_{\delta_x}(A) = P_{\delta_x}(A \cap \text{extinction })$ and thus by (4.4) it follows that

$$\begin{aligned}\exp(-\phi(x)) &= P_{\delta_x}(A|\text{extinction }) \cdot P_{\delta_x}(\text{extinction }) = \\ &= P_{\delta_x}(A|\text{extinction }) \cdot \exp(-w(x)).\end{aligned}$$

Letting $\tilde{P}_{\delta_x}$ correspond to the quadruple $(L, \beta - 2\alpha w, \alpha; D)$, it follows that

$$\tilde{P}_{\delta_x}(A) \equiv P_{\delta_x}(A|\text{extinction }) = \exp(-(\phi(x) - w(x))).$$

By Theorem 4.1, $\phi - w = \tilde{\phi} > 0$, where $\tilde{\phi}$ is defined for $\tilde{P}_{\delta_x}$ as ϕ is defined for P_{δ_x} in Theorem 4.1.

Conversely, assume that $\phi > w$ on $D \setminus \bar{D}_0$. We will show that (7.15) holds by showing that $w^* = 0$, where w^* is as in the paragraph before (7.15). Make an h-transform with $h = w$. Then $w^{*,h} = \frac{w^*}{h}$ is the w-function for the quadruple $(L_0^w, \alpha w - 2\alpha\phi, \alpha w; D \setminus \bar{D}_0)$. Since, by assumption, $\alpha w - 2\alpha\phi < -\alpha w$, it follows by the construction of w in Theorem 3.1 and the parabolic maximum principle that $w^{*,h} \le \tilde{w}$, where $\tilde{w}$ is the w-function for the quadruple $(L_0^w, -\alpha w, \alpha w; D \setminus \bar{D}_0)$. Also by the construction of w in Theorem 3.1 and the parabolic maximum principle, $\tilde{w} \le w_0$, where w_0 is the w-function for the quadruple $(L_0^w, -\alpha w, \alpha w; D)$. Applying the last part of Theorem 3.1 to the quadruple $(L_0^w, \alpha w, \alpha w; D)$, which corresponds to $P^w(\cdot)$ and whose w-function is 1, it follows, that the quadruple $(L_0^w, -\alpha w, \alpha w; D)$ corresponds to $P^w(\cdot \mid \text{extinction })$. Thus, $w_0 = 0$. We conclude then that $w^* = 0$.

To complete the proof we have to show that the recurrence property for the support does not depend on the choice of D_0. By a simple argument it is enough to show the following. If $D_0^1 \subset\subset D_0^2 \subset\subset D$ and (7.16) holds for D_0^1 then (7.16) holds for D_0^2, as well. Let $A_i \equiv \{X(t, D_0^i) = 0,\ \forall t > 0\}$, $i = 1, 2$. Let ϕ_i denote the function ϕ defined for D_0^i in $D \setminus D_0^i, i = 1, 2$. If (7.16) holds for D_0^1, then $\inf_{x \in D \setminus \hat{D}_1} \frac{\phi_1(x)}{w(x)} = 0$. By the minimal growth property of ϕ_2 (see Proposition 7.3), it follows that there exists a $K > 0$ such that $K\phi_1(x) > \phi_2(x)$ on $D \setminus D^*$, where $D_0^2 \subset\subset D^* \subset\subset D$. It follows that $\inf_{x \in D \setminus D_0^2} \frac{\phi_2(x)}{w(x)} = 0$. $\square$

Proof of Theorem 4.3. If $d = 1$, then the statement is a simple consequence of the well-known integral test for recurrence/transience in one dimension (Theorem 5.1.1. in [13]).
For $d \ge 2$ we use a mini-max variational criterion for transience/recurrence given in [13]. Let $\{D_n\}_{n=1}^\infty$ be a sequence of smooth domains such that D_1 is simply connected, $D_n \subset\subset D_{n+1}$ and $D = \bigcup_{n=1}^\infty D_n$. Let $\Omega_n = D_n \setminus \bar{D}_1$. For $f \in C^{2,\eta}(\partial D_1)$, define

$$\mu_n^{(f)}(a,b) = \inf_{\substack{g \in W^{1,2}(\Omega_n) \\ g = e^f \ on \ \partial D_1, g = 0 \ on \ \partial D_n \\ (dist(x, \partial D_n))^{-1} g \in L^\infty(\Omega_n)}} \ \sup_{\substack{h \in W^{1,2}(\Omega_n, g^2\, dx) \\ h = f \ on \ \partial D_1}} D(g,h), \tag{7.17}$$

where

$$D(g,h) = \frac{1}{2} \int_{\Omega_n} dx\, g^2 \left[\left(\frac{\nabla g}{g} - a^{-1} b \right) a \left(\frac{\nabla g}{g} - a^{-1} b \right) - (\nabla h - a^{-1} b) a (\nabla h - a^{-1} b) \right].$$

By Theorem 6.6.1. in [13], the diffusion corresponding to L $(\hat{L})$ is recurrent on D if and only if

$$\lim_{n \to \infty} \mu_n^{(f)}(a,b) = 0 \quad \left(\lim_{n \to \infty} \mu_n^{(f)}(a, b + a\nabla Q) = 0\right) \tag{7.18}$$

for some (or equivalently, all) $f \in C^{2,\eta}(\partial D_1)$. The convergence to zero is thus independent of the particular choice of f.

Consider $\mu_n^{(f)}(a, b + a\nabla Q)$. Make the substitution $G \equiv g e^{-Q}, H \equiv h - Q$, and note that the boundary conditions $g = e^f, g = 0, h = f$ appearing in the mini-max formula become $G = e^{f-Q}, G = 0, H = f - Q$. Since Q and ∇Q are bounded on Ω_n, it follows that $g \in W^{1,2}(\Omega_n)$ if and only if $G \in W^{1,2}(\Omega_n)$ and that $h \in W^{1,2}(\Omega_n, g^2\,dx)$ if and only if $H \in W^{1,2}(\Omega_n, G^2\,dx)$. Also, the last condition on g under the infimum in (7.17) is clearly equivalent to the same condition on G. Therefore

$$\mu_n^{(f)}(a, b + a\nabla Q) = \inf_{\substack{G\in W^{1,2}(\Omega_n)\\ G=e^{f-Q} \text{ on } \partial D_1, G=0 \text{ on } \partial D_n\\ (dist(x,\partial D_n))^{-1}G\in L^\infty(\Omega_n)}} \sup_{\substack{H\in W^{1,2}(\Omega_n,G^2\,dx)\\ H=f-Q \text{ on } \partial D_1}} \frac{1}{2}\int_{\Omega_n} dx\, G^2 e^{2Q}\cdot$$

$$\cdot\left[\left(\frac{\nabla G}{G} - a^{-1}b\right) a \left(\frac{\nabla G}{G} - a^{-1}b\right) - (\nabla H - a^{-1}b)a(\nabla H - a^{-1}b)\right].$$

It follows that
(7.19)

$$\left(\inf_{x\in\Omega_n} e^{2Q(x)}\right)\mu_n^{(f-Q)}(a,b) \le \mu_n^{(f)}(a, b + a\nabla Q) \le \left(\sup_{x\in\Omega_n} e^{2Q(x)}\right)\mu_n^{(f-Q)}(a,b).$$

The theorem now follows from (7.18) and (7.19).

Proof of Theorem 4.4. a) Make an h-transform with $h = w$. Then $X^w(t)$ corresponds to $(L_0^w, \alpha w, \alpha w; D)$. Let $\tilde\phi$ and $\tilde w$ denote the ϕ-function of Theorem 4.1 and the w-function of Theorem 3.1 for the quadruple $(L_0^w, \alpha w, \alpha w; D)$. Under an h-transform, the w-function transforms to $\frac{w}{h}$; thus $\tilde w = 1$. By assumption, L_0^w corresponds to a recurrent diffusion on D; thus a simple modification of the proof of Theorem 3.7 then shows that $\tilde\phi > 1$. Therefore $\frac{\tilde\phi}{\tilde w} > 1$ and by Theorem 4.2, the support of $X^w(t)$ is recurrent. The same therefore holds for $X(t)$.
b) Part (b) follows from part (a) and Theorem 4.3.
c) If $\frac{\beta}{\alpha} \le c$, then $u \equiv c_0$ satisfies $Lu + \beta u - \alpha u^2 \le 0$, for any $c_0 \ge c$. Recall that $\phi = \lim_{n\to\infty}\phi_n$ where ϕ_n is the minimal positive solution to (4.2). Furthermore, $\phi_n = \lim_{m\to\infty} v_m^{(n)}$, where $v_m^{(n)}$ satisfies

$$\begin{aligned} Lv + \beta v - \alpha v^2 + \psi_n = 0 \quad &\text{in } D_m \\ v = 0 \text{ on } \partial D_m&, \end{aligned}$$

and $\{D_m\}_{m=1}^\infty$ and $\{\psi_n\}_{n=1}^\infty$ are as in Theorem 4.1. This was proved in [14] in the case $D = \mathbb{R}^d$ and $\alpha, \beta =$const; the same proof goes through in general. Thus, we have $\phi = \lim_{n\to\infty}\lim_{m\to\infty} v_m^{(n)}$. From this representation for ϕ, along with the elliptic maximum principle , it follows by comparison with $u = c_0$ that $\phi \le c_0$ on $D \setminus D_0$, for c_0 sufficently large. Thus $\inf_{x\in D\setminus D_0}\frac{\phi(x)}{w(x)} = 0$, and by Theorem 4.2 the support is transient. □

Proof of Corollary 4.1. By Theorem 4.4(c), w is bounded for the quadruple $(L, \beta, \alpha; D)$. Using the construction of w, along with the parabolic maximum principle , the w-function for the quadruple $(L, \hat\beta, \hat\alpha; D)$ is bounded as well. Thus, if the latter process survives with positive probability, then by Theorem 4.4(b) it has a recurrent support. □

Proof of Theorem 4.5. a) By Proposition 3.1, $w \equiv \frac{K}{\epsilon}$ for the process corresponding to $(L, K, \epsilon; D)$, and therefore by Theorem 4.4(b), it has a recurrent support. Part (a) now follows from Corollary 4.1.
b) By Theorem 4.3.3.(i) and Theorem 4.6.3.(i) in [13], $\lambda_c(L+\beta) > 0$ on D and therefore, by Theorem 4.9, there is no local extinction. Consequently, the process survives with positive probability and the proof now follows as in part (a). □

Proof of Theorem 4.6. We assume $d \geq 2$ (the proof for $d = 1$ is similar). First, assume that $(i) - (iv)$ hold. Consider the h-transformed process with $h = w$. The corresponding quadruple is $(L_0^w, \alpha w, \alpha w; D)$. Let $\tilde{\phi}$ and $\tilde{w}$ denote the ϕ-function and the w-function for this new quadruple. As in the proof of Theorem 4.4, we have $\tilde{w} \equiv 1$. We will show that $\inf_D \tilde{\phi} = 0$, which, by Theorem 4.2, proves that the support is transient. Note that by (iii) and Theorem 4.3, L_0^w corresponds to a transient diffusion on D. By (iv) and Theorem 3.6(ii), L_0^w corresponds to a conservative diffusion on D. Using (ii) and the invariance of $\lambda_{c,\infty}$ under h-transforms, we have

$$\lambda_{c,\infty}(L_0^w + \alpha w) = \lambda_{c,\infty}(L^w - \frac{Lw}{w} + \alpha w) = \lambda_{c,\infty}(L^w + \beta) = \lambda_{c,\infty}((L+\beta)^w) < 0.$$

Thus, we can choose a $\hat{D} \subset\subset D$ and an $\epsilon > 0$ such that there exists a $u > 0$ on $D \setminus \hat{D}$ satisfying

$$(L_0^w + \alpha w + \epsilon)u = 0 \text{ in } D \setminus \hat{D}. \tag{7.20}$$

Let $Y(t)$ denote the diffusion corresponding to L_0^w on D with probabilities $\{Q_x\}$ and define

$$\tau \equiv \inf\{t \geq 0 : Y(t) \in \hat{D}\}.$$

Then, using (7.20) and applying Itô's formula gives

(7.21)
$$0 \leq E_x u(Y(t \wedge \tau)) = u(x) - (\alpha w + \epsilon) E_x \int_0^{t \wedge \tau} u(Y(s))\, ds, \text{ for } x \in D \setminus \hat{D}.$$

We will show that $\inf_{D \setminus \hat{D}} u = 0$. Assume to the contrary that $\inf_{D \setminus \hat{D}} u > 0$. Since $Y(t)$ is conservative and transient, it follows that

$$0 < Q_x(Y(t) \in D \setminus \hat{D}, \forall t > 0), \ \forall x \in D \setminus \hat{D}. \tag{7.22.}$$

Letting $t \to \infty$ in (7.21), it follows by (7.22) that the righthand side tends to $-\infty$, which is a contradiction. Thus $\inf_{D \setminus \hat{D}} u = 0$, and using the minimality of $\tilde{\phi}$ on $D \setminus \hat{D}$, it follows from the elliptic maximum principle that $\inf_{D \setminus \hat{D}} \tilde{\phi} = 0$.

Now assume (iv') in place of (iv). Then an equation similar to (7.21) can be obtained, where $Y(t)$ corresponds to L and αw is replaced by $\beta \geq 0$. The rest of the proof is the same as before. $\square$

Proof of Theorem 4.7. Assume first that the support of $X(t)$ is recurrent. Then by Theorem 4.2, $w < \phi$ on $D \setminus \bar{D}_0$. By assumption, the compact support property holds; thus by Theorem 3.3, $w = w_{max}$. Therefore, if u solves (3.4), it follows that $u \leq w_{max} = w < \phi$ on $D \setminus \bar{D}_0$. Since ϕ is of minimal growth, the same is true for u.

Assume now that the support of $X(t)$ is transient. Then by Theorem 4.2, there is no $c > 0$ such that $c\phi > w$ on $D \setminus \bar{D}_0$; therefore w is not of minimal growth. $\square$

Proof of Theorem 4.8. Let $A \equiv \{X(t, \Omega) = 0, \ \forall t > 0\}$, let $C = \{$the range of $X(t)$ is compactly embedded$\}$ and let $E = \{X(t)$ becomes extinct$\}$. By Theorem 4.1,

$$P_{\delta_x}(A) = e^{-\phi(x)}, \text{ for } \ x \in D \setminus \bar{\Omega}, \tag{7.23}$$

where ϕ is the minimal positive solution to (4.6). Using arguments similar to the ones appearing in the proofs of Theorem 3.2 and Theorem 4.1, and using the uniqueness of the elliptic solution with boundary blow-up for annular domains of the type $\hat{\Omega} \setminus \bar{\Omega}$, where $\Omega \subset\subset \hat{\Omega} \subset\subset D$ (see [10]), one can show that

$$P_{\delta_x}(A \cap C) = e^{-\hat{w}}, \text{ for } \ x \in D \setminus \bar{\Omega}, \tag{7.24}$$

where $\hat{w}$ is the maximal positive solution to (4.6). ¿From (7.23) and (7.24) it follows that the uniqueness of the positive solution to (4.6) is equivalent to $P_{\delta_x}(A \cap C) = P_{\delta_x}(A)$ for $x \in D \setminus \bar{\Omega}$. On the other hand, the recurrence of the support is equivalent to $P_{\delta_x}(A \cap E) = P_{\delta_x}(A)$ for $x \in D \setminus \bar{\Omega}$. If the compact support property holds, then $P_{\delta_x}(C \triangle E) = 0$ by (3.8) and Theorem 3.3. Thus $P_{\delta_x}(A \cap C) = P_{\delta_x}(A \cap E)$. This proves the theorem except for the last statement. Take any function $u > 0$ satisfying

$$Lu + \beta u - \alpha u^2 = 0 \text{ in } \ D \setminus \bar{\Omega}.$$

By the maximality of $\hat{w}$, $u \leq \hat{w}$ on $D \setminus \bar{\Omega}$. On the other hand, since $\hat{w}$ is the unique solution to 4.6, it follows that $\hat{w} = \phi$, where ϕ is as in Theorem 4.1. By Proposition 7.3, ϕ is a solution of minimal growth at ∂D; thus $\hat{w}$ is a solution of minimal growth at ∂D, and consequently, so is u. $\square$

Appendix A. Construction of the $(L, \beta, \alpha; D)$-superprocess. Particle picture approximation and the martingale formulation for the superdiffusion.

Let $D \subseteq \mathbb{R}^d$ be a domain and let L be as in (1.1). Let $\alpha, \beta \in C^\eta(D), \eta \in (0, 1]$, satisfy $\alpha > 0$ and $\sup_D \beta < \infty$. Denote by T_t the semigroup corresponding to L on D with the Dirichlet boundary condition. Denote by $C_b^+(D)$ the space of non-negative, bounded, continuous functions on D. We will denote bounded pointwise convergence on $C_b^+(D)$ by $g_n \to g$.
We state and prove two lemmas.

Lemma A1. *(i) Let $g \in C_b^+(D)$ and consider the semilinear equation:*

$$\begin{aligned} u_t &= Lu + \beta u - \alpha u^2 \quad \text{on } D \times (0,\infty) \\ u(\cdot,0) &= g(\cdot). \end{aligned} \tag{A.1}$$

There exists a minimal nonnegative solution $u \in C^{2,1}(D \times (0,\infty)) \cap C(D \times [0,\infty))$ for (A.1) and $\sup_{0 \le s \le t} ||u(\cdot,s)||_\infty < \infty$, for all $t > 0$. Finally, when α and β are bounded, u is the unique function which solves the m i l d equation

$$u(\cdot,t) = T_t g + \int_0^t ds\ T_s \phi(u(\cdot,t-s)), \tag{A.1*}$$

with $\sup_{0 \le s \le t} ||u(\cdot,s)||_\infty < \infty$ for all $t > 0$, where $\phi(z) = \beta z - \alpha z^2$.

(ii) Let $V_t(g) \equiv u(\cdot,t)$. Then

$$V_{t+s}(g) = V_t V_s(g) \tag{A.2}$$

and the map $g \to V_t(g)$ is continuous with respect to bounded pointwise convergence on $C_b^+(D)$.

Proof of Lemma A1.

(i) We construct the minimal solution for (A.1). Let $\{D_n\}_{n=1}^\infty, D_n \subset\subset D_{n+1}$ be a sequence of smooth domains such that $\bigcup_{n=1}^\infty D_n = D$. Let $0 \le g_n \in C_c^2(D_n), g_n \uparrow g,\ , n = 1, 2, \ldots$. Denote by $T_t^{(n)}$ the strongly continuous semigroup corresponding to L on D_n with the Dirichlet boundary condition. Let $\mathcal{B}_0(D_n) = \{u \in C(\overline{D}_n); u = 0 \ on\ \partial D_n\}$. Since $\alpha(x)$ and $\beta(x)$ are bounded on D_n, ϕ is locally Lipschitz continuous on $C_b^+(D_n)$. Thus by Theorem 6.1.4 in [11] it follows that there is a $t_{max} \le \infty$ such that $(A.1^*)$ has a unique solution u_n on $[0, t_{max})$ with D replaced by D_n and T_s replaced by $T_s^{(n)}$, for every $n \in \mathbb{N}$. Since the map $\phi : \mathcal{B}_0(D_n) \to \mathcal{B}_0(D_n)$ is continuously differentiable and since $g_n \in \mathcal{D}(L, D_n)$, the domain of the infinitesimal generator of the semigroup $T_t^{(n)}$, applying Theorem 6.1.5 in [11] , it follows that $u_n \in C^{2,1}(D_n \times (0, t_{max}))$, $u_n(\cdot,t) \in \mathcal{B}_0(D_n)$, for $t \in (0, t_{max})$, and u_n satisfies $(A.1)$ on $(0, t_{max})$ with g replaced by g_n, T_s replaced by $T_s^{(n)}$ and D replaced by D_n. Also, $0 \le u_n$ (see the argument in [8], p.115). Let $B \ge \max\{||g||, \sup_{x \in D} \beta(x)\}$ and consider the function $\hat{u}(t) = Be^{Bt}$. Using the parabolic maximum principle, it is easy to show that $u_n(\cdot,t) \leqq \hat{u}(t), t > 0$, for every $n \in \mathbb{N}$. Therefore by Theorem 6.1.4 in [11] again, it follows that $t_{max} = \infty$ for every $n \in \mathbb{N}$. By the parabolic maximum principle u_n is monotone increasing in n. Define $u = \lim_{n\to\infty} u_n \le \hat{u}$. Using an argument like the one used in [14] (where it is shown that the function v appearing in the proof of Theorem 6 of that paper satisfies equation (5.2) of that paper), it can be shown that u is in fact a solution for $(A.1)$. The minimality of u follows from the parabolic maximum principle. For the last statement in (i), assume that α and β are bounded. Using bounded convergence, it is easy to see that u satisfies $(A.1^*)$ as well. Using Gronwall's inequality (see Theorem 1.4. in [16]), it can be shown that u is the unique solution to $(A.1^*)$ satisfying $\sup_{0\le s\le t}||u(\cdot,s)||_\infty<\infty$, for every $t>0$.

(ii) It is easy to see that (A.2) is a consequence of the minimality.We now prove the continuity of the map $g\to V_t(g)$ with respect to bounded pointwise convergence on $C_b^+(D)$. Let $0<t$ be fixed. Let $u=V_t(g), u^{(n)}=V_t(g_n)$, and assume that $g_n\to g$. Let $h_n=g-g_n$. Then $f^{(n)}\equiv u-u^{(n)}$ satisfies

$$Lf^{(n)}+\beta^{(n)}f^{(n)}-f_t^{(n)}=0$$

$$f^{(n)}(\cdot,0)=h_n(\cdot),$$

where $\beta^{(n)}=\beta-\alpha(u+u^{(n)})$. Since $\beta^{(n)}<\beta\leqq B$, it follows from the Feynman-Kac formula that $|f^{(n)}(x,t)|\le e^{Bt}E_x|h_n(Y(t))|$. Since $h_n\to 0$, it follows by bounded convergence that $f^{(n)}\to 0$. $\square$

Lemma A2. *Let V_t be as in Lemma A1 (ii). Then*

a. $V_t(0)=0$,

b. *V_t is continuous on $C_b^+(D)$,*

c. *V_t is negative semidefinite on $C_b^+(D)$; that is*

$$\sum_{i,j=1}^n\lambda_i\lambda_jV_t(f_i+f_j)\le 0\ ,\ if\ \sum_i^n\lambda_i=0,\ \forall n\in\mathbb{N}, f_i\in C_b^+(D).$$

Proof of Lemma A2. (a) and (b) follow trivially from Lemma A1. We now show (c). Let $V_t^{(n)}(g)=u_n(\cdot,t)$ for $g\in C_b^+(D)$, where u_n is as in the proof of Lemma A1. Since u_n satisfies $(A.1^*)$, using the argument in [4, p.1215], it follows that $V_t^{(n)}$ is negative semidefinite on $C_b^+(D)$ for every $n\in\mathbb{N}$. Since $u=\lim_{n\to\infty}u_n$, V_t is also negative semidefinite on $C_b^+(D)$. $\square$

Theorem A1. *Let $\alpha>0$, $\beta\le B<\infty$ and let V_t correspond to the operator $Lu+\beta u-\alpha u^2$. Let $\mathcal{L}_t(\cdot)=e^{-V_t(\cdot)}$ on $C_b^+(D)$ and let*

$(A.3)$
$$\mathcal{L}(t,\mu,g)\equiv\exp(-\int_D V_tg(x)\,\mu(dx)),\quad g\in C_b^+(D),\ \mu\in\mathcal{M}_F(D).$$

Then $\mathcal{L}(t,\mu,g)$ is a Laplace-transition functional; that is, there exists a unique finite measure valued Markov-process, $X(t)$, such that

$(A.4)$
$$\mathcal{L}(t,\mu,g)=E_\mu\exp(-<X(t),g>)\ for\ g\in C_b^+(D),\mu\in\mathcal{M}_F(D).$$

Proof of Theorem A1. By Lemma A.2, $\mathcal{L}_t$ satisfies:

a. $\mathcal{L}_t(0) = 1$,

b. $\mathcal{L}_t$ is continuous on $C_b^+(D)$with respect to bounded convergence,

c. $\mathcal{L}_t \geq 0$,

d. $\mathcal{L}_t$ is positive definite; that is

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j \mathcal{L}_t(f_i + f_j) \geq 0 \text{ for all } \lambda_i \in \mathbb{R},\ 1 \leq i \leq n \text{ and } n \in \mathbb{N}.$$

(For the proof of property d., see [2], page 74.)

Using (a)-(d), and following the proof of Corollary A.6 in [6] it follows that for every $x \in D$ and $t > 0$ fixed, there exists a unique probability measure $P^{x,t}$ on $\mathcal{M}_F(D)$ satisfying

$$\mathcal{L}_t(g)(x) = \int_{\mathcal{M}_F(D)} \exp(- < \nu, g >)\, P^{x,t}(d\nu) \text{ for } g \in C_b^+(D).$$

(Although Corollary A.6 in [6] asserts the result for nonnegative bounded measurable functions , the proof works for $C_b^+(D)$ as well.)

In order to complete the proof, note that $V_0 = I$ by definition, and $V_{t+s} = V_t V_s$ by Lemma A1(ii). $\square$

We call the measure valued process constructed in Theorem A1 the $(L, \beta, \alpha; D)$ *-superprocess.*

Let $Y(t)$ be the diffusion process corresponding to the generalized martingale problem for L on D (see 1.13 in [13]). The diffusion, which lives on $D^* = D \cup \{\Delta\}$, the one-point compactification of D, enters Δ and remains there forever once it leaves D. For each positive integer n, consider N_n particles, each of mass $\frac{1}{n}$, starting at points $x_i^{(n)} \in D, i = 1, 2, \ldots, N_n$, and performing independent branching diffusion according to the motion process $Y(t)$, with branching rate $cn, c > 0$, and branching distribution $\{p_k^{(n)}(x)\}_{k=0}^{\infty}$, where

$$(A.5)\qquad \begin{aligned} e_n(x) &\equiv \sum_{k=0}^{\infty} k p_k^{(n)}(x) = 1 + \frac{\gamma(x)}{n}, \\ v_n^2(x) &\equiv \sum_{k=0}^{\infty} (k-1)^2 p_k^{(n)}(x) = m(x) + o(1) \ \textit{as}\ n \to \infty, \text{ uniformly in } x; \end{aligned}$$

$m, \gamma \in C^{\eta}(D), \eta \in (0, 1]$ and $m(x) > 0$. Let $\mu_n = \frac{1}{n} \sum_{i=1}^{N_n} \delta_{x_i^{(n)}}$. Let $N_n(t)$ denote the number of particles alive at time t and denote their positions by $\{X_i^n(t)\}_{i=1}^{N_n(t)}$. Denote by $\mathcal{M}_F(D)$ $(\mathcal{M}_F(D^*))$ the space of finite measures on D (D^*). Define an $\mathcal{M}_F(D^*)$- valued process $X_n(t)$ by $X_n(t) = \frac{1}{n}\sum_{i=1}^{N_n(t)} \delta_{X_i^n(t)}$. Denote by $P^{(n)}_{\mu_n}$ the probability measure on $D([0,\infty), \mathcal{M}_F(D^*))$ induced by $X_n(t)$. The notation $w{-}lim$ denotes the limit in the weak topology.

Theorem A2. *Assume that $m(x)$ and $\gamma(x)$ are bounded from above. Let $w - \lim_{n\to\infty} \mu_n = \mu \in \mathcal{M}_F(D)$. Then there exists a $P^*_\mu \in C([0,\infty), \mathcal{M}_F(D^*))$ such that $P^*_\mu = w - \lim_{n\to\infty} P^{(n)}_{\mu_n}$. Define $P_\mu \in C([0,\infty), \mathcal{M}_F(D))$ by $P_\mu(\cdot) = P^*_\mu(\cdot \cap D)$ and let $X(t)$ be the process corresponding to P_μ. Then $X(t)$ is an $(L, \beta, \alpha; D)$-superprocess, where L corresponds to $Y(t)$ on D, $\beta = c\gamma(x)$ and $\alpha(x) = \frac{1}{2}cm(x)$. Furthermore*

$(A.6)$

$$M^f(t) \equiv < f, X(t) > - < f, X(0) > - \int_0^t < (L+\beta)f, X(s) > ds$$

is a martingale for $f \in C_c^2(D)$, with increasing process

$$< M_t(f) > = 2\int_0^t < X(s), \alpha f^2 > ds. \tag{A.7}$$

Assume in addition, that $Y(t)$ is conservative on D. Then D^* and $C_c^2(D)$ in the statement above can be replaced by D and $C^2_{const}(D) \equiv \{f \in C^2(D) : \exists\ \Omega \subset\subset D$ such that $f \equiv const$ on $D \setminus \Omega\}$ respectively.

Before proving the theorem, we need a lemma.

Lemma A3. *Let $\beta \le B$, let $\mu \in \mathcal{M}_F(D)$ be such that $n\mu$ is integer valued, and let $\tau \le t$ be a stopping time. Then*

$$E^{(n)}_\mu < 1, X(\tau) > \le < \mu, 1 > e^{Bt} \tag{A.8}$$

and

$$E^{(n)}_\mu \int_0^\tau < X(s), \beta > ds \le < \mu, 1 > (e^{Bt} - 1), n = 1, 2, \ldots. \tag{A.9}$$

Proof. Define $\beta(\Delta) = \alpha(\Delta) = Lf(\Delta) = 0$ and define

$$C^2_{const}(D^*) \equiv \{f \in C^2(D) \cap C(D^*) : \exists\ \Omega \subset\subset D \text{ such that } \ f \equiv const \text{ on } D^* \setminus \Omega\}.$$

Note that $C^2_{const}(D^*)$ separates points in $\mathcal{M}_F(D^*)$ and is closed under addition. A modification of the argument on p.60 in [16] shows that if $f \in C^2_{const}(D^*)$, then

$(A.10)$

$$M_t(f) \equiv < f, X(t) > - < f, X(0) > - \int_0^t < (L+\beta)f, X(s) > ds$$

is a $P_\mu^{(n)}$-martingale, $n = 1, 2, \ldots$, with increasing process

$$< M^n(f) >_t = \int_0^t < X(s), (cm + \delta_n) f^2 > ds \longrightarrow 2 \int_0^t < X(s), \alpha f^2 > ds$$

as $n \to \infty$, where $\sup_{x \in D} |\delta_n(x)| \to 0$ as $n \to \infty$. From the above martingale property and the optional sampling theorem, it follows that

$$\begin{aligned} E_\mu^{(n)} < X(\tau), 1 > &= < \mu, 1 > + E_\mu^{(n)} \int_0^\tau < X(s), \beta > ds \\ &\leq < \mu, 1 > + B \cdot E_\mu^{(n)} \int_0^t < X(s), 1 > ds. \end{aligned}$$

Let $Z_\mu^{(n)}(\tau) \equiv E_\mu^{(n)} < X(\tau), 1 >$. Then $Z_\mu^{(n)}(\tau) \leq Z_\mu^{(n)}(0) + B \int_0^t Z_\mu^{(n)}(s) ds$ and (A.8) follows by Gronwall's inequality.

The above calculation along with (A.8) now gives

$$E_\mu^{(n)} \int_0^\tau < X(s), \beta > ds = E_\mu^{(n)} < 1, X(\tau) > - < \mu, 1 > \leq < \mu, 1 > (e^{Bt} - 1).$$

This proves (A.9).

Proof of Theorem A2. We are going to use a modification of the argument given in [16], where the statement is proved for the case $\beta \equiv 0, \alpha = const$ and $Y(t)$ conservative. First, we show that $\{P_{\mu_n}{}^{(n)}\}_{n=1}^\infty$ is tight on $D([0, \infty), \mathcal{M}_F(D^*))$. Using a modification of Theorem 2.3 in [16], it will suffice to show that the following three claims are true:

(*i*)
$$\lim_{K \to \infty} \sup_n P_{\mu_n}^{(n)} \left(\sup_{t \leq T} < X(t), 1 > \geq K \right) = 0, \text{ for all } T > 0,$$

(*ii*) For all $n \in N, \epsilon > 0,$ and $K \in R,$ and for all $f \in C^2_{const}(D^*),$ there exists a $\delta = \delta(n, \epsilon, K, f)$ such that

$$P_{\mu_n}^{(n)} \left(\int_{\tau_n}^{\tau_n + \delta} | < X(s), (L + \beta) f > | ds \geq K \right) < \epsilon, \text{ for every stopping time } \tau_n \leq n.$$

(*iii*) For all $n \in N, \epsilon > 0,$ and $K \in R,$ and for all $f \in C^2_{const}(D^*),$ there exists a $\delta = \delta(n, \epsilon, K, f)$ such that

$$P_{\mu_n}^{(n)} \left(\int_{\tau_n}^{\tau_n + \delta} | < X(s), \alpha f^2 > | \geq K \right) < \epsilon, \text{ for every stopping time } \tau_n \leq n.$$

By the martingale inequality and Lemma A3, we have

$$P_{\mu_n}^{(n)}(\sup_{t\le T}[< X(t),1> - \int_0^t <\beta, X(s)> ds] > K)$$
$$\le \frac{1}{K} E_{\mu_n}^{(n)} | < X(T), 1> - \int_0^T <\beta, X(s)> ds|$$
$$\le \frac{1}{K}(E_{\mu_n}^{(n)} < X(T),1> + |E_{\mu_n}^{(n)} \int_0^T <\beta, X(s)> ds|)$$
$$\le \frac{1}{K}[<\mu_n, 1> e^{BT} + <\mu_n,1> (e^{BT}-1)]$$
$$= \frac{1}{K} \cdot <\mu_n, 1> (2e^{BT}-1).$$

Since by Gronwall's inequality, the event $\{\exists t \le T \, : < X(t), 1 >> Ke^{BT}\}$ is contained in the event $\left\{\exists t \le T \, : < X(t), 1> - \int_0^t < \beta, X(s) > ds > K\right\}$, we have

$$P_{\mu_n}^{(n)}(\sup_{t\le T} < X(t), 1 >> L)$$
$$\le P_{\mu_n}^{(n)}(\sup_{t\le T}[< X(t),1> - \int_0^t <\beta, X(s)> ds] > Le^{-BT})$$
$$\le \frac{e^{BT}}{L} \cdot <\mu_n, 1> (2e^{BT}-1),$$

for all $T > 0, L > 0$, and $n \in \mathbb{N}$. Since $<\mu_n, 1> \to <\mu, 1>$ as $n \to \infty$, (i) follows.

We prove now (ii). Let $f \in C^2_{const}(D)$. Then by Lemma A3 again,

$$E_{\mu_n}^{(n)} \int_{\tau_n}^{\tau_n+\delta} | < X(s), (L+\beta)f > | \, ds = E_{\mu_n}^{(n)} E_{X_{\tau_n}} \int_0^\delta | < X(s), (L+\beta)f > | \, ds$$
$$\le \frac{e^{B\delta}-1}{B} \cdot E_{\mu_n}^{(n)} < X_{\tau_n}, 1 > \cdot ||(L+\beta)f||_\infty$$
$$\le \frac{e^{B\delta}-1}{B} \cdot <\mu_n, 1> \cdot e^{Bn} \cdot ||(L+\beta)f||_\infty.$$

Thus, by Chebyshev's inequality

$$P_{\mu_n}^{(n)}(\int_{\tau_n}^{\tau_n+\delta} | < X(s), (L+\beta)f > | ds \ge K)$$
$$\le \frac{(e^{B\delta}-1)e^{Bn}}{BK} \cdot <\mu_n, 1> \cdot ||(L+\beta)f||_\infty.$$

This proves (ii). The calculation for (iii) is essentially the same. This completes the proof of tightness.

Let $P^{(n_j)}_{\mu_{n_j}} \to P^*$ weakly on $D([0,\infty),\mathcal{M}_F(D^*))$, where $\{n_j\}$ is a subsequence. We show that P^* is uniquely determined. Let $G_f(\nu) = F(< f,\nu >)$ for $f \in C^2_{const}(D^*), F \in C^2_b(\mathbb{R})$ and $\nu \in \mathcal{M}_F(D^*)$. Also, let $G'_f(\nu) = F'(< f,\nu >)$ and $G''_f(\nu) = F''(< f,\nu >)$. Then, similarly as in [16], one obtains that the generator $\mathbb{L}^{(n)}$ of the particle system at the nth level satisfies:

$$\mathbb{L}^{(n)}G_f(\nu) = G'_f(\nu) < Lf,\nu > +n^{-1}G''_f(\nu) < Lf^2 - 2fLf,\nu >$$
$$+\sum_{i=1}^{n} cn^2 < p_i \cdot (G_f(\nu + \frac{i-1}{n}\delta.) - G_f(\nu)),\nu >$$

Using a Taylor expansion (cf. [16], p.62), we have

$$\mathbb{L}^{(n)}G_f(\nu) = G'_f(\nu) < Lf,\nu > +O(n^{-1})$$
$$+ cn^2(G'_f(\nu)n^{-1} < (e_n - 1)f,\nu > +\frac{1}{2}G''_f(\nu)n^{-2} < mf^2,\nu > +o(n^{-2})).$$

Using that $cn(e_n - 1) = \beta, cm = 2\alpha$, it follows that

$$\mathbb{L}G_f(\nu) \equiv \lim_{n\to\infty} \mathbb{L}^{(n)}G_f(\nu) = G'_f(\nu) < (L+\beta)f,\nu > +G''_f(\nu) < \alpha f^2,\nu > .$$

Then, using Lemma A3, a similar calculation to the one in [16, pp.62-63] shows that

$$G_f(X(t)) - G_f(X(0)) - \int_0^t \mathbb{L}G_f(X(s))\,ds \tag{A.11}$$

is a P^*-martingale.

Define $(\hat{V}_t(f))$ for $f \in C(D^*)$ and $t \geq 0$ as follows :

$$(\hat{V}_t(f))(x) = (V_t(f))(x), x \in D$$
$$(\hat{V}_t(f))(\Delta) = f(\Delta),$$

where V_t is the semigroup defined in Lemma A1. Then, using (A.11) and Lemma 1.5 of [16], it follows that for $f \in C^2_{const}(D^*)$

$$N^f(t) \equiv \exp(- < X(t), \hat{V}_{T-t}f >), 0 \leq t \leq T,$$

is a martingale under P^*, for all $T > 0$. Let $0 \leq s < t = T$ and $\mu,\nu \in \mathcal{M}_F(D^*)$. Then using the martingale property, we obtain

$$E_\nu\left(e^{-<X(t),f>} \,|\, X(s) = \mu\right) = E_\nu\left(N^f(t) \,|\, X(s) = \mu\right) = \exp(- < \mu, \hat{V}_{t-s}f >).$$

Since $C(D^*)$ separates points in D^*, it follows that P^* is uniquely determined on the space $D([0,\infty),\mathcal{M}_F(D^*))$.

That $M^f(t)$ is a P^*-martingale for $f \in C^2_{const}(D^*)$, follows from Theorem 1.3 of [16], where the statement is proved for the case $\beta \equiv 0, \alpha = const$ and $Y(t)$ conservative; the proof goes through for the general case without difficulties.

Denote $X(t) \equiv X^*(t) \mid_D$. Then applying the log-Laplace equation for functions in $C_c^+(D)$, it follows that $P(X(t) \in \cdot)$ corresponds to the quadruple $(L, \beta, \alpha; D)$. Also, $M^f(t)$ is a martingale for $f \in C_c^2(D)$ under P. The branching term $\beta(x)z - \alpha(x)z^2$ guarantees that $X(t)$ is actually supported on $C([0,\infty),\mathcal{M}_F(D))$ (cf. [16, Theorem 1]). Finally, consider the case when $Y(t)$ is conservative on D. Then in the proof above one could work just as well with D instead of D^*. To see that the resulting process coincides with the one obtained by working first on D^* and then restricting the process to D, it is enough to note that the log-Laplace functionals coincide on $C_c^+(D)$, and therefore on $C_b^+(D)$ as well. $\square$

Appendix B: Summary of results in criticality theory.

Let
$$L = \frac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla + V$$
be a strictly elliptic operator defined on a domain $D \subseteq \mathbb{R}^d$ with smooth coefficients. Then there exists a corresponding diffusion process $X(t)$ on D that solves the generalized martingale problem for $L - V$ on D (see [13], Chapter 1.). The process lives on $D^* = D \cup \Delta$ with Δ playing the role of a cemetery state. We denote by P_x and E_x the corresponding probabilities and expectations, and define the transition measure $p(t,x,dy)$ for L by
$$p(t,x,B) = E_x\left(\exp\left(\int_0^t V(X(s))\,ds\right); X(t) \in B\right),$$
for measurable $B \subseteq D$.

Definition. If
$$\int_0^\infty p(t,x,B)\,dt = E_x\int_0^\infty \exp(\int_0^t V(X(s))\,ds) 1_B(X(t))\,dt < \infty,$$
for all $x \in D$ and all $B \subset\subset D$, then
$$G(x,dy) = \int_0^\infty p(t,x,dy)\,dt$$
is called the *Green's measure* for L on D. If the above condition fails, then the Green's measure for L on D is said not to exist.

In the former case, $G(x,dy)$ possesses a density, $G(x,dy) = G(x,y)dy$, which is called the *Green's function* for L on D.

For $\lambda \in R$ define

$$C_{L-\lambda}(D) \;=\; \{\; u \in C^2(D) \;:\; (L-\lambda)u = 0 \text{ and } u > 0 \text{ in } D \;\}.$$

The operator $L - \lambda$ on D is called *subcritical* if the Green's function exists for $L - \lambda$ on D; in this case $C_{L-\lambda}(D) \neq \emptyset$. If the Green's function does not exist for $L - \lambda$ on D, but $C_{L-\lambda}(D) \neq \emptyset$, then the operator $L - \lambda$ on D is called *critical.* In this case $C_{L-\lambda}(D)$ is one-dimensional. The unique function (up to a constant multiple) in $C_{L-\lambda}(D)$ is called the *ground state* of L on D. Finally, if $C_{L-\lambda}(D) = \emptyset$, then $L - \lambda$ on D is called *supercritical.*

A handy alternative way of characterizing subcritical operators is as follows :

L on D is subcritical if and only if there exists a $\phi \in C^{2,\eta}(D), \eta \in (0,1]$ satisfying $\phi > 0$, $L\phi \leq 0$ and $L\phi \not\equiv 0$.

If $V \equiv 0$, then L is not supercritical on D since the function $f \equiv 1$ satisfies $Lf = 0$ on D. In this case L is subcritical or critical on D according to whether the corresponding diffusion process, $X(t)$, is transient or recurrent on D . If $V \leq 0$ and $V \not\equiv 0$, then L is subcritical on D.

In terms of the solvability of inhomogeneous Dirichlet problems, subcriticality guarantees that the equation $Lu = -f$ in D has a positive solution u for every $0 \lneq f \in C_c^\eta(D)$. If subcriticality does not hold, then there are no positive solutions for any $0 \lneq f \in C_c^\eta(D)$.

One of the two following possibilities holds :

1) There exists a number $\lambda_c(D) \in R$ such that $L - \lambda$ on D is subcritical for $\lambda > \lambda_c(D)$, supercritical for $\lambda < \lambda_c(D)$, and either subcritical or critical for $\lambda = \lambda_c(D)$.

2) $L - \lambda$ on D is supercritical for all $\lambda \in R$, in which case we define $\lambda_c(D) = \infty$.

Definition. The number $\lambda_c(D) \in (-\infty, \infty]$ is called the *generalized principal eigenvalue* for L on D .

Note that $\lambda_c(D) \;=\; \inf\,\{\lambda \in R \;:\; C_{L-\lambda}(D) \neq \emptyset\}$. Also, if V is bounded from above, then (1) holds. The generalized principal eigenvalue coincides with the classical principal eigenvalue (that is, with the supremum of the real part of the spectrum) if D is bounded with a smooth boundary and the coefficients of L are smooth up to ∂D. Also, if L is symmetric with respect to a reference density ρ , then $\lambda_c(D)$ equals the supremum of the spectrum of the self-adjoint operator on $L^2(D, \rho\, dx)$ obtained from L via the Friedrichs' extension theorem.

The generalized principal eigenvalue is monotone nondecreasing as a function of the domain. It is continuous with respect to monotone increasing sequences of domains.

For $D \subseteq \mathbb{R}^d,\ d \geq 2$, let $\{D_n\}_{n=1}^\infty$ be an increasing sequence of bounded domains satisfying $D = \bigcup_{n=1}^\infty D_n$ and define the *generalized principal eigenvalue at* ∞ by

$$\lambda_{c,\infty} = \lim_{n\to\infty} \lambda_c(D \setminus \bar{D}_n).$$

Since λ_c is monotone nondecreasing in D, $\lambda_{c,\infty}$ is independent of $\{D_n\}_{n=1}^\infty$. If L is symmetric with respect to a reference density ρ , then $\lambda_{c,\infty}$ is equal to the supremum of the essential spectrum of the self-adjoint operator on $L^2(D, \rho\, dx)$ obtained from L via the Friedrichs' extension theorem.

If $d = 1$ and $D = (a, b), a \in [-\infty, \infty), b \in (-\infty, \infty], a < b$, define the *generalized principal eigenvalue at* $\pm\infty$ by

$$\lambda_{c,+\infty} = \lim_{n\to\infty} \lambda_c((b_n, b)) \text{ and } \lambda_{c,-\infty} = \lim_{n\to\infty} \lambda_c((a, a_n)),$$

where $a_n \downarrow a, b_n \uparrow b$.

Let $h \in C^{2,\eta}(D)$ satisfy $h > 0$ in D. The operator L^h defined by

$$L^h f = \frac{1}{h} L(hf)$$

is called the *h-transform* of the operator L. Written out explicitly, one has

$$L^h f = L_0 + a\frac{\nabla h}{h} \cdot \nabla + \frac{Lh}{h},$$

where $L_0 = \frac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla$. All the properties defined above are invariant under h-transforms.

For further elaboration and proofs see [13, Chapter 4.].

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