OPTIMIZING THE DRIFT IN A DIFFUSIVE SEARCH FOR A RANDOM STATIONARY TARGET

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Abstract. Let $a \in \mathbb{R}$ denote an unknown stationary target with a known distribution $\mu \in \mathcal{P}(\mathbb{R})$, the space of probability measures on $\mathbb{R}$. A diffusive searcher $X(\cdot)$ sets out from the origin to locate the target. The time to locate the target is $T_a = \inf\{t \geq 0 : X(t) = a\}$. The searcher has a given constant diffusion rate $D > 0$, but its drift b can be set by the search designer from a natural admissible class $\mathcal{D}_\mu$ of drifts. Thus, the diffusive searcher is a Markov process generated by the operator $L = \frac{D}{2}\frac{d^2}{dx^2} + b(x)\frac{d}{dx}$. For a given drift b, the expected time of the search is

$$\int_{\mathbb{R}} (E_0^{(b)} T_a)\, \mu(da). \tag{0.1}$$

Our aim is to minimize this expected search time over all admissible drifts $b \in \mathcal{D}_\mu$. For measures μ that satisfy a certain balance condition between their restriction to the positive axis and their restriction to the negative axis, a condition satisfied, in particular, by all symmetric measures, we can give a complete answer to the problem. We calculate the above infimum explicitly, we classify the measures for which the infimum is attained, and in the case that it is attained, we calculate the minimizing drift explicitly. For measures that do not satisfy the balance condition, we obtain partial results.

1. Introduction and Statement of Results

A number of recent papers have considered a stochastic search model for a stationary target $a \in R^d$, which might be random and have a known distribution attached to it, whereby a searcher sets off from a fixed point, say the origin, and performs Brownian motion with diffusion constant D.

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The searcher is also armed with a (possibly space dependent) exponential resetting time, so that if it has failed to locate the target by that time, then it begins its search anew from the origin. One may be interested in several statistics, the most important one being the expected time to locate the target. (In dimension one, the target is considered "located" when the process hits the point a, while in dimensions two and higher, one chooses an $\epsilon > 0$ and the target is said to be "located" when the process hits the ϵ-ball centered at a.) Without the resetting, this expected time is infinite. When the rate of the exponential clock is constant, the expected time to locate the target is finite; furthermore, this jump-Brownian motion process possesses an invariant probability density, call it ν. See, for example, [1, 2, 3, 9]. For related models, see [4, 6, 7] as well as the references in all of the above articles.

It is well known that the Brownian motion with diffusion constant D and with drift $\frac{D}{2}\frac{\nabla\nu}{\nu}$, that is the diffusion process generated by $\frac{D}{2}\Delta + \frac{D}{2}\frac{\nabla\nu}{\nu}\cdot\nabla$, also has invariant probability density ν. In [3], for the case of constant resetting rate in one dimension, it was shown that the expected time to locate a target at the deterministic point $a \in \mathbb{R}$ for the jump-Brownian motion process is less than the expected time for the corresponding (non-jumping) diffusion process with the same invariant measure (generated by $\frac{D}{2}\frac{d^2}{dx^2} + \frac{D}{2}\frac{\nu'}{\nu}\frac{d}{dx}$) to locate the target. The above is partial motivation for the problem we consider in this paper; we believe it is also of some independent interest.

Let $a \in \mathbb{R}$ denote an unknown stationary target with a known distribution $\mu \in \mathcal{P}(\mathbb{R})$, the space of probability measures on $\mathbb{R}$. A diffusive searcher $X(\cdot)$ sets out from the origin to locate the target. The time to locate the target is $T_a = \inf\{t \geq 0 : X(t) = a\}$. We assume that the diffusive searcher has a given constant diffusion rate $D > 0$, but that its drift b can be set by the search designer from a natural admissible class $\mathcal{D}_\mu$ of drifts, which we define below. Thus, the searcher is a Markov diffusion process generated by the operator $L = \frac{D}{2}\frac{d^2}{dx^2} + b(x)\frac{d}{dx}$. We will denote probabilities and expectations with respect to $X(\cdot)$ by $P_0^{(b)}$ and $E_0^{(b)}$. For a given drift b, the expected time of the search is

$$\int_{\mathbb{R}} (E_0^{(b)} T_a)\, \mu(da). \tag{1.1}$$

Our aim is to minimize this expected search time over all admissible drifts $b \in \mathcal{D}_\mu$. We note that this same problem was recently considered in the physics literature [5]; for more on this, see Remark 1 after Theorem 2.

We now discuss the influence of the drift, which will lead us to the definition of the admissible class $\mathcal{D}_\mu$ of drifts. In order to avoid trivialities, we will assume that the support of μ has a non-empty intersection with both open half-lines. (Otherwise, if say, μ is supported in $[0,\infty)$, then $\int_{\mathbb{R}} (E_0^{(b)} T_a)\, \mu(da)$ is a decreasing function of the drift b and converges to 0 as the drift converges pointwise to $+\infty$.) For convenience only, we will assume that the origin is not an atom of the distribution μ. We write μ in the form

(1.2)

$$\mu = (1-p)\mu_- + p\mu_+, \text{ where } p \in (0,1),\ \mu_- \text{ is a probability measure on } (-\infty,0)$$
$$\text{and } \mu_+ \text{ is a probability measure on } (0,\infty).$$

Define

$$\begin{aligned} A_-(\mu) &= \inf\{x \in (-\infty,0) : \mu_-\big((-\infty,x]\big) > 0\}, \\ A_+(\mu) &= \sup\{x \in (0,\infty) : \mu_+\big([x,\infty)\big) > 0\}. \end{aligned} \tag{1.3}$$

If $A_-(\mu) > -\infty$ ($A_+(\mu) < \infty$), then there is no point in searching to the left of $A_-(\mu)$ (to the right of $A_+(\mu)$). If $A_-(\mu) > -\infty$ ($A_+(\mu) < \infty$) and the diffusion can reach $A_-(\mu)$ ($A_+(\mu)$), then we consider the diffusion with reflecting boundary at $A_-(\mu)$ ($A_+(\mu)$). In terms of the generator L, the reflecting boundary at $A_-(\mu)$ ($A_+(\mu)$) is equivalent to imposing the Neumann boundary condition $u'(A_-(\mu)) = 0$ ($u'(A_+(\mu)) = 0$). (At least heuristically, the reflecting boundary at $A_-(\mu)$ ($A_+(\mu)$) may be thought of as imposing a drift of $+\infty$ on $(-\infty, A_-(\mu))$ ($-\infty$ on $(A_+(\mu),\infty)$), so we include the boundary condition as part of our drift condition below.) The above discussion leads us to define the following condition on the drift b:

(1.4)

i. b is piecewise continuous and locally bounded on $(A_-(\mu), A_+(\mu))$.

Also, if $A_-(\mu) > -\infty$ *is an atom for* μ, *then* b *is locally bounded on* $[A_-(\mu), A_+(\mu))$,

and if $A_+(\mu) < \infty$ *is an atom for* μ, *then* b *is locally bounded on* $(A_-(\mu), A_+(\mu)]$.

ii. If $A_-(\mu) > -\infty$ $(A_+(\mu) < \infty)$ and the diffusion can reach $A_-(\mu)$ $(A_+(\mu))$, then the diffusion is reflected at $A_-(\mu)$ $(A_+(\mu))$.

Remark. In particular, if μ has atoms at both $A_-(\mu)$ and $A_+(\mu)$, then the drifts satisfying (1.4) are bounded on $(A_-(\mu), A_+(\mu))$.

As is well-known, the expected hitting time $E_0^{(b)}T_a$ is finite for all $a \in (A_-(\mu), A_+(\mu))$ if and only if the diffusion $X(\cdot)$ is positive recurrent. Positive recurrence for drifts satisfying (1.4) is equivalent to the condition

$$\int_{A_-(\mu)}^{A_+(\mu)} dx \exp(\frac{2}{D}\int_0^x b(y)dy) < \infty. \tag{1.5}$$

(See [8].) We can now define the class of admissible drifts.

The Class $\mathcal{D}_\mu$ of Admissible Drifts:

$$\begin{aligned} &\mathcal{D}_\mu \text{ is the class of drifts } b \\ &\text{satisfying (1.4) and (1.5).} \end{aligned} \tag{1.6}$$

Let

$$\overline{\mu}_-(x) = \mu_-((-\infty, x)), \text{ for } x \le 0, \ \ \overline{\mu}_+(x) = \mu_+((x, \infty)), \text{ for } x \ge 0, \tag{1.7}$$

denote the tails of μ_- and μ_+.

We begin with the following result.

Theorem 1. *Let the target distribution* μ *satisfy* $\mu = (1-p)\mu_- + p\mu_+$ *as in* (1.2), *let* $A_-(\mu)$ *and* $A_+(\mu)$ *be as in* (1.3) *and let* $\overline{\mu}_-(x)$ *and* $\overline{\mu}_+(x)$ *be as in* (1.7). *Let the class of admissible drifts* $\mathcal{D}_\mu$ *be as in* (1.6). *If* $\int_{-\infty}^0 \overline{\mu}_-^{\frac12}(x)dx = \infty$ *and* $\int_0^\infty \overline{\mu}_+^{\frac12}(x)dx = \infty$, *then* $\int_{\mathbb{R}}(E_0^{(b)}T_a)\,\mu(da) = \infty$, *for all* $b \in \mathcal{D}_\mu$.

Remark. Note of course that $\int_0^\infty \overline{\mu}_+^{\frac{1}{2}}(x)dx = \int_0^{A_+(\mu)} \overline{\mu}_+^{\frac{1}{2}}(x)dx$ and $\int_{-\infty}^0 \overline{\mu}_-^{\frac{1}{2}}(x)dx = \int_{A_-(\mu)}^0 \overline{\mu}_-^{\frac{1}{2}}(x)dx$, and that the first integral (second integral) is always finite if $A_+(\mu) < \infty$ $(A_-(\mu) > -\infty)$.

The following simple proposition gives a sufficient moment condition for integrals of the above type to be finite.

Proposition 1. *Let ν be a probability measure on $(0,\infty)$ and let $\overline{\nu}(x) = \nu((x,\infty))$. If $\int_0^\infty x^2|\log x|^{1+\epsilon}\nu(dx) < \infty$, for some $\epsilon > 0$, then $\int_0^\infty \overline{\nu}^{\frac{1}{2}}(x)dx < \infty$. The condition $\int_0^\infty x^2|\log x|^{1-\epsilon}\nu(dx) < \infty$, for all $\epsilon \in (0,1)$, is not sufficient for the finiteness of $\int_0^\infty \overline{\nu}^{\frac{1}{2}}(x)dx$.*

Proof. For $\epsilon > 0$,

$$\int_0^\infty \overline{\nu}^{\frac{1}{2}}(x)dx \le 2 + C\Big(\int_2^\infty x|\log x|^{1+\epsilon}\,\overline{\nu}(x)\Big)^{\frac{1}{2}},$$

where $C = \big(\int_2^\infty \frac{1}{x|\log x|^{1+\epsilon}}dx\big)^{\frac{1}{2}} < \infty$. An integration by parts shows that the integral on the right hand side above is finite if $\int_0^\infty x^2(|\log x|)^{1+\epsilon}\nu(dx) < \infty$. This proves the first claim in the proposition. For the second claim, let ν be a distribution that satisfies $\overline{\nu}(x) = \frac{1}{x^2(|\log x|)^2}$, for $x \ge 2$. Then $\int_0^\infty x^2|\log x|^{1-\epsilon}\nu(dx) < \infty$, for all $\epsilon \in (0,1)$, but $\int_0^\infty \overline{\nu}^{\frac{1}{2}}(x)dx = \infty$. $\square$

In the case that $\int_{-\infty}^0 \overline{\mu}_-^{\frac{1}{2}}(x)dx$ and $\int_0^\infty \overline{\mu}_+^{\frac{1}{2}}(x)dx$ are finite, the following condition on the target distribution μ will play a seminal role.

Square Root Balance Condition. The target distribution $\mu = (1-p)\mu_- + p\mu_+$ is such that the integrals $\int_0^\infty \overline{\mu}_+^{\frac{1}{2}}(x)dx$ and $\int_{-\infty}^0 \overline{\mu}_-^{\frac{1}{2}}(x)dx$ are finite and satisfy

$$\frac{\int_0^\infty \overline{\mu}_+^{\frac{1}{2}}(x)dx}{\int_{-\infty}^0 \overline{\mu}_-^{\frac{1}{2}}(x)dx} = \frac{(1-p)\log(1-p)}{p\log p}. \tag{1.8}$$

Remark. A symmetric target distribution (the case in which $\overline{\mu}_+(x) = \overline{\mu}_-(-x)$, for $x \in (0,\infty)$, and $p = \frac{1}{2}$) always satisfies the square root balance condition.

When the target distribution satisfies the square root balance condition, we can give a complete answer to the optimization problem.

Theorem 2. *Let the target distribution μ satisfy $\mu = (1-p)\mu_- + p\mu_+$ as in* (1.2)*, let $A_-(\mu)$ and $A_+(\mu)$ be as in* (1.3) *and let $\overline{\mu}_-(x)$ and $\overline{\mu}_+(x)$ be as in* (1.7)*. Let the class of admissible drifts $\mathcal{D}_\mu$ be as in* (1.6)*. Assume also that the target distribution μ satisfies the square root balance condition* (1.8)*. Then*

i.

$$\inf_{b\in\mathcal{D}_\mu}\int_{\mathbb{R}}(E_0^{(b)}T_a)\,\mu(da) = \frac{2}{D}\Big(\frac{1-p}{|\log p|}\,(\int_{-\infty}^0\overline{\mu}_-^{\frac12}(x)dx)^2+\frac{p}{|\log(1-p)|}\,(\int_0^\infty\overline{\mu}_+^{\frac12}(x)dx)^2\Big). \tag{1.9}$$

In particular, in the case of a symmetric target distribution,

$$\inf_{b\in\mathcal{D}_\mu}\int_{\mathbb{R}}(E_0^{(b)}T_a)\,\mu(da)=\frac{2}{D\log 2}\,(\int_0^\infty\overline{\mu}_+^{\frac12}(x)dx)^2. \tag{1.10}$$

ii. The infimum in (i) is attained if and only if the restriction of μ to $(A_-(\mu),A_+(\mu))$ is absolutely continuous with a piecewise continuous, locally bounded density. (μ may possess an atom at $A_-(\mu)$ and/or at $A_+(\mu)$.) This infimim is attained uniquely at the drift

$$b_0(x)=\begin{cases} D\Big(\frac14\frac{\overline{\mu}_-{}'(x)}{\overline{\mu}_-(x)}-\frac{|\log p|}{2\int_{-\infty}^0\overline{\mu}_-^{\frac12}(y)dy}\,\overline{\mu}_-^{\frac12}(x)\Big), & A_-(\mu)<x<0;\\ D\Big(\frac14\frac{\overline{\mu}_+{}'(x)}{\overline{\mu}_+(x)}+\frac{|\log(1-p)|}{2\int_0^\infty\overline{\mu}_+^{\frac12}(y)dy}\,\overline{\mu}_+^{\frac12}(x)\Big), & 0<x<A_+(\mu).\end{cases} \tag{1.11}$$

If $\int_{A_-(\mu)}\overline{\mu}_-^{-\frac12}(x)dx=\infty$ ($\int_{A_+(\mu)}\overline{\mu}_+^{-\frac12}(x)dx=\infty$), then this drift prevents the diffusion $X(\cdot)$ from reaching $A_-(\mu)$ $\big(A_+(\mu)\big)$. Otherwise the diffusion $X(\cdot)$ can reach $A_-(\mu)$ $\big(A_+(\mu)\big)$, and consequently the diffusion is considered with reflection at $A_-(\mu)$ ($A_+(\mu)$).

iii. For those μ for which the infimum in (i) is not attained, the infimum is approached by a sequence $\{b_n\}_{n=1}^\infty$ of drifts, with b_n given by (1.11) *with $\mu=(1-p)\mu_-+p\mu_+$ replaced by $\mu_n=(1-p)\mu_{-;n}+p\mu_{+;n}$, where μ_n satisfies the square root balance condition* (1.8)*, is of the type described in (ii) and converges weakly to μ.*

Remark 1. After this paper was competed and placed on the Mathematics ArXiv, I was directed to [5] by one of its coauthors. That paper, which appears in the physics literature, treats the same problem considered here. In particular, in the case that μ is symmetric and possesses a density, the authors found that b_0 from (1.11) (with $p = \frac{1}{2}$ and $\mu_+(x) = \mu_-(-x)$) is a critical point of the map $b \to \int_{\mathbb{R}} (E_0^{(b)} T_a)\, \mu(da)$, and they calculated the corresponding expected search time, obtaining the expression on the righthand side of (1.10). They stated that this search time is optimal.

Remark 2. Let

$$\mathrm{EV}(\mu_-) := \int_{-\infty}^{0} x\mu_-(dx),\ \mathrm{EV}(\mu_+) := \int_{0}^{\infty} x\mu_+(dx)$$

denote respectively the expected values of random variables distributed according to μ_- and according to μ_+. Since $|\mathrm{EV}(\mu_-)| = \int_{-\infty}^{0} \overline{\mu}_-(x)dx$ and $\mathrm{EV}(\mu_+) = \int_0^\infty \overline{\mu}_+(x)dx$, it follows from part (i) of the theorem that

$$\inf_{b\in\mathcal{D}_\mu} \int_{\mathbb{R}} (E_0^{(b)} T_a)\, \mu(da) \ge \frac{2}{D}\Big(\frac{1-p}{|\log p|}(\mathrm{EV}(\mu_-))^2 + \frac{p}{|\log(1-p)|}(\mathrm{EV}(\mu_+))^2\Big),$$

with equality if and only if μ_- and μ_+ are the degenerate probability measures $\delta_{A_-(\mu)}$ and $\delta_{A_+(\mu)}$ respectively. In particular, in the case that the target distribution μ is symmetric, then $\mathrm{AvgDist}(\mu) := \mathrm{EV}(\mu_+)$ is the expected distance of the target to the origin, and

$$\inf_{b\in\mathcal{D}_\mu} \int_{\mathbb{R}} (E_0^{(b)} T_a)\, \mu(da) \ge \frac{2}{D\log 2}(\mathrm{AvgDist}(\mu))^2, \tag{1.12}$$

with equality if and only if the target distribution is $\mu = \frac{1}{2}\delta_{-A} + \frac{1}{2}\delta_A$, where $A = -A_-(\mu) = A_+(\mu)$. In section 2, it is shown that for a number of families of symmetric distributions, the ratio of $\inf_{b\in\mathcal{D}_\mu} \int_{\mathbb{R}} (E_0^{(b)} T_a)\, \mu(da)$ to $(\mathrm{AvgDist}(\mu))^2$ is constant within each family, that is, independent of the particular parameter.

Remark 3. Note that in part (ii), if μ_+ does not have an atom at $A_+(\mu)$ and has a density that is differentiable at $A_+(\mu)$, then this density vanishes at least to order one, and thus $\overline{\mu}_+$ vanishes at least to order two. Thus, $\int_{A_+(\mu)} \overline{\mu}_+^{-\frac{1}{2}}(x)dx = \infty$, and the diffusion with optimal drift b_0 cannot reach $A_+(\mu)$. However, if μ_+ has an atom at $A_+(\mu)$, or if it doesn't have an atom at $A_+(\mu)$ and its density vanishes to an order less than one at $A_+(\mu)$, then

$\int_{A_+(\mu)} \overline{\mu}_+^{-\frac{1}{2}}(x)dx < \infty$, and the diffusion with optimal drift can reach $A_+(\mu)$. The same considerations hold at $A_-(\mu)$.

Remark 4. For sufficiently nice target distributions $\mu = (1-p)\mu_- + p\mu_+$ with $A_-(\mu) = -\infty$ and $A_+(\mu) = \infty$, one can choose a drift $b = b_\mu$ so that the diffusion process has μ as its invariant measure. This drift, non-optimal for our problem, will be

$$b_\mu(x) := \begin{cases} \frac{D}{2}\frac{\bar{\mu}''_-(x)}{\bar{\mu}'_-(x)}, & x < 0; \\ \frac{D}{2}\frac{\bar{\mu}''_+(x)}{\bar{\mu}'_+(x)}, & x > 0. \end{cases}$$

It is known that $\int_{\mathbb{R}} (E_0^{(b_\mu)} T_a)\,\mu(da) = \int_{\mathbb{R}} (E_x^{(b_\mu)} T_a)\,\mu(da)$, for all $x \in \mathbb{R}$, and this constant value, which will be finite if and only if $\pm\infty$ are both entrance boundaries for the diffusion, is called *Kemeny's constant* [10].

In section 2 we illustrate Theorem 2 with a number of examples.

We now turn to the case that the target distribution does not satisfy the square root balance condition (1.8). Here we have only partial results.

Theorem 3. *Let the target distribution μ satisfy $\mu = (1-p)\mu_- + p\mu_+$ as in* (1.2)*, let $A_-(\mu)$ and $A_+(\mu)$ be as in* (1.3) *and let $\overline{\mu}_-(x)$ and $\overline{\mu}_+(x)$ be as in* (1.7)*. Let the class of admissible drifts $\mathcal{D}_\mu$ be as in* (1.6)*. Assume also that the target distribution does not satisfy the square root balance condition* (1.8)*, but that $\int_0^\infty \overline{\mu}_+^{\frac{1}{2}}(x)dx$ and $\int_{-\infty}^0 \overline{\mu}_-^{\frac{1}{2}}(x)dx$ are finite. Then*
i. $\inf_{b\in\mathcal{D}_\mu} \int_{\mathbb{R}} (E_0^{(b)} T_a)\,\mu(da)$ *is not attained.*
ii. If μ restricted to $(A_-(\mu), A_+(\mu))$ is absolutely continuous with a piecewise continuous, locally bounded density on $(A_-(\mu), A_+(\mu))$ (μ may possess an atom at $A_-(\mu)$ and/or at $A_+(\mu)$), then

$$\begin{aligned} &\inf_{b\in\mathcal{D}_\mu} \int_{\mathbb{R}} (E_0^{(b)} T_a)\,\mu(da) < \\ &\frac{2}{D}\Big(\frac{1-p}{|\log p|}\,\big(\int_{-\infty}^0 \overline{\mu}_-^{\frac{1}{2}}(x)dx\big)^2 + \frac{p}{|\log(1-p)|}\,\big(\int_0^\infty \overline{\mu}_+^{\frac{1}{2}}(x)dx\big)^2\Big) - \\ &\frac{2}{D}\Big(\frac{1-p}{|\log p|}\int_{-\infty}^0 \overline{\mu}_-^{\frac{1}{2}}(x)dx - \frac{p}{|\log(1-p)|}\int_0^\infty \overline{\mu}_+^{\frac{1}{2}}(x)dx\Big)^2, \end{aligned} \tag{1.13}$$

and$\int_{\mathbb{R}}(E_0^{(b)}T_a)\,\mu(da)$ *is equal to the righthand side of* (1.13) *when* b *is given by* (1.11)*.*

Remark 1. Note that the expression on the third line of (1.13) would be zero if the square root balance condition held, in which case the right hand side of (1.13) would be equal to the right hand side of (1.9). The right hand side of (1.13) can also be written as

$$\begin{aligned}&\frac{2}{D}\,\frac{1-p}{|\log p|}\,(1-\frac{1-p}{|\log p|})\Big(\int_{-\infty}^{0}\overline{\mu}_-^{\frac12}(x)dx\Big)^2+\\&\frac{2}{D}\,\frac{p}{|\log(1-p)|}\,(1-\frac{p}{|\log(1-p)|})\Big(\int_{0}^{\infty}\overline{\mu}_+^{\frac12}(x)dx\Big)^2+\\&\frac{4}{D}\,\frac{p(1-p)}{|\log(1-p)||\log p|}\,\Big(\int_{-\infty}^{0}\overline{\mu}_-^{\frac12}(x)dx\Big)\Big(\int_{0}^{\infty}\overline{\mu}_+^{\frac12}(x)dx\Big).\end{aligned}$$

It is easy to check that the coefficients of $\big(\int_{-\infty}^0\overline{\mu}_-^{\frac12}(x)dx\big)^2$ and $\big(\int_0^\infty\overline{\mu}_+^{\frac12}(x)dx\big)^2$ in the above expression are positive.

The above results suggest two open problems.

Open Problem 1. In the case that the square root balance condition fails, calculate $\inf_{b\in\mathcal{D}_\mu}\int_{\mathbb{R}}(E_0^{(b)}T_a)\,\mu(da)$.

Open Problem 2. Is $\inf_{b\in\mathcal{D}_\mu}\int_{\mathbb{R}}(E_0^{(b)}T_a)\,\mu(da)$ necessarily infinite in the case that one out of $\int_{-\infty}^0\overline{\mu}_-^{\frac12}(x)dx$ and $\int_0^\infty\overline{\mu}_+^{\frac12}(x)dx$ is infinite and the other is finite? If not, what can be said about $\inf_{b\in\mathcal{D}_\mu}\int_{\mathbb{R}}(E_0^{(b)}T_a)\,\mu(da)$?

It is natural to wonder about the corresponding problem in higher dimensions. Let the unknown stationary target $a\in\mathbb{R}^d$ be distributed according to a known distribution $\mu\in\mathcal{P}(\mathbb{R}^d)$, the space of probability measures on $\mathbb{R}^d$. Consider a diffusion process $X(\cdot)$ starting at 0 and generated by $\frac{D}{2}\Delta+b(x)\cdot\nabla$, and denote probabilities and expectations with respect to this process by $P_0^{(b)}$ and $E_0^{(b)}$. Let $\epsilon>0$ and define $\tau_{a;\epsilon}=\inf\{t\ge0:|X(t)-a|\le\epsilon\}$ One then wants to minimize $\int_{\mathbb{R}}(E_0^{(b)}\tau_{a;\epsilon})\mu(da)$ over a natural class of admissible drifts. In the two-dimensional case, resolve the drift into radial and angular components, r and θ, and write $b(x)\cdot\nabla=b_{\text{rad}}(r,\theta)\frac{\partial}{\partial r}+b_{\text{ang}}(r,\theta)\frac{1}{r}\frac{\partial}{\partial\theta}$. It is intuitively clear that if we let $b_{\text{rad}}(r,\theta)$ depend only on r and let

$b_{\text{ang}}(r,\theta)$ be equal to a constant b_{ang}, then for $|a|-\epsilon>0$, the quantity $\lim_{b_{\text{ang}}\to\infty} E_0^{(b)}\tau_\epsilon$ will just be equal to the expected hitting time of $|a|-\epsilon$ for the one-dimensional radial diffusion started from 0^+ and generated by $\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+b_{\text{rad}}(r)\frac{d}{dr}$. And this latter hitting time converges to 0 as the drift $b_{\text{rad}}(r)$ converges pointwise to ∞. Thus, in order to obtain something interesting, a restriction must be placed on the angular drift. Such a limitation doesn't seem to occur in higher dimensions. In any case, perhaps a good starting point would be to consider the class of radial drifts. Our intuition is that the higher the dimension, the more strongly toward the origin will point an optimal or near-optimal radial drift, since the higher the dimension, the more space there is to search at each fixed radius. Of course, the great difficulty with the multi-dimensional case is that there isn't an explicit formula for $E_0^{(b)}\tau_{a;\epsilon}$.

We conclude this introductory section with a sketch of our method of approach to the variational problem, $\inf_{b\in\mathcal{D}_\mu}\int_{\mathbb{R}}(E_0^{(b)}T_a)\,\mu(da)$, since it has a certain novelty to it. To proceed, we need the following proposition.

Proposition 2. *Let $\mu\in\mathcal{P}(\mathbb{R})$, and let $b\in\mathcal{D}_\mu$, where $\mathcal{D}_\mu$ is the class of admissible drifts as in* (1.6). *Then*

(1.14)
$$E_0^{(b)}T_a=\begin{cases}\frac{2}{D}\int_a^0 dx\exp(-\int_0^x\frac{2}{D}b(y)dy)\int_x^{A_+(\mu)}dz\exp(\int_0^z\frac{2}{D}b(t)dt), & A_-(\mu)\le a<0;\\ \frac{2}{D}\int_0^a dx\exp(-\int_0^x\frac{2}{D}b(y)dy)\int_{A_-(\mu)}^x dz\exp(\int_0^z\frac{2}{D}b(t)dt), & 0<a\le A_+(\mu).\end{cases}$$

Remark. The explicit formula for the hitting time in Proposition 2 is of course not new, but since we need it for a variety of situations—including the case in which the drift can blow up at the boundary, and including the case of reflection at the boundary, we will present its proof in section 5.

In light of Proposition 2, for $\mu=(1-p)\mu_-+p\mu_+$, we have

(1.15)
$$\frac{D}{2}\int_{\mathbb{R}}(E_0^{(b)}T_a)\,\mu(da)=$$
$$(1-p)\int_{A_-(\mu)}^0\mu_-(da)\Big[\int_a^0dx\exp(-\int_0^x\frac{2}{D}b(y)dy)\int_x^{A_+(\mu)}dz\exp(\int_0^z\frac{2}{D}b(t)dt)\Big]+$$
$$p\int_0^{A_+(\mu)}\mu_+(da)\Big[\int_0^adx\exp(-\int_0^x\frac{2}{D}b(y)dy)\int_{A_-(\mu)}^xdz\exp(\int_0^z\frac{2}{D}b(t)dt)\Big],$$

and after a Reimann-Stieltjes integration by parts, we obtain

(1.16)
$$\frac{D}{2}\int_{\mathbb{R}}(E_0^{(b)}T_a)\,\mu(da)=$$
$$(1-p)\int_{A_-(\mu)}^0da\,\overline{\mu}_-(a)\Big[\exp(-\int_0^a\frac{2}{D}b(y)dy)\int_a^{A_+(\mu)}dz\exp(\int_0^z\frac{2}{D}b(t)dt)\Big]+$$
$$p\int_0^{A_+(\mu)}da\,\overline{\mu}_+(a)\Big[\exp(-\int_0^a\frac{2}{D}b(y)dy)\int_{A_-(\mu)}^adz\exp(\int_0^z\frac{2}{D}b(t)dt)\Big].$$

In the case that $A_+(\mu)<\infty$ ($A_-(\mu)>-\infty$), the passage from (1.15) to (1.16) is true even if μ_+ (μ_-) has an atom at $A_+(\mu)$ ($A_-(\mu)$), or if $\int_0^{A_+(\mu)}dx\exp(-\int_0^x\frac{2}{D}b(y)dy)=\infty$ ($\int_{A_-(\mu)}^0dx\exp(-\int_0^x\frac{2}{D}b(y)dy)=\infty$). This is because in (1.7), $\overline{\mu}_+(x)$ ($\overline{\mu}_-(x)$) has been defined not to include $\mu_+(\{x\})$ ($\mu_-(\{x\})$). In the case that $A_+(\mu)=\infty$ ($A_-(\mu)=-\infty$), the passage from (1.15) to (1.16) is true for the following reason. (We explain it for $A_+(\mu)=\infty$.) We need to justify having ignored in the integration by parts the possible contribution

$$\lim_{A\to\infty}\overline{\mu}_+(A)\int_0^Adx\exp(-\int_0^x\frac{2}{D}b(y)dy)\int_{A_-(\mu)}^xdz\exp(\int_0^z\frac{2}{D}b(t)dt).\tag{1.17}$$

If the term $\int_0^\infty da\,\overline{\mu}_+(a)\Big[\exp(-\int_0^a\frac{2}{D}b(y)dy)\int_{A_-(\mu)}^adz\exp(\int_0^z\frac{2}{D}b(t)dt)\Big]$ on the right hand side of (1.16) is infinite, then nothing need be checked; thus, assume this integral is finite. Then we need to show that (1.17) is equal to 0. Since $\lim_{A\to\infty}\overline{\mu}_+(A)=0$, (1.17) is equal to

$$\lim_{A\to\infty}\overline{\mu}_+(A)\int_{A_0}^Adx\exp(-\int_0^x\frac{2}{D}b(y)dy)\int_{A_-(\mu)}^xdz\exp(\int_0^z\frac{2}{D}b(t)dt),$$

for any fixed $A_0>0$. We have

$$\overline{\mu}_+(A)\int_{A_0}^A dx\exp(-\int_0^x\frac{2}{D}b(y)dy)\int_{A_-(\mu)}^x dz\exp(\int_0^z\frac{2}{D}b(t)dt)\le$$
$$\int_{A_0}^A da\,\overline{\mu}_+(a)\Big[\exp(-\int_0^a\frac{2}{D}b(y)dy)\int_{A_-(\mu)}^a dz\exp(\int_0^z\frac{2}{D}b(t)dt)\Big]:=\delta(A_0,A).$$

By the integrability assumption, $\lim_{A_0\to\infty}\lim_{A\to\infty}\delta(A_0,A)=0$. We conclude from the above argument that (1.17) is indeed equal to 0.

We want to minimize the righthand side of (1.16) over $b\in\mathcal{D}_\mu$. There are two points of view that one can take, and it turns out that both of them are essential. One point of view is to consider the righthand side of (1.16) as a functional of b; we will call it G_1:

(1.18)
$$G_1(b)=(1-p)\int_{A_-(\mu)}^0 da\,\overline{\mu}_-(a)\Big[\exp(-\int_0^a\frac{2}{D}b(y)dy)\int_a^{A_+(\mu)} dz\exp(\int_0^z\frac{2}{D}b(t)dt)\Big]+$$
$$p\int_0^{A_+(\mu)} da\,\overline{\mu}_+(a)\Big[\exp(-\int_0^a\frac{2}{D}b(y)dy)\int_{A_-(\mu)}^a dz\exp(\int_0^z\frac{2}{D}b(t)dt)\Big].$$

For the other point of view, define the distribution function

$$F(x)=\frac{\int_{A_-(\mu)}^x dz\exp(\int_0^z\frac{2}{D}b(t)dt)}{\int_{A_-(\mu)}^{A_+(\mu)} dz\exp(\int_0^z\frac{2}{D}b(t)dt)},\tag{1.19}$$

and let $f(x)=F'(x)$ denote its density. Then the righthand side of (1.16) can be thought of as a functional of F; we call it $G_2(F)$. It is given by

(1.20)
$$G_2(F)=(1-p)\int_{A_-(\mu)}^0\overline{\mu}_-(a)\frac{F(A_+(\mu))-F(a)}{f(a)}da+p\int_0^{A_+(\mu)}\overline{\mu}_+(a)\frac{F(a)}{f(a)}da.$$

Of course, $F(A_+(\mu))=1$, but it is useful to write it as we have done in order to exploit the homogeneity. Indeed, note that now we can consider G_2 to be a functional of positive multiples of distribution functions of the type just described, and we have $G_2(cF)=G_2(F)$, for all $c>0$. We denote the domain of the functional G_2 by $\mathcal{D}(G_2)$ and specify it as follows:

(1.21)
$\mathcal{D}(G_2)$ is the set of positive multiples of the class of distributions functions F that can be written in the form (1.19), where $b\in\mathcal{D}_\mu$.

To search for critical points, the first point of view requires us to consider the condition

$$0=\lim_{\epsilon\to 0}\frac{G_1(b+\epsilon\beta)-G_1(b)}{\epsilon}, \tag{1.22}$$

for an appropriate wide class of drifts β. To isolate β in (1.22) requires numerous integration by parts. This eventually leads to an equation of the form $(1-p)\int_{A_-(\mu)}^0\beta(a)\Psi_-(a)da+p\int_0^{A_+(\mu)}\beta(a)\Psi_+(a)da=0$, for all β, where Ψ_- and Ψ_+ are expressions involving b. Thus $\Psi_-(a)=0$, for $A_-(\mu)<a<0$, and $\Psi_+(a)\equiv 0$, for $0<a<A_+(\mu)$. However, we did not find it tractable to solve these equations for b.

Since G_2 is homogeneous of order zero, to search for critical points via the second point of view we consider the condition

$$0=\lim_{\epsilon\to 0}\frac{G_2(F+\epsilon Q)-G_2(F)}{\epsilon} \tag{1.23}$$

(here ϵ takes on both positive and negative values), where Q is such that $F+\epsilon Q$ belongs to the domain $\mathcal{D}(G_2)$ of G_2. In fact, in order to ensure that we can interchange the order of the integration and the differentiation when we calculate (1.23) with G_2 given by (1.20), and also in order to ensure that $F+\epsilon Q$ is positive for small negative ϵ, we will actually restrict ourselves to distribution functions Q with densities compactly supported in $\big(A_-(\mu),A_+(\mu)\big)$. After integrating by parts several times to isolate the density $q:=Q'$ of Q, we obtain an equation of the form

$$(1-p)\int_{A_-(\mu)}^0 q(a)\Phi_-(a)da+p\int_0^{A_+(\mu)}q(a)\Phi_+(a)da=\Sigma(F,\overline{\mu}_-,\overline{\mu}_+),$$

where Φ_- is an expression involving F,F' and $\overline{\mu}_-$, Φ_+ is an expression involving F,F' and $\overline{\mu}_+$, and Σ is a constant involving F', $\overline{\mu}_-$ and $\overline{\mu}_+$. Since q is a general compactly supported density function, this leads to the equations $(1-p)\Phi_-(a)=\Sigma(F,\overline{\mu}_-,\overline{\mu}_+)$, for $A_-(\mu)<a<0$ and $p\Phi_+(a)=\Sigma(F,\overline{\mu}_-,\overline{\mu}_+)$, for $0<a<A_+(\mu)$. These equations for F turn out to be tractable. If μ satisfies the square root balance condition and is as in (ii) of Theorem 2, then there is a unique solution F_0 for which the corresponding

b_0 (obtained via $\frac{F_0''(x)}{F_0'(x)} = \frac{f_0'}{f_0}(x) = \frac{2}{D}b_0(x)$) is in $\mathcal{D}_\mu$; otherwise there is no solution, and thus there are no critical points.

When G_2 possesses a critical point F_0, how do we show that in fact G_2 attains its global minimum uniquely at F_0? (Or equivalently, how do we show that the global minimum of G_1 is attained uniquely at b_0, where b_0 corresponds to F_0 via (1.19)?) Uniqueness is immediate since there is only one critical point. Due to certain technical obstacles, we can only show directly that F_0 is the global minimum in the case of target measures μ for which $A_-(\mu)$ and $A_+(\mu)$ are finite and are atoms of the measure. The case of a general measure is obtained by approximating by measures as above. To prove that the critical point F_0 is the global minimum, it would be natural to take an arbitrary admissible F and consider $L_2(t) := G_2((1-t)F_0 + tF)$. We would like to show that G_2 is convex and that $L_2'(0) = 0$, from which it would follow that the global minimum is attained at F_0. However, we see no way to prove that G_2 is convex. On the other hand, it is very easy to show that G_1 is convex.

Proposition 3. *For all target distributions μ, the set $\mathcal{D}_\mu$ is convex and the functional G_1 on $\mathcal{D}_\mu$ is convex.*

The above result does not require that the target measure be of the special type mentioned above. However, we require this restriction to prove the following technical result.

Proposition 4. *Assume that $A_-(\mu)$ and $A_+(\mu)$ are finite, that μ has atoms at both $A_-(\mu)$ and $A_+(\mu)$, and that its restriction to $(A_-(\mu), A_+(\mu))$ is absolutely continuous with a piecewise continuous, locally bounded density. Assume also that μ satisfies the square root boundary condition* (1.8) *and let b_0 be as in* (1.11). *Let $b \in \mathcal{D}_\mu$, and define $L_1(t) = G_1((1-t)b_0 + tb)$, $0 \le t \le 1$, where G_1 is as in* (1.18). *Then $L_1'(0) = 0$.*

From the above two propositions, it follows immediately that when μ is as in Proposition 4, the critical point F_0 is in fact the global minimum.

The rest of the paper is organized as follows. In section 2, we illustrate Theorem 2 with a number of examples. The proof of Theorem 1 requires the result in Theorem 2-i, and the proof of Theorem 3 requires some of the proof of Theorem 2. Thus we first prove Theorem 2 in section 3, and then prove Theorems 1 and 3 in section 4. Of course, these result also depend on Propositions 2, 3 and 4. The first of these propositions is proved in section 5 and the next two are proved in section 6.

2. Some examples of Theorem 2

We give several examples to illustrate Theorem 2, restricting always to the case that the target distribution μ is symmetric. Recall that in the symmetric case, the infimum is given by (1.10). Recall also from Remark 2 after Theorem 2 that in the symmetric case, the expected distance of the target is equal to $\int_0^\infty \overline{\mu}_+(dx)$, and has been denoted by $\text{AvgDist}(\mu)$. Furthermore, by (1.12), the ratio $\frac{D\log 2}{2}\frac{\inf_{b\in\mathcal{D}_\mu}\int_{\mathbb{R}}(E_0^{(b)}T_a)\,\mu(da)}{(\text{AvgDist}(\mu))^2}$ is always greater or equal to 1, with equality only in the case of the distributions in example I below.

I. *Symmetric Degenerate Distribution*: $\mu=\frac12\delta_{-A}+\frac12\delta_A,\ A>0$

We have $\overline{\mu}_+(x)=1,\ x\in[0,A)$, and $\overline{\mu}_+(x)=0,\ x\ge A$. Thus,

$$\inf_{b\in\mathcal{D}_\mu}\int_{\mathbb{R}}(E_0^{(b)}T_a)\,\mu(da)=\frac{2}{D\log 2}A^2=\frac{2}{D\log 2}(\text{AvgDist}(\mu))^2.$$

The infimum is attained at the anti-symmetric drift b_0 satisfying

$$b_0(x)=\frac{D\log 2}{2A},\ 0<x<A.$$

Of course, the corresponding diffusion can reach $\pm A$, so we impose reflection at $\pm A$.

II. *Symmetric Uniform Distribution*: $\mu=\text{U}([-A,A]),\ A>0$

We have $\overline{\mu}_+(x)=1-\frac{x}{A},\ x\in[0,A)$, and $\overline{\mu}_+(x)=0,\ x>A$. One has $\int_0^\infty\overline{\mu}_+^{\frac12}(x)dx=\frac{1}{A^{\frac12}}\int_0^A(A-x)^{\frac12}dx=\frac23 A$. Also, $\text{AvgDist}(\mu)=\frac{A}{2}$. Thus,

$$\inf_{b\in\mathcal{D}_\mu}\int_{\mathbb{R}}(E_0^{(b)}T_a)\,\mu(da)=\frac{8}{9D\log 2}A^2=\frac{16}{9}\frac{2}{D\log 2}(\text{AvgDist}(\mu))^2.$$

The infimum is attained at the anti-symmetric drift b_0 satisfying

$$b_0(x)=D\big[-\frac{1}{4(A-x)}+\frac{3\log 2}{4A}(1-\frac{x}{A})^{\frac12}\big],\ x\in(0,A).$$

Despite the unbounded drift, the corresponding diffusion can reach $\pm A$; thus we impose reflection at $\pm A$.

III. *Symmetric Exponential Distribution*: $\mu=\frac12\text{Exp}(\lambda)+\frac12\big(-\text{Exp}(\lambda)\big),\ \lambda>0$

We have $\overline{\mu}_+(x)=e^{-\lambda}$, $x>0$, and $\text{AvgDist}(\mu)=\frac1\lambda$. Thus,

$$\inf_{b\in\mathcal{D}_\mu}\int_{\mathbb{R}}(E_0^{(b)}T_a)\,\mu(da)=\frac{8}{D\lambda^2\log 2}=4\frac{2}{D\log 2}(\text{AvgDist}(\mu))^2.$$

The infimum is attained at the anti-symmetric drift b_0 satisfying

$$b_0(x)=D\big(-\frac{\lambda}{4}+\frac{\lambda}{4}(\log 2)e^{-\frac{\lambda}{2}x}\big),\ x>0.$$

IV. *Symmetric Gaussian Distribution*: $\mu=N(0,\sigma^2)$

We have $\overline{\mu}_+(x)=\int_x^\infty\frac{\exp(-\frac{y^2}{2\sigma^2})}{\sqrt{2\pi}\sigma}dy=1-\Phi(\frac{x}{\sigma})$, where $\Phi(z)=\int_{-\infty}^z\frac{\exp(-\frac{y^2}{2})}{\sqrt{2\pi}}dy$. One has $\int_0^\infty\overline{\mu}_+^{\frac12}(x)dx=\sigma\int_0^\infty\big(1-\Phi(z)\big)^{\frac12}dz\approx 0.9219\,\sigma$. Also, $\text{AvgDist}(\mu)=\frac{\sigma}{\sqrt{2\pi}}$. Thus,

$$\inf_{b\in\mathcal{D}_\mu}\int_{\mathbb{R}}(E_0^{(b)}T_a)\,\mu(da)=\frac{2\sigma^2\big(\int_0^\infty\big(1-\Phi(z)\big)^{\frac12}dz\big)^2}{D\log 2}=$$

$$2\pi\big(\int_0^\infty\big(1-\Phi(z)\big)^{\frac12}dz\big)^2\frac{2}{D\log 2}(\text{AvgDist}(\mu))^2\approx 5.340\frac{2}{D\log 2}(\text{AvgDist}(\mu))^2.$$

The infimum is attained at the anti-symmetric drift b_0 satisfying

$$b_0(x)=D\big[-\frac14\frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma(1-\Phi(\frac{x}{\sigma}))}+\frac{\log 2}{2\sigma\int_0^\infty\big(1-\Phi(z)\big)^{\frac12}dz}\big(1-\Phi(\frac{x}{\sigma})\big)^{\frac12}\big],\ x>0.$$

V. *Symmetric Pareto Distribution*: $\mu=\frac12\text{Pareto}(\alpha,A_0)+\frac12(-\text{Pareto}(\alpha,A_0))$, where $A_0>0$, $\alpha>2$ and $\mu_+\sim\text{Pareto}(\alpha,A_0)$ is given by $\overline{\mu}_+(x)=\min(1,(\frac{x}{A_0})^{-\alpha}),\ x>0$.

One has $\int_0^\infty\overline{\mu}_+^{\frac12}(x)dx=A_0+\frac{2}{\alpha-2}$ and $\text{AvgDist}(\mu)=A_0+\frac{1}{\alpha-1}$. Thus

$$\inf_{b\in\mathcal{D}_\mu}\int_{\mathbb{R}}(E_0^{(b)}T_a)\,\mu(da)=\big(\frac{\alpha-1}{\alpha-2}\big)^2\Big(\frac{A_0(\alpha-2)+2}{A_0(\alpha-1)+1}\Big)^2\frac{2}{D\log 2}(\text{AvgDist}(\mu))^2.$$

The infimum is attained at the anti-symmetric drift b_0 satisfying

$$b_0(x)=\begin{cases} D\frac{\log 2}{2(A_0+\frac{2}{\alpha-2})}, \ x\in(0,A_0);\\ D\big(-\frac{\alpha}{4x}+\frac{\log 2}{2(A_0+\frac{2}{\alpha-2})}(\frac{x}{A_0})^{-\frac{\alpha}{2}}\big), \ x>A_0.\end{cases}$$

Note that this drift is only piecewise continuous, because $\overline{\mu}_+$ is only piecewise continuously differentiable.

Remark 1. Note that the ratio $\frac{D\log 2}{2}\frac{\inf_{b\in\mathcal{D}_\mu}\int_{\mathbb{R}}(E_0^{(b)}T_a)\,\mu(da)}{(\mathrm{AvgDist}(\mu))^2}$ is independent of the parameter for each of the families of distributions in examples I-IV above. For the family of Pareto distributions in example V, for fixed A_0, this ratio increases from 1^+ to ∞ as α decreases from ∞ to 2^+.

Remark 2. Note the asymptotic behavior as $x\to\infty$ of the minimizing drift $b_0(x)$ in examples III–V:

Exponential: $\lim_{x\to\infty}b(x)=-\frac{\lambda}{4}D$;

Gaussian: $b(x)\sim-\frac{x}{4\sigma^2}D$;

Pareto: $b(x)\sim-\frac{\alpha}{4x}D$.

3. Proof of Theorem 2

We begin with the long proof of part (ii). The proofs of the other two parts use the result of part (ii).

Proof of part (ii). Recalling (1.16)-(1.20) and recalling the definition of $\mathcal{D}(G_2)$ from (1.21), we search for critical points $F\in\mathcal{D}(G_2)$ of the functional $G_2(F)$. Without loss of generality, we may assume that F is a distribution; that is, $F(A_+(\mu))=1$. Let Q denote an arbitrary distribution function on $(A_-(\mu),A_+(\mu))$, with a density q that is continuous, piecewise continuously differentiable and compactly supported in $(A_-(\mu),A_+(\mu))$. Then $F+\epsilon Q$ belongs to the domain $\mathcal{D}(G_2)$ for all ϵ with sufficiently small absolute value. To prove this, one needs to find a $b_\epsilon\in\mathcal{D}_\mu$, the class of admissible drifts, such that

$$\frac{F(x)+\epsilon Q(x)}{1+\epsilon}=\frac{\int_{A_-(\mu)}^x dz\exp(\int_0^z\frac{2}{D}b_\epsilon(t)dt)}{\int_{A_-(\mu)}^{A_+(\mu)}dz\exp(\int_0^z\frac{2}{D}b_\epsilon(t)dt)}.$$

This can be solved directly for b_ϵ by differentiating, taking logarithms and then differentiating again. (The conditions above on q are dictated by the conditions on $b_\epsilon\in\mathcal{D}_\mu$.)

We call F a critical point if (1.23) holds for all such Q. A necessary condition for $\inf_{b\in\mathcal{D}_\mu}\int_{\mathbb{R}}(E_0^{(b)}T_a)\,\mu(da)$ to be attained at some particular b is that the corresponding F (via (1.19)) is critical for G_2. Indeed, if F is not critical, then for some ϵ with small absolute value, we will have $G_2(F+\epsilon Q)<G_2(F)$, or equivalently, $\int_{\mathbb{R}}(E_0^{(b)}T_a)\,\mu(da)=G_1(b)>G_1(b_\epsilon)=\int_{\mathbb{R}}(E_0^{(b_\epsilon)}T_a)\,\mu(da)$.

Now F will be critical, that is, (1.23) will hold for all such Q, if and only if

$$\begin{aligned}&(1-p)\int_{A_-(\mu)}^0\overline{\mu}_-(a)\Big(\frac{1-Q(a)}{f(a)}-\frac{(1-F(a))q(a)}{f^2(a)}\Big)da+\\&p\int_0^{A_+(\mu)}\overline{\mu}_+(a)\Big(\frac{Q(a)}{f(a)}-\frac{F(a)q(a)}{f^2(a)}\Big)da=0,\end{aligned}\tag{3.1}$$

for all such Q. Integration by parts gives

$$\int_{A_-(\mu)}^0\overline{\mu}_-(a)\frac{Q(a)}{f(a)}da=\int_{A_-(\mu)}^0 q(a)\Big(\int_a^0\frac{\overline{\mu}_-(x)}{f(x)}dx\Big)da\tag{3.2}$$

and

$$\int_0^{A_+(\mu)}\overline{\mu}_+(a)\frac{Q(a)}{f(a)}da=-\int_0^{A_+(\mu)}q(a)\Big(\int_0^a\frac{\overline{\mu}_+(x)}{f(x)}dx\Big)da+\int_0^{A_+(\mu)}\frac{\overline{\mu}_+(a)}{f(a)}da.\tag{3.3}$$

Substituting (3.2) and (3.3) into (3.1) gives

$$\begin{aligned}&(1-p)\int_{A_-(\mu)}^0 q(a)\Big[-\int_a^0\frac{\overline{\mu}_-(x)}{f(x)}dx-\frac{\overline{\mu}_-(a)(1-F(a))}{f^2(a)}\Big]da+\\&p\int_0^{A_+(\mu)}q(a)\Big[-\int_0^a\frac{\overline{\mu}_+(x)}{f(x)}dx-\frac{\overline{\mu}_+(a)F(a)}{f^2(a)}\Big]+\\&(1-p)\int_{A_-(\mu)}^0\frac{\overline{\mu}_-(a)}{f(a)}da+p\int_0^{A_+(\mu)}\frac{\overline{\mu}_+(a)}{f(a)}da=0.\end{aligned}\tag{3.4}$$

Now (3.4) will hold for all densities q of the type described above if and only if

(3.5)

$$\int_a^0 \frac{\overline{\mu}_-(x)}{f(x)}dx + \frac{\overline{\mu}_-(a)(1-F(a))}{f^2(a)} = \int_{A_-(\mu)}^0 \frac{\overline{\mu}_-(x)}{f(x)}dx + \frac{p}{1-p}\int_0^{A_+(\mu)} \frac{\overline{\mu}_+(x)}{f(x)}dx,$$

$a \in (A_-(\mu), 0)$;

$$\int_0^a \frac{\overline{\mu}_+(x)}{f(x)}dx + \frac{\overline{\mu}_+(a)F(a)}{f^2(a)} = \frac{1-p}{p}\int_{A_-(\mu)}^0 \frac{\overline{\mu}_-(x)}{f(x)}dx + \int_0^{A_+(\mu)} \frac{\overline{\mu}_+(x)}{f(x)}dx,$$

$a \in (0, A_+(\mu))$.

Denoting by C the constant on the right hand side of the first equation above, we have

$$\overline{\mu}_-(a) = \frac{f^2(a)}{1-F(a)}\big(C - \int_a^0 \frac{\overline{\mu}_-(x)}{f(x)}dx\big),\ a \in (A_-(\mu), 0). \tag{3.6}$$

From (1.19) and the fact that $b \in \mathcal{D}_\mu$, it follows that F is continuously differentiable on $(A_-(\mu), 0)$, and that f is continuous and piecewise continuously differentiable with a locally bounded derivative on $(A_-(\mu), 0)$. Thus, we deduce from (3.6) that $\overline{\mu}_-$ is continuous and piecewise continuously differentiable with locally bounded derivative on $(A_-(\mu), 0)$. The same analysis shows that $\overline{\mu}_+$ is continuous and piecewise continuously differentiable with locally bounded derivative on $(0, A_+(\mu))$. We have thus shown that a necessary condition for the existence of a critical point is that the restriction of μ to $(A_-(\mu), A_+(\mu))$ is absolutely continuous with a density that is piecewise continuous and locally bounded. In addition, μ might possibly possess an atom at $A_-(\mu)$ and/or at $A_+(\mu)$.

We now continue our analysis under the assumption that μ satisfies the above noted necessary condition for a critical point. Then $\overline{\mu}_-$ and $\overline{\mu}_+$ are continuous and piecewise continuously differentiable with a locally bounded derivative. In the analysis that follows, we implicitly assume that $\overline{\mu}_-$ and $\overline{\mu}_+$ are continuously differentiable. However everything still goes through under the weaker assumption that they are continuous and piecewise continuously differentiable with locally bounded derivative. (See example V in section 2 for a case where $\overline{\mu}_-$ and $\overline{\mu}_+$ are only piecewise continuously differentiable.) Differentiating (3.5) gives

$$2\,\frac{\overline{\mu}_-(a)}{f(a)}+2\,\frac{\overline{\mu}_-(a)(1-F(a))f'(a)}{f^3(a)}-\frac{\overline{\mu}'_-(a)(1-F(a))}{f^2(a)}=0,\ a\in(A_-(\mu),0);$$
$$-\,2\,\frac{\overline{\mu}_+(a)}{f(a)}+2\,\frac{\overline{\mu}_+(a)F(a)f'(a)}{f^3(a)}-\frac{\overline{\mu}'_+(a)F(a)}{f^2(a)}=0,\ a\in(0,A_+(\mu).$$

Multiplying through by f, and noting that $f=F'$ and $f'=F''$, we can rewrite the above equations as

$$\begin{aligned}&2\,\overline{\mu}_-+2\,\frac{\overline{\mu}_-(1-F)F''}{(F')^2}-\frac{\overline{\mu}'_-(1-F)}{F'}=0,\quad\text{on }(A_-(\mu),0);\\&-2\,\overline{\mu}_++2\,\frac{\overline{\mu}_+FF''}{(F')^2}-\frac{\overline{\mu}'_+F}{F'}=0\ \ \text{on }(0,A_+(\mu).\end{aligned}\tag{3.7}$$

Since

$$\Big(\frac{F}{F'}\Big)'=1-\frac{FF''}{(F')^2}\quad\text{and}\quad\Big(\frac{1-F}{F}\Big)'=-1-\frac{(1-F)F''}{(F')^2},$$

it follows that (3.7) is equivalent to

$$\begin{aligned}&2\,\Big(\frac{1-F}{(1-F)'}\Big)'+\frac{\overline{\mu}'_-}{\overline{\mu}_-}\,\frac{1-F}{(1-F)'}=0\ \ \text{on }(A_-(\mu),0);\\&2\,\Big(\frac{F}{F'}\Big)'+\frac{\overline{\mu}'_+}{\overline{\mu}_+}\,\frac{F}{F'}=0\text{ on }(0,A_+(\mu)).\end{aligned}\tag{3.8}$$

We now work with the second equation in (3.8). Substituting $H=\frac{F}{F'}$, we obtain the linear equation

$$2H'+\frac{\overline{\mu}'_+}{\overline{\mu}_+}\,H=0.$$

Solving for H gives $H=\text{const.}\,\overline{\mu}_-^{-\frac12}$. Thus, $\frac{F'}{F}=\text{const.}\,\overline{\mu}_-^{\frac12}$, and solving for F gives

$$F(a)=\exp(-k_1\int_a^{A_+(\mu)}\overline{\mu}_+^{\frac12}(x)dx),\ a\in(0,A_+(\mu)),\tag{3.9}$$

for some $k_1>0$, and thus

$$f(a)=k_1\overline{\mu}_+^{\frac12}(a)\exp(-k_1\int_a^{A_+(\mu)}\overline{\mu}_+^{\frac12}(x)dx),\ a\in(0,A_+(\mu)).\tag{3.10}$$

Analyzing the first equation in (3.8) similarly, we arrive at

$$F(a)=1-\exp(-k_2\int_{A_-(\mu)}^{a}\overline{\mu}_-^{\frac12}(x)dx),\ a\in(A_-(\mu),0).\tag{3.11}$$

for some $k_2 > 0$, and thus

$$f(a) = k_2\overline{\mu}_-^{\frac{1}{2}}(a)\exp(-k_2\int_{A_-(\mu)}^a \overline{\mu}_-^{\frac{1}{2}}(x)dx),\ \ a \in (A_-(\mu), 0). \tag{3.12}$$

Since F and f are continuous at $a = 0$, it follows from (3.9)-(3.12) that k_1 and k_2 must satisfy

$$\begin{aligned}
&\exp(-k_1\int_0^{A_+(\mu)} \overline{\mu}_+^{\frac{1}{2}}(a)da) + \exp(-k_2\int_{A_-(\mu)}^0 \overline{\mu}_-^{\frac{1}{2}}(a)da) = 1;\\
&k_1\exp(-k_1\int_0^{A_+(\mu)} \overline{\mu}_+^{\frac{1}{2}}(a)da) = k_2\exp(-k_2\int_{A_-(\mu)}^0 \overline{\mu}_-^{\frac{1}{2}}(a)da),
\end{aligned}$$

or equivalently,

$$\begin{aligned}
&\exp(-k_1\int_0^{A_+(\mu)} \overline{\mu}_+^{\frac{1}{2}}(a)da) = \frac{k_2}{k_1+k_2};\\
&\exp(-k_2\int_{A_-(\mu)}^0 \overline{\mu}_-^{\frac{1}{2}}(a)da) = \frac{k_1}{k_1+k_2}.
\end{aligned} \tag{3.13}$$

When a function F, satisfying (3.9) and (3.11), with k_1, k_2 satisfying (3.13), is substituted into the left hand sides of (3.5), it will render these expressions constant in a, since F has been obtained by solving the differential equation obtained by setting to 0 the derivative of the left hand side in each of the two equations in (3.5). However, in order to conclude that such an F is indeed a critical point of G_2, we still need to verify that (3.5) holds. Since the left hand sides are constant in a, it suffices to verify these the equations at $a = 0$ (actually as $a \to 0^-$ for the first equation and as $a \to 0^+$ for the second one). This yields the requirement

$$\begin{aligned}
&\frac{1-F(0)}{f^2(0)} = \int_{A_-(\mu)}^0 \frac{\overline{\mu}_-(a)}{f(a)}da + \frac{p}{1-p}\int_0^{A_+(\mu)} \frac{\overline{\mu}_+(a)}{f(a)}da,\\
&\frac{F(0)}{f^2(0)} = \frac{1-p}{p}\int_{A_-(\mu)}^0 \frac{\overline{\mu}_-(a)}{f(a)}da + \int_0^{A_+(\mu)} \frac{\overline{\mu}_+(a)}{f(a)}da.
\end{aligned} \tag{3.14}$$

We now use (3.9)-(3.12) to write (3.14) exclusively in terms of μ_+, μ_-, k_1, k_2 and p. Using (3.9)-(3.12), we have

$$
\begin{aligned}
\frac{F(0)}{f^2(0)} &= \frac{1}{k_1^2}\exp(k_1\int_0^{A^+(\mu)}\overline{\mu}_+^{\frac12}(a)da),\\
\frac{1-F(0)}{f^2(0)} &= \frac{1}{k_2^2}\exp(k_2\int_{A^-(\mu)}^0\overline{\mu}_-^{\frac12}(a)da).
\end{aligned}
\tag{3.15}
$$

Using (3.10), we have

$$
\begin{aligned}
&\int_0^{A_+(\mu)}\frac{\overline{\mu}_+(a)}{f(a)}da = \frac{1}{k_1}\int_0^{A_+(\mu)}\overline{\mu}_+^{\frac12}(a)\exp(k_1\int_a^{A_+(\mu)}\overline{\mu}_+^{\frac12}(x)dx) =\\
&\frac{1}{k_1^2}\Big(\exp(k_1\int_0^{A_+(\mu)}\overline{\mu}_+^{\frac12}(a)da)-1\Big),
\end{aligned}
\tag{3.16}
$$

and similarly, using (3.12), we obtain

$$
\int_{A_-(\mu)}^0\frac{\overline{\mu}_-(a)}{f(a)}da = \frac{1}{k_2^2}\Big(\exp(k_2\int_{A_-(\mu)}^0\overline{\mu}_-^{\frac12}(a)da)-1\Big).
\tag{3.17}
$$

From (3.15)-(3.17), the requirement in (3.14) can be written as

$$
\begin{aligned}
\frac{1-p}{p} &= \frac{k_2^2}{k_1^2}\Big(\exp(k_1\int_0^{A_+(\mu)}\overline{\mu}_+^{\frac12}(a)da)-1\Big),\\
\frac{p}{1-p} &= \frac{k_1^2}{k_2^2}\Big(\exp(k_2\int_{A_+(\mu)}^0\overline{\mu}_-^{\frac12}(a)da)-1\Big).
\end{aligned}
\tag{3.18}
$$

Thus, we have shown that F is a critical point if and only if it satisfies (3.9) and (3.11), where k_1 and k_2 satisfy (3.13) and (3.18). However, the pair k_1, k_2 is over-determined by (3.13) and (3.18). From these two equations, it follows that

$$
\begin{aligned}
\frac{k_1}{k_1+k_2} &= p\,;\\
\frac{k_2}{k_1+k_2} &= 1-p.
\end{aligned}
\tag{3.19}
$$

Substituting this back into (3.13) gives

$$
\begin{aligned}
&\exp(-k_1\int_0^{A_+(\mu)}\overline{\mu}_+^{\frac12}(a)da) = 1-p;\\
&\exp(-k_2\int_{A_-(\mu)}^0\overline{\mu}_-^{\frac12}(a)da) = p,
\end{aligned}
$$

or equivalently

$$k_1 = \frac{|\log(1-p)|}{\int_0^{A_+(\mu)} \overline{\mu}_+^{\frac{1}{2}}(a)da} \quad \text{and} \quad k_2 = \frac{|\log p|}{\int_{A_-(\mu)}^0 \overline{\mu}_-^{\frac{1}{2}}(a)da}. \tag{3.20}$$

But (3.19) and (3.20) hold simultaneously if and only if μ satisfies the square root balance condition (1.8).

We have now shown that if μ satisfies the square root balance condition and if the restriction of μ to $(A_-(\mu), A_+(\mu))$ is absolutely continuous with a piecewise continuous, locally bounded density, then G_2 possesses a unique critical point, call it F_0, while otherwise G_2 has no critical points. This critical point F_0 is given by (3.9) and (3.11), where k_1, k_2 are as in (3.19):

(3.21)

$$F_0(a) = \exp\Big(-\frac{|\log(1-p)|}{\int_0^{A_+(\mu)} \overline{\mu}_+^{\frac{1}{2}}(x)dx}\int_a^{A_+(\mu)} \overline{\mu}_+^{\frac{1}{2}}(x)dx\Big),\ a \in (0, A_+(\mu)),$$

$$F_0(a) = 1 - \exp\Big(-\frac{|\log p|}{\int_{A_-(\mu)}^0 \overline{\mu}_-^{\frac{1}{2}}(x)dx}\int_{A_-(\mu)}^a \overline{\mu}_-^{\frac{1}{2}}(x)dx\Big),\ a \in (A_-(\mu), 0).$$

Recall that distributions F are connected to drifts b via (1.19); thus $b = \frac{D}{2}\frac{F''}{F'} = \frac{D}{2}\frac{f'}{f}$. Using this with (3.21), it follows that the drift b_0 associated with F_0 is given by (1.11).

We now show b_0 constitutes the unique global minimum of G_1. Uniqueness is immediate. Indeed, if b_1 is also the global minimum, then F_1 would be a critical point for G_2, where F_1 corresponds to b_1 via (1.19); however F_0 is the unique critical point of G_2.

We turn to showing that the global minimum occurs at b_0. Recall that we are assuming that the restriction of μ to $(A_-(\mu), A_+(\mu))$ is absolutely continuous with piecewise continuous, locally bounded density. First assume that μ possesses atoms at $A_-(\mu)$ and $A_+(\mu)$ as in Proposition 4. Let $b \in \mathcal{D}_\mu$, and define L_1 as in Proposition 4. Then it follows from that proposition and Proposition 3 that $G_1(b) \geq G_1(b_0)$.

Now assume that μ possesses an atom at $A_-(\mu)$ but not at $A_+(\mu)$. (The cases in which μ possesses an atom at $A_+(\mu)$ but not at $A_-(\mu)$, or in which μ possesses an atom neither at $A_-(\mu)$ nor at $A_+(\mu)$ are treated similarly.) For each $n \in \mathbb{N}$, approximate μ_+ by $\mu_{+;n}$, defined as follows. If $A_+(\mu) < \infty$, let $\mu_{+,n}$ restricted to $(0, A_+(\mu) - \frac{1}{n})$ coincide with μ_+ restricted to $(0, A_+(\mu) - \frac{1}{n})$. Also, let $\mu_{+;n}$ have an atom of mass $\mu_+\big(A_+(\mu) - \frac{1}{n}, A_+(\mu)\big)$ at $A_+(\mu) - \frac{1}{n}$. If $A_+(\mu) = \infty$, let $\mu_{+,n}$ restricted to $(0, n)$ coincide with μ_+ restricted to $(0, n)$. Also, let $\mu_{+;n}$ have an atom of mass $\mu_+\big((n, \infty)\big)$ at n. Then $\mu_{+;n}$ converges weakly to μ_+, and since the integrands are monotone, we have

$$\lim_{n\to\infty} \int_0^\infty \mu_{+;n}^{\frac{1}{2}}(x)dx = \int_0^\infty \mu_+^{\frac{1}{2}}(x)dx. \tag{3.22}$$

Now define $\mu_n = (1-p_n)\mu_- + p_n\mu_{+;n}$, where p_n is defined so that μ_n satisfies the square root balance condition (1.8). Note that according to the previous paragraph, μ_n is a measure of the type for which the critical b, call it b_n, is in fact the global minimum of G_1. It is given by (1.11), with $\overline{\mu}_+$ replaced by $\overline{\mu}_{+;n}$. Substituting this drift in (1.16), or equivalently in (1.18), and performing the routine calculation gives

$$\begin{aligned} &\inf_{b\in\mathcal{D}_{\mu_n}} \int_{\mathbb{R}} (E_0^{(b)} T_a)\mu_n(da) = \\ &\frac{2}{D}\Big(\frac{1-p_n}{|\log p_n|}\big(\int_{-\infty}^0 \overline{\mu}_-^{\frac{1}{2}}(x)dx\big)^2 + \frac{p_n}{|\log(1-p_n)|}\big(\int_0^\infty \overline{\mu}_{+;n}^{\frac{1}{2}}(x)dx\big)^2\Big). \end{aligned} \tag{3.23}$$

By (3.22) and the fact that μ satisfies the square root balance condition, it follows that

$$\lim_{n\to\infty} p_n = p. \tag{3.24}$$

Since $E_0^{(b)} T_a$ is an increasing function of $a \in (0, \infty)$, by the construction of μ_n we have

$$\int_{\mathbb{R}} (E_0^{(b)} T_0)\mu(da) \ge \int_{\mathbb{R}} (E_0^{(b)} T_0)\mu_n(da), \text{ for any drift } b. \tag{3.25}$$

The critical drift b_0 that we have found is given by (1.11). Substituting this drift in (1.16) and performing the routine calculation gives

$$\begin{aligned} &\int_{\mathbb{R}} (E_0^{(b_0)} T_a)\, \mu(da) = \\ &\frac{2}{D}\Big(\frac{1-p}{|\log p|}\big(\int_{-\infty}^0 \overline{\mu}_-^{\frac{1}{2}}(x)dx\big)^2 + \frac{p}{|\log(1-p)|}\big(\int_0^\infty \overline{\mu}_+^{\frac{1}{2}}(x)dx\big)^2\Big). \end{aligned} \tag{3.26}$$

From (3.22)-(3.26), we conclude that b_0 is indeed the global minimum of G_1.

To complete the proof of part (ii), it remains to prove the statements that follow (1.11). The function $u(a)=\int_0^a dx\exp(-\frac{2}{D}\int_0^x b_0(t)dt)$ is harmonic for the diffusion generator $\frac{D}{2}\frac{d^2}{dx^2}+b_0(x)\frac{d}{dx}$. Thus, by Ito's formula it follows that

$$P_0^{(b_0)}(\tau_{a_1}<\tau_{a_2})=\frac{u_0(0)-u_0(a_2)}{u_0(a_1)-u_0(a_2)},\ \text{for } A_-(\mu)<a_1<0<a_2<A_+(\mu).$$

Substituting for b_0 above from (1.11), we have

$$u_0(a)=\begin{cases}-\int_a^0\overline{\mu}_-^{-\frac12}(x)\exp\big(\frac{|\log p|}{\int_{-\infty}^0\overline{\mu}_-^{\frac12}(x)dx}\int_0^x\overline{\mu}_-^{\frac12}(y)dy\big),\ A_-(\mu)<a<0;\\ \int_0^a\overline{\mu}_+^{-\frac12}(x)\exp(-\frac{|\log(1-p)|}{\int_0^\infty\overline{\mu}_+^{\frac12}(x)dx}\int_0^x\overline{\mu}_+^{\frac12}(y)dy),\ 0<a<A_+(\mu).\end{cases}$$

Then $\lim_{a_1\to A_-(\mu)^+}u_0(a_1)$ is infinite or finite depending on whether $\int_{A_-(\mu)}\overline{\mu}^{-\frac12}(x)dx$ is infinite or finite. In the former case, $\lim_{a_1\to A_-(\mu)^+}P_0^{(b_0)}(\tau_{a_1}<\tau_{a_2})$ is equal to 0, and in the latter case it is positive. Thus, in the former case, the diffusion cannot reach $A_-(\mu)$. In the latter case, either the diffusion can reach $A_-(\mu)$, or else with positive probability the diffusion approaches $A_-(\mu)$ as $t\to\infty$. But this latter scenario is ruled out since it would mean that $E_0^{(b_0)}T_a=\infty$, for $a\in(0,A_+(\mu))$. The exact same argument holds with regard to $\overline{\mu}_+$ and $A_+(\mu)$. □

Proof of part (i). If μ is such that the infimum is attained, as specified in part (ii), then substituting the optimal drift from (1.11) in (1.16), and performing the routine calculation shows that (1.9) holds.

Now assume that $\mu=(1-p)\mu_-+p\mu_+$ satisfies the square root balance condition and is such that the infimum is not attained, as specified in part (ii). For each $n\in\mathbb{N}$, define measures $\mu_{+;n,+}$ and $\mu_{+;n,-}$ on $(0,\infty)$ as follows. For $k=0,1,\cdots$, let $\mu_{+;n,+}$, when restricted to $(\frac{k+1}{n},\frac{k+2}{n})$, be uniform with total mass equal to $\mu_+((\frac{k}{n},\frac{k+1}{n}])$. For $k=2,\cdots$, let $\mu_{+;n,-}$, when restricted to $(\frac{k-1}{n},\frac{k}{n})$, be uniform with total mass equal to $\mu_+((\frac{k}{n},\frac{k+1}{n}])$. Also, let $\mu_{+;n,-}$, when restricted to $(0,\frac1n)$, be uniform with total mass equal to $\mu_+((0,\frac2n])$ Define respectively $\mu_{-;n,+}$ and $\mu_{-;n,-}$ in a parallel fashion for μ_- as $\mu_{+;n,+}$ and $\mu_{+;n,-}$ were defined for μ_+. Then $\mu_{+;n,+}$ and $\mu_{+;n,-}$ both converge weakly to μ_+, and $\mu_{-;n,+}$ and $\mu_{-;n,-}$ both converge weakly to μ_-, as $n\to\infty$. Therefore, since all the integrands are monotone, we have

$$\begin{aligned}&\lim_{n\to\infty}\int_0^\infty \overline{\mu}^{\frac12}_{+;n,+}(x)dx=\lim_{n\to\infty}\int_0^\infty \overline{\mu}^{\frac12}_{+;n,-}(x)dx=\int_0^\infty \overline{\mu}^{\frac12}_{+}(x)dx,\\ &\lim_{n\to\infty}\int_{-\infty}^0 \overline{\mu}^{\frac12}_{-;n,+}(x)dx=\lim_{n\to\infty}\int_{-\infty}^0 \overline{\mu}^{\frac12}_{-;n,-}(x)dx=\int_{-\infty}^0 \overline{\mu}^{\frac12}_{-}(x)dx.\end{aligned}\tag{3.27}$$

Now define

$$\mu_{+,n}=(1-p_{+,n})\mu_{-;n,+}+p_{+,n}\,\mu_{+;n,+}\,;$$

$$\mu_{-,n}=(1-p_{-,n})\mu_{-;n,-}+p_{-,n}\,\mu_{+;n,-}\,,$$

where $p_{+,n}$ and $p_{-,n}$ are defined so that $\mu_{+,n}$ and $\mu_{-,n}$ satisfy the square root balance condition (1.8). By (3.27) and the fact that μ satisfies the square root balance condition, it follows that

$$\lim_{n\to\infty}p_{+,n}=\lim_{n\to\infty}p_{-,n}=p.\tag{3.28}$$

By part (ii), the measures $\mu_{+,n}$ and $\mu_{-,n}$ are of the type for which the infimum is attained; let $b_{+,n}$ and $b_{-,n}$ denote the corresponding drifts for which the infimum is attained. Then

$$\begin{aligned}&\inf_{b\in\mathcal{D}_{\mu_{+,n}}}\int_{\mathbb{R}}(E_0^{(b)}T_a)\,\mu_{+,n}(da)=\int_{\mathbb{R}}(E_0^{(b_{+,n})}T_a)\,\mu_{+,n}(da)=\\ &\frac{2}{D}\Big(\frac{1-p_{+,n}}{|\log p_{+,n}|}\big(\int_{-\infty}^0\overline{\mu}^{\frac12}_{-;n,+}(x)dx)^2+\frac{p_{+,n}}{|\log(1-p_{+,n})|}\big(\int_0^\infty\overline{\mu}^{\frac12}_{+;n,+}(x)dx)^2\Big);\\ &\inf_{b\in\mathcal{D}_{\mu_{-,n}}}\int_{\mathbb{R}}(E_0^{(b)}T_a)\,\mu_{-,n}(da)=\int_{\mathbb{R}}(E_0^{(b_{-,n})}T_a)\,\mu_{-,n}(da)=\\ &\frac{2}{D}\Big(\frac{1-p_{-,n}}{|\log p_{-,n}|}\big(\int_{-\infty}^0\overline{\mu}^{\frac12}_{-;n,-}(x)dx)^2+\frac{p_{-,n}}{|\log(1-p_{-,n})|}\big(\int_0^\infty\overline{\mu}^{\frac12}_{+;n,-}(x)dx)^2\Big).\end{aligned}\tag{3.29}$$

Since $E_0^{(b)}T_a$ is an increasing function of $a\in(0,\infty)$ and a decreasing function of $a\in(-\infty,0)$, it follows from the construction that

$$\begin{cases}\int_0^\infty(E_0^{(b)}T_a)\mu_+(da)\le\int_0^\infty(E_0^{(b)}T_a)\mu_{+;n,+}(da),\\ \int_{\frac1n}^\infty(E_0^{(b)}T_a)\mu_{+;n,-}(da)\le\int_{\frac2n}^\infty(E_0^{(b)}T_a)\mu_+(da),\\ \int_{-\infty}^0(E_0^{(b)}T_a)\mu_-(da)\le\int_{-\infty}^0(E_0^{(b)}T_a)\mu_{-;n,+}(da),\\ \int_{-\infty}^{-\frac1n}(E_0^{(b)}T_a)\mu_{-;n,-}(da)\le\int_{-\infty}^{-\frac2n}(E_0^{(b)}T_a)\mu_-(da),\end{cases}\quad\text{for any drift } b.\tag{3.30}$$

The proof of part (i) now follows from (3.27)-(3.30). □

Proof of part (iii). In the proof of part (i) above, we proved the statement in part (iii) for two particular sequences $\{b_n\}_{n=1}^\infty$; namely for what we called $\{b_{+,n}\}_{n=1}^\infty$ and $\{b_{-,n}\}_{n=1}^\infty$. We leave it to the reader to do the routine analysis to show that the result holds more generally as stated in part (iii). □

4. Proofs of Theorems 1 and 3

Proof of Theorem 1. For the proof of part (i) of Theorem 2, we constructed the measures $\mu_{-,n} = (1-p_{-,n})\mu_{-;n,-} + p_{-,n}\,\mu_{+;n,-}$. For the proof here we consider the measures

$$\mu_{-,n,\mathrm{tr}} := (1-p_{-,n,\mathrm{tr}})\mu_{-;n,-,\mathrm{tr}} + p_{-,n,\mathrm{tr}}\,\mu_{+;n,-,\mathrm{tr}}, \tag{4.1}$$

where $\mu_{+;n,-,\mathrm{tr}}$ and $\mu_{-;n,-,\mathrm{tr}}$ are appropriately truncated versions of $\mu_{+;n,-}$ and $\mu_{-;n,-}$, and $p_{-,n,\mathrm{tr}}$ is chosen so that $\mu_{-,n,\mathrm{tr}}$ satisfies the square root balance condition.

The truncated version, $\mu_{+;n,-,\mathrm{tr}}$ of $\mu_{+;n,-}$, is defined as follows. Let $\mu_{+;n,-,\mathrm{tr}}$, restricted to $(0,n]$ coincide with $\mu_{+;n,-}$ on $(0,n]$, and let $\mu_{+;n,-,\mathrm{tr}}$, restricted to $(n,n+\frac{1}{n})$ be uniform with total mass equal to $\mu_+((n+\frac{1}{n},\infty))$. The truncated version, $\mu_{-;n,-,\mathrm{tr}}$ of $\mu_{-;n,-}$, is defined in the exact parallel fashion on $(-\infty,0)$.

The measures $\mu_{-;n,-\mathrm{tr}}$ and $\mu_{+;n,-\mathrm{tr}}$ converge weakly to $\mu_{-;n}$ and to $\mu_{+;n}$; thus,

$$\begin{aligned} &\lim_{n\to\infty}\int_{-\infty}^0 \overline{\mu}^{\frac{1}{2}}_{-;n,-,\mathrm{tr}}(x)dx = \int_{-\infty}^0 \overline{\mu}^{\frac{1}{2}}_{-}(x)dx = \infty;\\ &\lim_{n\to\infty}\int_0^\infty \overline{\mu}^{\frac{1}{2}}_{+;n,-,\mathrm{tr}}(x)dx = \int_0^\infty \overline{\mu}^{\frac{1}{2}}_{+}(x)dx = \infty. \end{aligned} \tag{4.2}$$

By part (ii) of Theorem 1, the measure $\mu_{-,n,\mathrm{tr}}$ is of the type for which the infimum is attained; let $b_{-,n,\mathrm{tr}}$ denote the corresponding drift for which the infimum is attained. Then

(4.3)

$$\inf_{b\in\mathcal{D}_{\mu_{-,n,\mathrm{tr}}}}\int_{\mathbb{R}}(E_0^{(b)}T_a)\,\mu_{-,n,\mathrm{tr}}(da)=\int_{\mathbb{R}}(E_0^{(b_{-,n,\mathrm{tr}})}T_a)\,\mu_{-,n,\mathrm{tr}}(da)=$$

$$\frac{2}{D}\Big(\frac{1-p_{-,n,\mathrm{tr}}}{|\log p_{-,n,\mathrm{tr}}|}\big(\int_{-\infty}^0\overline{\mu}^{\frac12}_{-;n,-,\mathrm{tr}}(x)dx)^2+\frac{p_{-,n,\mathrm{tr}}}{|\log(1-p_{-,n,\mathrm{tr}})|}\big(\int_0^\infty\overline{\mu}^{\frac12}_{+;n,-,\mathrm{tr}}(x)dx)^2\Big).$$

As in (3.30), we have

(4.4)

$$\begin{cases}\int_{\frac1n}^\infty(E_0^{(b)}T_a)\mu_{+;n,-,\mathrm{tr}}(da)\le\int_{\frac2n}^\infty(E_0^{(b)}T_a)\mu_+(da),\\ \int_{-\infty}^{-\frac1n}(E_0^{(b)}T_a)\mu_{-;n,-,\mathrm{tr}}(da)\le\int_{-\infty}^{-\frac2n}(E_0^{(b)}T_a)\mu_-(da),\end{cases}\quad\text{for any drift } b.$$

In our construction, we have no control over $p_{-,n,\mathrm{tr}}\in(0,1)$. Note that

$$\lim_{p\to1}\frac{1-p}{|\log p|}=\lim_{p\to0}\frac{p}{|\log(1-p)|}=1;\quad\lim_{p\to0}\frac{1-p}{|\log p|}=\lim_{p\to1}\frac{p}{|\log(1-p)|}=0.$$

Keeping this in mind, Theorem 1 now follows from (4.2)-(4.4). □

Proof of Theorem 3. By assumption, μ does not satisfy the square root balance condition. The proof of part (ii) of Theorem 2 revealed that in such a case, there are no critical points. Thus, the infimum is not attained, proving part (i). For part (ii), one substitutes the drift from (1.11) into (1.16) and performs a routine calculation. One obtains the expression in Remark 1 after the statement of the theorem. A bit of algebra converts this to the expression on the right hand side of (1.13). Thus, for this choice of drift b, $\int_{\mathbb{R}}(E_0^{(b)}T_a)\,\mu(da)$ is given by the right hand side of (1.13), and now the inequality in (1.13) follows from part (i). □

5. Proof of Propostion 2

We will prove the proposition for $a\in(0,A_+(\mu))$; a similar proof holds for $a\in(A_-(\mu),0)$. First assume that $A_-(\mu)>-\infty$ and that the diffusion cannot reach $A_-(\mu)$; that is $P_0(T_{A_-(\mu)}<\infty)=0$. This is equivalent to

$$\int_{A_-(\mu)}^0 dx\exp(-\frac{2}{D}\int_0^x b(y)dy)\int_x^0 dy\exp(\frac{2}{D}\int_0^y b(z)dz)=\infty. \tag{5.1}$$

(See [8, Theorem 5.1.5], where this is referred to as the diffusion not exploding.) Since (1.5) holds, (5.1) is equivalent to

$$\int_{A_-(\mu)}^{0} dx \exp(-\frac{2}{D}\int_0^x b(y)dy) = \infty. \tag{5.2}$$

Since $P_0(T_{A_-(\mu)} < \infty) = 0$, we have $E_0T_a = \lim_{n\to\infty} E_0T_a \wedge T_{A_-(\mu)+\frac{1}{n}}$. Define $u_n(x) = E_xT_a \wedge T_{A_-(\mu)+\frac{1}{n}}$, for $x \in [A_-(\mu) + \frac{1}{n}, a]$. By Ito's formula, u_n solves the differential equation

$$\begin{aligned}&\frac{D}{2}u_n'' + b(x)u_n' = -1,\ x \in (A_-(\mu) + \frac{1}{n}, a);\\ &u_n(A_-(\mu + \frac{1}{n})) = u_n(a) = 0.\end{aligned} \tag{5.3}$$

Writing the differential equation in the form

$$\frac{D}{2}\Big(\exp\big(\int_a^x \frac{2}{D}b(y)dy\big)u_n'(x)\Big)' = -\exp(\int_a^x \frac{2}{D}b(y)dy),$$

integrating twice and using the boundary conditions, we obtain

$$\begin{aligned}&u_n(x) = -u_n'(a)\int_a^x dy \exp(-\int_a^y \frac{2}{D}b(t)dt)-\\ &\frac{2}{D}\int_x^a dy\exp(-\int_a^y \frac{2}{D}b(t)dt)\int_y^a \exp(\int_a^z \frac{2}{D}b(t)dt),\end{aligned} \tag{5.4}$$

where

$$u_n'(a) = -\frac{\frac{2}{D}\int_{A_-(\mu)+\frac{1}{n}}^a dy\exp(-\int_a^y \frac{2}{D}b(t)dt)\int_y^a dz\exp(\int_a^z \frac{2}{D}b(t)dt)}{\int_{A_-(\mu)+\frac{1}{n}}^a dy \exp(-\int_a^y \frac{2}{D}b(t)dt)}.$$

By (5.2),

$$\lim_{n\to\infty} u_n'(a) = -\frac{2}{D}\int_{A_-(\mu)}^a dz\exp(\frac{2}{D}\int_a^z b(t)dt). \tag{5.5}$$

From (5.4) and (5.5) it follows that

$$E_0T_a = \lim_{n\to\infty} u_n(0) = \frac{2}{D}\int_0^a dx\exp(-\int_a^x \frac{2}{D}b(y)dy)\int_{A_-(\mu)}^x dz\exp(\int_a^z \frac{2}{D}b(t)dt).$$

The number a appearing twice as a lower limit of an integral on the right hand side above can be changed to any other value without changing the value of the right hand side. Changing a to 0 gives the formula for E_0T_a appearing in the statement of the proposition.

In the case that $A_-(\mu)=-\infty$, the assumption (1.5) ensures that (5.2) holds. Then (1.5) and (5.2) ensure that (5.1) holds. Thus, $P_0(T_{-\infty}<\infty)=0$, and consequently, $E_0T_a=\lim_{n\to\infty}E_0T_a\wedge T_{-n}$. One now proceeds as in the case treated above, replacing $A_-(\mu)+\frac{1}{n}$ by $-n$.

In the case that $A_-(\mu)>-\infty$ and that (5.1) does not hold, one has $P_0(T_{A_-(\mu)}<\infty)>0$. In this case, we are considering the diffusion with reflection at $A_-(\mu)$. Let $u(x)=E_xT_a$. Then by Ito's formula, u satisfies

$$\begin{aligned}&\frac{D}{2}u''+b(x)u'=-1,\ x\in(A_-(\mu),a);\\&u'(A_-(\mu))=u(a)=0.\end{aligned}\tag{5.6}$$

Solving this similarly but more simply than we solved the above equations, we obtain the formula for E_0T_a appearing in the statement of the proposition. □

6. Proofs of Propositions 3 and 4

Proof of Proposition 3. We first show that the set $\mathcal{D}_\mu$ is convex. Let $b,\beta\in\mathcal{D}_\mu$. We need to show that $(1-t)b+t\beta\in\mathcal{D}_\mu$, for $t\in(0,1)$; that is, that $(1-t)b+t\beta$ satisfies (1.4) and (1.5). Now (1.4) holds trivially. For (1.5), we use Hölder's inequality with $p=\frac{1}{1-t}$ and $q=\frac{1}{t}$ to obtain

$$\int_{A_-(\mu)}^{A_+(\mu)}dx\exp(\frac{2}{D}\int_0^x\big((1-t)b+t\beta\big)(y)dy)\le$$
$$\Big(\int_{A_-(\mu)}^{A_+(\mu)}dx\exp(\frac{2}{D}\int_0^x b(y)dy)\Big)^{1-t}\Big(\int_{A_-(\mu)}^{A_+(\mu)}dx\exp(\frac{2}{D}\int_0^x\beta(y)dy)\Big)^t<\infty.$$

We now prove that G_1 is convex. Recall the definition of G_1 from (1.18). We will show that

$$H(b):=(1-p)\int_{A_-(\mu)}^0 da\,\overline{\mu}_-(a)\Big[\exp(-\int_0^a\frac{2}{D}b(y)dy)\int_a^{A_+(\mu)}dz\exp(\int_0^z\frac{2}{D}b(s)ds)\Big]$$

is convex. The same proof works for the second term in G_1. We rewrite H as

$$H(b)=(1-p)\int_{A_-(\mu)}^0 da\,\overline{\mu}_-(a)\int_a^{A_+(\mu)}dz\exp(\int_a^z\frac{2}{D}b(s)ds).\tag{6.1}$$

It follows by the convexity of the function e^x on all of $\mathbb{R}$ that

$$\begin{aligned}(6.2)\qquad &\exp\Big(\int_a^z \frac{2}{D}\big((1-t)b+t\beta\big)(s)ds\Big) \le (1-t)\exp\Big(\int_a^z \frac{2}{D}b(s)ds\Big)+\\ &t\exp\Big(\int_a^z \frac{2}{D}\beta(s)ds\Big),\ 0\le t\le 1.\end{aligned}$$

Substituting (6.2) into (6.1) gives $H\big((1-t)b+t\beta\big)\le(1-t)H(b)+tH(\beta)$, for $0\le t\le 1$, proving the convexity. $\square$

Proof of Proposition 4. Since we are assuming that μ has an atom at $A_-(\mu)$ and at $A_+(\mu)$, if follows from the definition of $\mathcal{D}(\mu)$ that any $b\in\mathcal{D}(\mu)$ is bounded on $(A_-(\mu),A_+(\mu))$ (as noted in the remark following (1.4)). Define

$$\hat F_\epsilon(a)=\int_{A_-(\mu)}^a dz\exp\Big(\int_0^z\frac{2}{D}\big((1-\epsilon)b_0+\epsilon b\big)(t)dt\Big)\quad\text{and}\quad \hat f_\epsilon(a)=\hat F'_\epsilon(a).$$

Recalling G_1 from (1.18), and recalling that G_2 from (1.20) has been defined for positive multiples of distribution functions, we can write

$$\begin{aligned}(6.3)\qquad &G_1\big((1-\epsilon)b_0+\epsilon b\big)\big)=(1-p)\int_{A_-(\mu)}^0\overline{\mu}_-(a)\frac{\hat F_\epsilon(A_+(\mu))-\hat F_\epsilon(a)}{\hat f_\epsilon(a)}+\\ &p\int_0^{A_+(\mu)}\overline{\mu}_+(a)\frac{\hat F_\epsilon(a)}{\hat f_\epsilon(a)}=G_2(\hat F_\epsilon).\end{aligned}$$

Also define

(6.4)

$$\hat Q(a)=\lim_{\epsilon\to0^+}\frac{\hat F_\epsilon(a)-\hat F_0(a)}{\epsilon}=\int_{A_-(\mu)}^a dz\Big(\int_0^z\frac{2}{D}(b-b_0)(t)dt\Big)\exp\Big(\int_0^z\frac{2}{D}b_0(t)dt\Big);$$

$$\hat q(a)=\hat Q'(a).$$

The second equality in the first line of (6.4) follows from the bounded convergence theorem and the assumptions in the statement of the proposition.

Using (6.3) and (6.4) we have, similar to (3.1),

$$
\begin{aligned}
&L'(0^+) = \lim_{\epsilon\to 0^+}\frac{L_1(\epsilon)-L_1(0)}{\epsilon} = \\
&\lim_{\epsilon\to 0^+}\frac{G\big((1-\epsilon)b_0+\epsilon b\big)-G_1(b_0)}{\epsilon} = \lim_{\epsilon\to 0^+}\frac{G_2(\hat F_\epsilon)-G_2(\hat F_0)}{\epsilon} = \\
&(1-p)\int_{A_-(\mu)}^0 \overline{\mu}_-(a)\Big(\frac{\hat Q(A_+(\mu))-\hat Q(a)}{f_0(a)}-\frac{(1-F_0(a))\hat q(a)}{f_0^2(a)}\Big)da+ \\
&p\int_0^{A_+(\mu)}\overline{\mu}_+(a)\Big(\frac{\hat Q(a)}{f_0(a)}-\frac{F_0(a)\hat q(a)}{f_0^2(a)}\Big)da.
\end{aligned}
\tag{6.5}
$$

The second equality above follows from the bounded convergence theorem and the assumptions in the statement of the proposition.

We need to consider the cases $\hat Q(A_+(\mu))\neq 0$ and $\hat Q(A_+(\mu))=0$ separately. First consider the case $\hat Q(A_+(\mu))\neq 0$. Define $\bar Q(a)=\frac{\hat Q(a)}{\hat Q(A_+(\mu))}$ and $\bar q(a)=\bar Q'(a)$. Then the right hand side of (6.5) will be equal to 0 if and only if

$$
\begin{aligned}
&(1-p)\int_{A_-(\mu)}^0 \overline{\mu}_-(a)\Big(\frac{1-\bar Q(a)}{f_0(a)}-\frac{(1-F_0(a))\bar q(a)}{f_0^2(a)}\Big)da+ \\
&p\int_0^{A_+(\mu)}\overline{\mu}_+(a)\Big(\frac{\bar Q(a)}{f_0(a)}-\frac{F_0(a)\bar q(a)}{f_0^2(a)}\Big)da=0.
\end{aligned}
\tag{6.6}
$$

Recall that since F_0 is the critical point of G_2, (3.1) holds with F_0 and f_0 substituted for F and f. The only difference between (3.1), with F_0 and f_0 substituted for F and f, and (6.6) is that $\hat Q$ and $\hat q$ appear in (6.6) while Q and q appear in (3.1), where Q is a distribution function with compactly supported density q. However, the analysis from (3.1) to (3.5) goes through just the same for $\bar Q$, since $\bar Q(A_+(\mu))=1$. (Neither the monotonicity of Q nor the compact support of q was used there; only the fact that $Q(A_+(\mu))=1$.) Thus, the right hand side of (6.5) is equal to 0, proving the proposition.

Now consider the case $\hat Q(A_+(\mu))=0$. Because $\hat Q(A_+(\mu))=0$, whereas in (3.1) one had $Q(A_+(\mu))=1$, the analysis that showed that the left hand side of (3.1) is equal to the left hand side of (3.4), when applied to the last two lines of (6.5), shows that

$$
\begin{aligned}
&(1-p)\int_{A_-(\mu)}^{0}\overline{\mu}_-(a)\Big(\frac{\hat{Q}(A_+(\mu))-\hat{Q}(a)}{f_0(a)}-\frac{(1-F_0(a))\hat{q}(a)}{f_0^2(a)}\Big)da+\\
&p\int_0^{A_+(\mu)}\overline{\mu}_+(a)\Big(\frac{\hat{Q}(a)}{f_0(a)}-\frac{F_0(a)\hat{q}(a)}{f_0^2(a)}\Big)da=\\
&(1-p)\int_{A_-(\mu)}^{0}\hat{q}(a)\Big[-\int_a^0\frac{\overline{\mu}_-(x)}{f_0(x)}dx-\frac{\overline{\mu}_-(a)(1-F_0(a))}{f_0^2(a)}\Big]da+\\
&p\int_0^{A_+(\mu)}\hat{q}(a)\Big[-\int_0^a\frac{\overline{\mu}_+(x)}{f_0(x)}dx-\frac{\overline{\mu}_+(a)F_0(a)}{f_0^2(a)}\Big].
\end{aligned}
\tag{6.7}
$$

(The right hand side of (6.7) corresponds to the first two lines of (3.4). The two terms on the third line of (3.4) do not appear now because $\hat{Q}(A_+(\mu))=0$.) Because F_0 is critical, it follows from (3.5) that

$$
\begin{aligned}
&(1-p)\Big[\int_a^0\frac{\overline{\mu}_-(x)}{f_0(x)}dx+\frac{\overline{\mu}_-(a)(1-F_0(a))}{f_0^2(a)}\Big]=C_1,\ \ a\in(A_-(\mu),0);\\
&p\Big[-\int_0^a\frac{\overline{\mu}_+(x)}{f_0(x)}dx-\frac{\overline{\mu}_+(a)F_0(a)}{f_0^2(a)}\Big]=C_1,\quad a\in(0,A_+(\mu)),
\end{aligned}
\tag{6.8}
$$

where $C_1=(1-p)\int_{A_-(\mu)}^0\frac{\overline{\mu}_-(a)}{f_0(a)}da+p\int_0^{A_+(\mu)}\frac{\overline{\mu}_+(a)}{f_0(a)}da$. Substituting (6.8) into the right hand side of (6.7), we conclude that the right hand side of (6.7) is equal to $C_1\int_{A_-(\mu)}^0\hat{q}(a)da+C_1\int_0^{A_+(\mu)}\hat{q}(a)da=C_1\int_{A_-(\mu)}^{A_+(\mu)}\hat{q}(a)da=C_1\hat{Q}(A_+(\mu))=0$. Thus, the left hand side of (6.7), which is the right hand side of (6.5), is equal to 0. □

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