POSITIVE SOLUTIONS OF REACTION DIFFUSION EQUATIONS WITH SUPER-LINEAR ABSORPTION: UNIVERSAL BOUNDS, UNIQUENESS FOR THE CAUCHY PROBLEM, BOUNDEDNESS OF STATIONARY SOLUTIONS

Ross G. Pinsky

Technion-Israel Institute of Technology
Department of Mathematics
Haifa, 32000, Israel
e-mail: pinsky@math.technion.ac.il

Abstract.

Consider classical solutions $u \in C^2(R^n \times (0,\infty)) \cap C(R^n \times [0,\infty))$ to the parabolic reaction diffusion equation

$$\begin{aligned}&u_t = Lu + f(x,u),\ (x,t) \in R^n \times (0,\infty);\\&u(x,0) = g(x) \geq 0,\ x \in R^n;\\&u \geq 0,\end{aligned}$$

where

$$L = \sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i}$$

is a non-degenerate elliptic operator, $g \in C(R^n)$ and the reaction term f converges to $-\infty$ at a super-linear rate as $u \to \infty$. The first result in this paper is a parabolic Osserman-Keller type estimate. We give a sharp minimal growth condition on f, independent of L, in order that there exist a universal, a priori upper bound for all solutions to the above Cauchy problem—that is, in order that there exist a finite function $M(x,t)$ on $R^n \times (0,\infty)$ such that $u(x,t) \leq M(x,t)$, for all solutions to the Cauchy problem. Assuming now in addition that $f(x,0) = 0$, so that $u \equiv 0$ is a solution to the Cauchy problem, we show that under a similar growth condition, an intimate relationship exists between two seemingly disparate phenomena—namely, uniqueness for the Cauchy problem with initial data $g = 0$ and the nonexistence of unbounded, stationary solutions to the corresponding elliptic problem. We also give a generic sufficient condition guaranteeing uniqueness for the Cauchy problem.

1991 *Mathematics Subject Classification.* 35K15, 35K55.

Key words and phrases. semilinear parabolic and elliptic equations, uniqueness for the Cauchy problem, reaction-diffusion equations, universal bounds, stationary solutions.

The research was supported by the Fund for the Promotion of Research at the Technion

Typeset by $\mathcal{AMS}$-TEX

1. Introduction and statement of results. Consider classical solutions $u \in C^2(R^n \times (0,\infty)) \cap C(R^n \times [0,\infty))$ to the parabolic reaction diffusion equation

$$\begin{aligned} &u_t = Lu + f(x,u),\ (x,t) \in R^n \times (0,\infty);\\ &u(x,0) = g(x) \geq 0,\ x \in R^n;\\ &u \geq 0, \end{aligned} \tag{1.1}$$

where

$$L = \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i},$$

with $a_{i,j}, b_i \in C^\alpha(R^n)$, $\alpha \in (0,1]$, and $\{a_{i,j}\}$ strictly elliptic; that is, $\sum_{i,j=1}^{n} a_{i,j}(x)\nu_i\nu_j > 0$, for all $x \in R^n$ and $\nu \in R^n - \{0\}$. We assume that $g \in C(R^n)$. We require that the reaction term f be locally Lipschitz in x and in u and converge to $-\infty$ at a super-linear rate as $u \to \infty$, for each $x \in R^n$. This latter requirement will be made more precise below.

Our first result is a sharp minimal growth condition on f, independent of L, in order that there exist a universal, a priori upper bound for all solutions to the Cauchy problem (1.1)—that is, in order that there exist a finite function $M(x,t)$ on $R^n \times (0,\infty)$ such that $u(x,t) \leq M(x,t)$, for all solutions to (1.1). This result may be thought of as a parabolic counterpart to the celebrated Osserman-Keller estimate for elliptic equations. (See Remark 1 following Example 1.) An estimate of the above type guarantees the existence of a solution to (1.1) with initial condition $g \equiv \infty$, as well as a solution to (1.1) in a ball (instead of R^n) with infinite Dirichlet data on the boundary.

After this result, we will always assume that $f(x,0) = 0$, so that $u \equiv 0$ is a solution to (1.1). We show that under a growth condition similar to the above one, an intimate relationship exists between two seemingly disparate phenomena—namely, uniqueness for the Cauchy problem (1.1) with initial data $g = 0$ and the nonexistence of unbounded, stationary solutions (or subsolutions) to the corresponding elliptic problem. We also give a generic sufficient condition guaranteeing uniqueness for the Cauchy problem.

For $R>0$, define

$$F_R(u) = \sup_{|x|\le R} f(x,u).$$

We will always assume that

$$\text{(F-1)} \qquad \sup_{u>0} F_R(u) < \infty, \text{ for all } R>0.$$

Theorem 1 and Example 1 below show that the following assumption on F_R is a sharp condition for the existence of such a universal a priori upper bound on all solutions to (1.1). Let $\log^{(n)} x$ denote the n-th iterate of $\log x$ so that $\log^{(1)} x = \log x, \log^{(2)} x = \log\log x$, etc.

$$\text{(F-2)} \qquad \begin{gathered}\text{For each } R>0, \text{ there exist an } m\ge 0 \text{ and an } \epsilon>0 \text{ such that}\\ \lim_{u\to\infty} \frac{F_R(u)}{u(\prod_{i=1}^m \log^{(i)} u)^2(\log^{(m+1)} u)^{2+\epsilon}} = -\infty,\end{gathered}$$

where by convention, $\prod_{i=1}^0 \log^{(i)} = 1$.

Remark. F_R will satisfy (F-1) and (F-2) if, for instance, $f(x,u) = V(x)u - \gamma(x)u^p$, for $p>1$, or if f is appropriately defined for small u and satisfies $f(x,u) = V(x)u - \gamma(x)u(\prod_{i=1}^m \log^{(i)} u)^2(\log^{(m+1)} u)^{2+\epsilon}$, for large u, where $V(x)$ is bounded on compacts and γ is positive and bounded away from 0 on compacts.

Theorem 1. *Assume that (F-1) and (F-2) hold. Then there exists a continuous function $M(x,t)$ on $R^n\times(0,\infty)$ such that every solution u to the Cauchy problem (1.1) satisfies $u(x,t)\le M(x,t)$, for all $x\in R^n$ and all $t\ge 0$.*

The following example shows that condition (F-2) is sharp.

Example 1. Let $L=\frac{d^2}{dx^2}$ and $f(x,u) = -u\left((\log u)^2+\log u\right)$, for $u\ge 1$. Then for each $l\in R$, $u_l(x) = \exp(\exp(x+l))$ solves (1.1) (as a stationary solution). Since $\lim_{l\to\infty} u_l(x)=\infty$, there is no universal a priori upper bound for all non-negative solutions of (1.1) for this choice of f. Alternatively, if we let $f(x,u) = -u\left((\log u)^2(\log\log u)^2 + \log u \log\log u\right)$, for $u\ge e$, then $u_l(x) = \exp(\exp(\exp(x+l)))$ solves (1.1). More generally, letting $u_l(x)$ denote the $(m+1)$-th iterate of the exponent function with argument $x + l$, then $Lu_l + f(u_l) = 0$, where the function f satisfies $f(u) < -u(\prod_{i=1}^{m} \log^{(i)} u)^2$, for large u

Remark 1. Under the condition

$$\int^{\infty} (\int_0^x g(z)dz)^{-\frac{1}{2}} < \infty, \tag{1.2}$$

the Osserman-Keller bound gives a precise universal, a priori upper bound on every solution u to $Lu + f(u) = 0$ in a domain $D \subset R^n$, where $-f(s) \geq g(s)$, for large s, and $g > 0$ is monotone nondecreasing. See [8] and [12] for the case $L = \Delta$, and see [1] for the case of general L. Condition (F-2) may be thought of as the "poor man's" version of the Osserman-Keller integral condition, which leads us to state the following conjecture:

Conjecture. Theorem 1 holds under the Osserman-Keller integral condition (1.2).

Remark 2. Consider the ordinary differential equation

$$v' = f(v),\ v(0) = c \geq 0, \tag{1.3}$$

where f is a locally Lipschitz function satisfying $f(0) = 0$ and $\lim_{u\to\infty} f(u) = -\infty$. The unique solution v_c to (1.3) satisfies $v_c \geq 0$ and is increasing as a function of its initial condition c. It is well-known and straight forward to show that $\lim_{c\to\infty} v_c(t) = \infty$, if $\int^{\infty} \frac{1}{-f(u)} du = \infty$, while $v_\infty(t) \equiv \lim_{c\to\infty} u_c(t) < \infty$, for $t > 0$, if $\int^{\infty} \frac{1}{-f(u)} du < \infty$. Thus, if the above integral is finite, v_∞ serves as a universal a priori upper bound for all solutions to (1.3), while if the above integral is infinite, there is no such finite function. In particular then, for the ordinary differential equation (1.3), a universal a priori upper bound on solutions exists when $f(u) = -u(\prod_{i=1}^{m} \log^{(i)} u)(\log^{(m+1)} u)^{1+\epsilon}$, but not when $f(u) = -u(\prod_{i=1}^{m} \log^{(i)} u)$. Comparing this with Theorem 1 and Example 1, one sees that *the introduction of spatial diffusion and drift slightly increases the minimal super-linearity threshold for the existence of a universal a priori upper bound.*

Define now

$$F(u) = \sup_{x\in R^n} f(x,u),$$

and consider the spatially uniform versions of conditions (F-1) and (F-2):

$(F\text{-}1')$ $$\sup_{u>0} F(u) < \infty;$$

$(F\text{-}2')$
$$\lim_{u\to\infty} \frac{F(u)}{u(\prod_{i=1}^m \log^{(i)} u)^2(\log^{(m+1)} u)^{2+\epsilon}} = -\infty, \text{for some } m \geq 0 \text{ and some } \epsilon > 0.$$

Consider also the following condition:

(F-3) $$f(x,0) = 0 \text{ and } F(u) \text{ is Lipschitz at } u = 0.$$

Remark. F will satisfy (F-1′), (F-2′) and (F-3) if, for instance, f is as in the remark following (F-2) with V bounded and γ positive and bounded away from 0.

The above conditions turn out to be critical for certain other important phenomena. Consider the associated elliptic equation corresponding to stationary solutions of (1.1):

(1.4) $$LW + f(x,W) = 0,\ x \in\ R^n; \qquad W \geq 0.$$

A function w will be called a stationary subsolution if

(1.5) $$Lw + f(x,w) \geq 0,\ x \in\ R^n; \qquad w \geq 0.$$

We will sometimes need one of the following two technical conditions on f:

(F-4a)
$G(u) \equiv \sup_{x\in R^n} \sup_{v\geq u}(f(x,v) - f(x,v-u))$ is locally Lipschitz, is negative for large u and satisfies $\int^\infty \frac{1}{-G(u)} du < \infty$.

(F-4b) $H(u) \equiv \sup_{x\in R^n} \sup_{v\geq 0}(f(x,u+v) - f(x,v))$ satisfies (F-1′) and (F-2′).

Remark. Note that if $f(x, \cdot)$ is concave for each $x \in R^n$ and F satisfies (F-1′), (F-2′) and (F-3), then both (F-4a) and (F-4b) hold. Indeed, by concavity, the supremum over v is attained in (F-4a) at $v = u$ and in (F-4b) at $v = 0$, giving $G(u) = H(u) = F(u) - F(0) = F(u)$. Furthermore, the integral condition in (F-4a) holds for any function satisfying (F-2′).

Theorem 2. *Assume that (F-1′), (F-2′), (F-3) and (F-4a) hold.*
i. If the trivial solution $u = 0$ is the only solution to the Cauchy problem (1.1) with initial data $g = 0$, then all stationary subsolutions w to (1.5) are bounded. More specifically,

$$w(x) \le c_0, \text{ for all } x \in R^n,$$

where c_0 is the largest root of the equation $G(u) = 0$, and G is as in (F-4a).
ii. Assume that $G(u) < 0$, for $u > 0$, where G is as in part (i). Then the trivial solution $u = 0$ is the only solution to the Cauchy problem (1.1) with initial data $g = 0$ if and only if there are no nontrivial solutions to the stationary equation (1.4) (or equivalently, if and only if there are no nontrivial stationary subsolutions to (1.5)).
In both parts of the theorem, if $f(x, \cdot)$ is concave for each x, then (F-4a) is superfluous and $G = F$.

Remark. Note that if under the conditions in Theorem 2-i, one can exhibit an unbounded stationary subsolution to (1.5), then Theorem 2-i guarantees the existence of a nontrivial solution to the Cauchy problem (1.1) with 0 initial data. Examples 2 and 3 below are applications of this, as is Theorem 5-i at the end of this section. Similarly, in the case that $G(u) < 0$, for $u > 0$, if one can exhibit a nontrivial stationary subsolution to (1.5), then Theorem 2-ii guarantees the existence of a nontrivial solution to the Cauchy problem (1.1) with 0 initial data. An application of this is given in the final sentence of the second paragraph after the remark following the open problem that appears later in this section. These examples illustrate the utility of Theorem 2—it is much easier to construct appropriate stationary subsolutions to (1.5) than to construct a nontrivial solution to (1.1) with initial data $g = 0$. By Theorem 2, with appropriate conditions on f, the existence of the latter is guaranteed by the existence of the former.

The next result gives conditions for uniqueness of the Cauchy problem (1.1) with initial data $g = 0$ and also for general initial data. Consider the following growth assumption on the coefficients of L:

$$\text{(L-1)} \qquad \begin{aligned} &\sum_{i,j=1}^{n} a_{ij}(x)\nu_i\nu_j \leq C|\nu|^2(1+|x|^2) \\ &|b(x)| \leq C(1+|x|), \end{aligned}$$

for some $C > 0$.

Theorem 3. *If (F-1′), (F-2′) (F-3) and (L-1) hold, then the trivial solution $u \equiv 0$ is the only solution to the Cauchy problem (1.1) with initial data $g = 0$. If in addition, (F-4b) holds, then there is a unique solution to the Cauchy problem (1.1) for each $g \in C(R^n)$.*

Remark 1. We emphasize that in the context of this paper, uniqueness for the Cauchy problem (1.1) means uniqueness with regard to *all* classical, nonnegative solutions. If one works only with, say, mild solutions, then the situation can be quite different. For example, there is a unique mild solution for $u_t = \Delta u - \gamma(x)u^p$, for $p > 1$ and bounded $\gamma \geq 0$ [13]; yet if γ decays sufficiently rapidly, uniqueness fails in the sense of all classical solutions. For details, see the next to the last paragraph before Theorem 5. This example also shows that one can not replace (F-2′) by (F-2) in Theorem 3.

Remark 2. The first uniqueness result of the type considered in Theorem 3 goes back to [4], where the case $L = \Delta$ and $f(u) = -u^p$ was studied. In the case of general L and nonlinearity of the form $f(u) = V(x)u - \gamma(x)u^p, p > 1$, see [6].

As an immediate corollary of Theorems 2 and 3, we obtain the following theorem.

Theorem 4. *Assume that (F-1′), (F-2′), (F-3), (F-4a) and (L-1) hold.*

i. All stationary subsolutions to (1.5) are bounded.

ii. Assume in addition that the function G from condition (F-4a) satisfies $G(u) < 0$, *for* $u > 0$ *(which will occur in particular if* $f(x,\cdot)$ *is concave for each* x *and* $F(u) < 0$, *for* $u > 0$*). Then there are no nontrivial stationary subsolutions to (1.5).*

Remark. Example 1 above shows that condition (F-2′) is sharp for Theorem 4.

We elaborate now on Theorems 3 and 4. We begin with two examples which demonstrate that condition (L-1) is sharp for both of these theorems, and demonstrate the utility of Theorem 2 as well.

Example 2. When $L = (1+x^2)^{1+\epsilon}\frac{d^2}{dx^2}$ and $f(x,u) = -2u^{1+\epsilon}$, for some $\epsilon > 0$, then (1.4) possesses the unbounded solution $u(x) = 1 + x^2$. By Theorem 2, it then follows that there exists a nontrivial solution to (1.1) with initial data $g = 0$.

Example 3. When $L = \frac{d^2}{dx^2} + |x|^{1+2\epsilon}\operatorname{sgn}(x)\frac{d}{dx}$ and $f(x,u) = -u^{1+\epsilon}$, for some $\epsilon > 0$, then (1.5) possesses the unbounded subsolution $u(x) = \lambda x^2$, for $\lambda > 0$ sufficiently small. By Theorem 2, it then follows that there exists a nontrivial solution to (1.1) with initial data $g = 0$.

Theorems 2-4 and Examples 2-3 suggest that under conditions (F-1′), (F-2′), (F-3) and the technical condition (F-4a), which is always satisfied if $f(x,\cdot)$ is concave, there may well be an equivalence between uniqueness for the Cauchy problem (1.1) with initial data $g = 0$ and nonexistence of unbounded solutions to the stationary equation (1.4). (Note that this equivalence (without the unboundedness requirement) has been established in Theorem 2-ii under the additional condition $G(u) < 0$, for $u > 0$.) We state this formally:

Open Problem. *Under conditions (F-1′), (F-2′) and (F-3), and perhaps some other technical conditions, is there an equivalence between uniqueness of the Cauchy* *problem for (1.1) with initial data $g = 0$ and nonexistence of unbounded solutions to the stationary equation (1.4)?*

Remark. The growth condition (F-2′) seems to be necessary in order for the equivalence alluded to above to have a chance of occurring. Indeed, on the one hand, considering that uniqueness holds for positive solutions to the linear Cauchy problem $u_t = \Delta u - u$ and, by Theorem 3, also for the Cauchy problem $u_t = \Delta u + f(u)$, when f approaches $-\infty$ sufficiently fast so as to satisfy (F-2′), it seems extremely likely that uniqueness also holds for $u_t = \Delta u + f(u)$ when f approaches $-\infty$ at a super-linear rate that does not satisfy (F-2′). But on the other hand, Example 1 shows that there are unbounded stationary solutions to $u'' + f(u) = 0$ for certain super-linear f not satisfying (F-2′).

Regarding Theorem 4-ii, it follows in particular that if $f(x, \cdot)$ is concave for each x, $F(u) < 0$, for $u > 0$, F satisfies conditions (F-2′) and (F-3), and the operator L satisfies condition (L-1), then there are no nontrivial solutions to (1.4). In general, the question of existence/nonexistence for (1.4) is delicate and can hinge greatly on the particular form of L and f. One generic result in the literature concerns the case that $f(\cdot, u)$ and the coefficients of L are periodic in x. Assume that for some $M_0 > 0$, $f(x, u) \leq 0$, for all $x \in R^n$ and all $u \geq M_0$. Let λ_0 denote the principal eigenvalue for the operator $L + \frac{\partial f}{\partial u}(x, 0)$ with periodic boundary conditions. If $\lambda_0 > 0$, then (1.4) possesses a nontrivial periodic solution, while if $\lambda_0 \leq 0$ and $\frac{f(x,u)}{u}$ is decreasing in u for each $x \in R^n$, then (1.4) does not possess a nontrivial *bounded* solution [3].

Consider now the well-studied case $L = \alpha(x)\Delta$ and $f(x, u) = -u^p$, with $p > 1$. If $n \geq 2$, then (1.4) possesses a nontrivial solution if $\lim_{|x|\to\infty} \frac{\alpha(x)}{(1+|x|)^{2+\epsilon}} > 0$, for some $\epsilon > 0$, and does not possess a nontrivial solution if $\lim_{|x|\to\infty} \frac{\alpha(x)}{(1+|x|)^2} < \infty$. For $n = 1$, the same result holds with the exponent 2 replaced by $1 + p$. For $n \geq 3$, this result goes back to [9] and [11], and it is shown in [9] that in the case of existence there are in fact an infinite number of bounded solutions. The n-dimensional analog of Example 2 above shows that there is also an unbounded solution. For $n = 1, 2$, the above result were proven in [5] and later apppeared with a different proof in [6] (which also re-derives the result for $n \geq 3$). Note that by Theorem 2, nonexistence of nontrivial solutions to (1.4) continues to hold for α in the above nonexistence range when the nonlinearity $-u^p$ is replaced by $f(x, u) = -u(\log(u + 1))^{2+\epsilon}$ or $f(x, u) = -u(\log(u + 1))^2(\log\log(u + e))^{2+\epsilon}$, etc., for some $\epsilon > 0$. Also, note that when α is in the above *existence* range, then by Theorem 2, there is a nontrivial solution to the Cauchy problem $u_t = \alpha\Delta u - u^p$ with initial condition $g = 0$.

The combination of conditions (L-1) and (F-2′) in Theorem 3 raises an interesting question. We recall that for the linear equation $u_t = Lu - u$, uniqueness is known to hold when b satisfies the condition in (L-1) and when a satisfies a two-sided bound of the form $c(1 + |x|^\gamma)|\nu|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\nu_i\nu_j \leq C(1 + |x|^\gamma)|\nu|^2$, for some $\gamma \in [0, 2]$. It is not known whether uniqueness holds for the linear problem under condition (L-1)! (For more about uniqueness in the linear case, see [7] and references therein.) It would be interesting to understand the interplay between very weak super-linearities (that is, those for which (F-2′) does not hold) and the condition required on the coefficients of the elliptic operator L in order to insure uniqueness.

We wish to emphasize an important point with regard to the connection between non-existence of unbounded solutions to (1.4) and uniqueness for the Cauchy problem (1.1) with initial condition $g = 0$. If one takes a scalar function $\alpha(x) > 0$ and replaces L and f by αL and αf respectively, then of course (1.4) remains unchanged. However, making the same change in the parabolic equation (1.1) can affect the question of uniqueness. Indeed, in [6], it was shown that if $L = \Delta$ and $f(x, u) = \frac{1}{\alpha}u^p$ with $p > 1$, then there exists a nontrivial solution to (1.1) with initial data $g = 0$ if $\alpha(x) \geq C\exp(|x|^{2+\epsilon})$, for some $\epsilon > 0$ and $C > 0$, and there doesn't exist such a solution if $\alpha(x) \leq C\exp(|x|^2)$, for some $C > 0$. (This is the example alluded to in Remark 1 after Theorem 3.) On the other hand, for $L = \alpha\Delta$ and $f(x, u) = u^p$, it follows from Theorem 3 and from the final sentence in the second paragraph after the remark following the open question that the existence of a nontrivial solution to (1.1) with initial data $g = 0$ depends on whether $\alpha(x) \geq C(1 + |x|)^{l+\epsilon}$ or $\alpha(x) \leq C(1 + |x|)^l$, where $l = 2$, if $n \geq 2$, and $l = 1 + p$, if $n = 1$. What allows for the connection between uniqueness for the Cauchy problem (1.1) with initial condition $g = 0$ and nonexistence of solutions to the stationary equation (1.4) is the assumption (F-2′) on $F(u) = \sup_{x\in R^n} f(x, u)$. In the above example, this assumption requires one to consider $L = \alpha\Delta$ and $f(x, u) = u^p$, rather than $L = \Delta$ and $f(x, u) = \frac{1}{\alpha}u^p$.

We conclude this section with another application of Theorem 2, this time when the domain is the punctured space. Consider the following semilinear equation in the punctured space $R^d - \{0\}$:

$$
\begin{aligned}
&u_t = \Delta u - u^p \text{ in } (R^d - \{0\}) \times (0, \infty);\\
&u(x, 0) = 0 \text{ in } R^d - \{0\};\\
&u \geq 0 \text{ in } (R^d - \{0\}) \times [0, \infty).
\end{aligned}
\tag{1.6}
$$

Theorem 5. *Let $p > 1$ and $d \geq 2$.*
i. If $d < \frac{2p}{p-1}$, then there exists a nontrivial solution to (1.6).
ii. If $d \geq \frac{2p}{p-1}$, then there is no nontrivial solution to (1.6).

Remark. Theorem 5 actually follows from a much more general result in [2], with a much more sophisticated proof.

The proof of Theorem 5-i is a simple consequence of Theorem 2.

Proof of Theorem 5-i. Since the problem is radially symmetric, it suffices to show that uniqueness fails for the radially symmetric equation

$$
\begin{aligned}
&u_t = u_{rr} + \frac{d-1}{r}u_r - u^p,\ r \in (0, \infty),\ t > 0;\\
&u(r, 0) = 0,\ r \in (0, \infty);\\
&u \geq 0,\ r \in (0, \infty),\ t \geq 0.
\end{aligned}
\tag{1.7}
$$

By assumption, we have $d < \frac{2p}{p-1}$. Thus, the function $W(r) = Cr^{-\frac{2}{p-1}}$, where $C^{p-1} = \frac{2}{p-1}(\frac{2p}{p-1} - d)$, is a positive, stationary solution of the parabolic equation

$u_t = u_{rr} + \frac{d-1}{r} u_r - u^p$ in $(0,\infty)$. Now Theorem 2 was stated for equations with domain R^d, $d \geq 1$, whereas the domain here is $(0,\infty)$. One can check that the proof also holds in a half space, or alternatively, one can make the change of variables $z = \frac{1}{x} - x$, which converts the problem to all of R. Thus, by Theorem 2-ii, the fact that there exists a nontrivial, positive, stationary solution guarantees that uniqueness does not hold for (1.7) $\square$

For completeness, in section five we give a direct proof of Theorem 5-(ii), using the technique of super solutions. The proof is rather delicate. Theorems 1 and 3 are also proved by constructing appropriate super solutions, the proof of Theorem 3 being the much more delicate of the two. The proofs of Theorems 1 and 3 are given in sections two and three respectively. The proof of Theorem 2 uses a gamut of techniques and is given in section four.

2. Proof of Theorem 1. We begin with a standard maximum principle.

Proposition 1. *Let $D \subset R^n$ be a bounded domain and let $0 \leq u_1, u_2 \in C^{2,1}(D \times (0,\infty)) \cap C(\bar{D} \times [0,\infty))$ satisfy*

$$Lu_1 + f(x,u_1) - \frac{\partial u_1}{\partial t} \leq Lu_2 + f(x,u_2) - \frac{\partial u_2}{\partial t}, \ \textit{for } (x,t) \in D \times (0,\infty),$$

$$u_1(x,t) \geq u_2(x,t), \ \textit{for } (x,t) \in \partial D \times (0,\infty)$$

and

$$u_1(x,0) \geq u_2(x,0), \quad \textit{for } x \in D.$$

Then $u_1 \geq u_2$ in $D \times (0,\infty)$.

Proof. Let $W = u_1 - u_2$ and define $V(x,t) = \frac{f(x,u_1(x,t)) - f(x,u_2(x,t))}{W(x,t)}$, if $W(x,t) \neq 0$, and $V(x,t) = 0$ otherwise. Since f is locally Lipschitz in u, V is bounded in $D \times [0,T]$, for any $T > 0$. We have $LW + VW - \frac{\partial W}{\partial t} \leq 0$ in $D \times (0,\infty)$, $W(x,0) \geq 0$ in D, and $W(x,t) \geq 0$ on $\partial D \times (0,\infty)$. Thus, by the standard linear parabolic maximum principle, $u_1 \geq u_2$. $\square$

We record the following result, mentioned in Remark 2 after Example 1.

Lemma 1. *Let $G(u)$ be locally Lipschitz and satisfy $\lim_{u\to\infty} G(u) = -\infty$. For $c \geq 0$, let $v_c(t)$ denote the solution to*

$$v' = G(v), \; t > 0; \qquad v(0) = c. \tag{2.1}$$

i. If $\int^\infty \frac{1}{-G(u)} du < \infty$, then $v_\infty(t) \equiv \lim_{c\to\infty} v_c(t) < \infty$, for all $t > 0$, and v_∞ solves (2.1) with $c = \infty$.
ii. If $\int^\infty \frac{1}{-G(u)} du = \infty$, then $\lim_{c\to\infty} v_c(t) = \infty$, for all $t \geq 0$.

Proof. We omit the straight forward proof of this standard result. $\square$

We now give the proof of Theorem 1. It suffices to show that for some $T_0 > 0$ and each $R > 0$, there exists a continuous function $M_R(x,t)$ on $\{|x| < R\} \times (0, T_0]$ such that every solution u to (1.1) satisfies $u(x,t) \leq M_R(x,t) < \infty$, for $|x| < R$ and $t \in (0, T_0]$. The reason it is enough to consider only $t \in (0, T_0]$ is that if $u(x,t)$ is a solution to (1.1), then $u(x, T_0 + t)$ is a solution to (1.1) with the initial condition $g(\cdot)$ replaced by $u(\cdot, T_0)$.

We will assume that F_R satisfies (F-2) with $m = 0$. At the end of the proof, we describe the simple change needed in the case that $m \geq 1$. In particular then, there exist an $\epsilon > 0$ and a $u_0 > 1$ such that

$$F_R(u) \leq -u(\log u)^{2+\epsilon} \equiv Q(u), \text{ for } u \geq u_0. \tag{2.2}$$

Since $\int^\infty \frac{1}{-Q(u)} du < \infty$, it follows from Lemma 1 that there exists a $T_0 > 0$ and a function $v_\infty(t)$ satisfying

$$v'_\infty = Q(v_\infty), \; t \in (0, T_0]; \qquad v_\infty(0) = \infty; \qquad v_\infty(t) > 1, \; t \in (0, T_0]. \tag{2.3}$$

Define

$$\phi_R(x) = \exp((R^2 - |x|^2)^{-l}),$$

with l satisfying $l\epsilon > 2$. Finally, choose K so that $\exp(K) > u_0$ and define

$$M_R(x,t) = \exp((K(t+1))\phi_R(x) + v_\infty(t).$$

Since $M_R(x,t) > u_0$, it follows from (2.2) that $f(x, M_R) \le F_R(M_R) \le Q(M_R)$. Since $Q(u)$ is concave for $u \ge 1$ and $Q(1) = 0$, it follows from the mean value theorem that $Q(b+a) - Q(b) < Q(a)$ for $1 \le a \le b$. Thus, since $\exp((K(t+1))\phi_R(x), v_\infty(t) > 1$, we have $Q(M_R(x,t)) < Q(\exp(K(t+1))\phi_R(x)) + Q(v_\infty(t))$. Using these facts along with (2.3), we obtain

(2.4)

$$\begin{aligned}&LM_R + f(x, M_R) - (M_R)_t \le \exp(K(t+1))L\phi_R + Q(\exp(K(t+1))\phi_R) + Q(v_\infty)\\ &- K\exp(K(t+1))\phi_R - v'_\infty =\\ &\exp(K(t+1))L\phi_R + Q(\exp(K(t+1))\phi_R) - K\exp(K(t+1))\phi_R <\\ &\exp(K(t+1))\left(L\phi_R - (R^2-|x|^2)^{-(2+\epsilon)l}\phi_R - K\phi_R\right), \text{ for } |x| < R \text{ and } t \in (0, T_0].\end{aligned}$$

We have

(2.5)

$$\begin{aligned}\frac{L\phi_R(x)}{\phi_R(x)} &= \left(4l^2(R^2-|x|^2)^{-2l-2} + 4l(l+1)(R^2-|x|^2)^{-l-2}\right)\sum_{i,j=1}^n a_{i,j}(x)x_ix_j\\ &+ 2nl(R^2-|x|^2)^{-l-1} + 2l(R^2-|x|^2)^{-l-1}\sum_{i=1}^n x_ib_i(x).\end{aligned}$$

The right hand side of (2.5) is bounded for $|x|$ in any ball of radius less than R. Furthermore, on the right hand side of (2.5), the dominating term as $|x| \to R$ is $4l^2(R^2-|x|^2)^{-2l-2}$. Thus, since $l\epsilon > 2$, it follows that the right hand side of (2.4) is negative if K is chosen sufficiently large. Using this with Proposition 1 and the fact that $M_R(x,0) = \infty$ and $M_R(x,t) = \infty$, for $|x| = R$, we conclude that any solution u to (1.1) satisfies $u(x,t) \le M_R(x,t)$, for $|x| < R$ and $t \in (0, T_0]$. This completes the proof of the theorem under the assumption that $m = 0$ in (F-2).

When $m > 0$ one simply replaces the test function $\phi_R(x)$ as above by $\phi_R(x) = \exp^{(m+1)}((R^2-|x|^2)^{-l})$, where $\exp^{(j)}$ denotes the j-th iterate of the exponential function. Everything goes through in a similar fashion.

$\square$

3. Proof of Theorem 3. By assumption, $F(u) = \sup_{x\in R^n} f(x,u)$ satisfies (F-2′). As we did in the proof of Theorem 1, we will assume that $m = 0$ in (F-2′). At the appropriate point in the proof, we describe the simple change needed in the case that $m \geq 1$.

We first consider the case with initial condition $g = 0$. By conditions (F-2′) and (F-3), it follows that there exist $C_0, \epsilon > 0$ and $M_0 > 1$ such that

$$\begin{aligned} &f(x,u) \leq F(u) \leq C_0 u, \ \text{for } u \leq M_0; \\ &f(x,u) \leq F(u) \leq -u(\log u)^{2+\epsilon}, \ \text{for } u \geq M_0. \end{aligned} \tag{3.1}$$

Fix $R > 1$ and $T \in (0,\infty)$. Define

$$\phi_R(x) = \exp((\frac{1+|x|^2}{R^2-|x|^2})^l),$$

with l satisfying $l\epsilon > 2$, and define

$$\psi_R(x,t) = (\phi_R(x) - 1)\exp(K(t+1)),$$

with $K > 0$. A direct calculation reveals that

$$L\psi_R = \exp(K(t+1))\phi_R(x)\left[W_1 + W_2 + W_3 + W_4 + W_5\right], \tag{3.2}$$

where

$$\begin{aligned} W_1 &= 4l^2(1+|x|^2)^{2l-2}(R^2-|x|^2)^{-2l-2}(R^2+1)^2\sum_{i,j=1}^n a_{i,j}(x)x_ix_j; \\ W_2 &= 4l(l-1)(1+|x|^2)^{l-2}(R^2-|x|^2)^{-l-2}(R^2+1)^2\sum_{i,j=1}^n a_{i,j}(x)x_ix_j; \\ W_3 &= 4l(1+|x|^2)^{l-1}(R^2-|x|^2)^{-l-2}(R^2+1)\sum_{i,j=1}^n a_{i,j}(x)x_ix_j; \\ W_4 &= 2nl(1+|x|^2)^{l-1}(R^2-|x|^2)^{-l-1}(R^2+1); \\ W_5 &= 2l(1+|x|^2)^{l-1}(R^2-|x|^2)^{-l-1}(R^2+1)\sum_{i=1}^n x_ib_i. \end{aligned}$$

We also have

$$\frac{\partial \psi_R}{\partial t} = K\psi_R(x). \tag{3.3}$$

We claim that for K sufficiently large and independent of R (but not independent of T in (3.4-b))

$$\begin{aligned} &\exp(K(t+1))\phi_R(x)W_i - \frac{1}{5}(K - C_0)\psi_R(x,t) \leq 0, \\ &\text{if } \psi_R(x,t) \leq M_0, \ \text{for } |x| < R \text{ and } t \in (0,T]; \end{aligned} \tag{3.4-a}$$

$$\begin{aligned} &\exp(K(t+1))\phi_R(x)W_i - \frac{1}{5}K\psi_R(x,t) - \frac{1}{5}\psi_R(x,t)(\log\psi_R(x,t))^{2+\epsilon} \leq 0, \\ &\text{if } \psi_R(x,t) \geq M_0, \ \text{for } |x| < R \text{ and } t \in (0,T], \end{aligned} \tag{3.4-b}$$

for $i = 1,2,3,4,5$

From (3.1)-(3.4), it follows that for sufficiently large K, independent of R,

$$L\psi_R - \frac{\partial \psi_R}{\partial t} + f(x,\psi_R) \leq 0, \ \text{for } |x| < R \text{ and } t \in (0,T]. \tag{3.5}$$

Since $\psi_R(x,0) \geq 0$ and $\lim_{|x|\to R}\psi_R(x,t) = \infty$, it follows from (3.5) and the maximum principle in Proposition 1 that any solution u to (1.1) with initial condition $g = 0$ must satisfy the bound

$$u(x,t) \leq (\exp((\frac{1+|x|^2}{R^2 - |x|^2})^l) - 1)\exp(K(t+1)), \ \text{for } |x| < R \text{ and } t \in (0,T]. \tag{3.6}$$

Since K doesn't depend on R, letting $R \to \infty$ in (3.6) gives $u(x,t) \equiv 0$ for $x \in R^n$ and $t \in (0,T]$. Now letting $T \to \infty$ gives $u(x,t) \equiv 0$ in $R^n \times (0,\infty)$, completing the proof.

When $m > 0$, one replaces the test function $\phi_R(x)$ as above by $\phi_R(x) = \exp^{(m+1)}((\frac{1+|x|^2}{R^2-|x|^2})^l)$, where $\exp^{(j)}$ denotes the j-th iterate of the exponential function, and then, in the definition of $\psi_R(x,t)$, one replaces $\phi_R(x) - 1$ by $\phi_R(x) - \exp^{(m+1)}(0)$. The resulting calculations are similar to the present case.

It thus remains to prove (3.4) for K independent of R. We will prove (3.4) for W_1. The proofs for $W_i, i \geq 2$, are similar. Consider first (3.4-a). We will always assume that $K \geq C_0$. Recall the definitions of ϕ_R and ψ_R. If $\psi_R(x,t) \leq M_0$, then a fortiori $\phi_R(x) \leq M_0 + 1$ and $(\frac{1+|x|^2}{R^2-|x|^2})^l \leq \log(M_0+1) \equiv L_0^l$. Also, we have $\psi_R(x,t) \geq \phi_R(x) - 1 \geq (\frac{1+|x|^2}{R^2-|x|^2})^l$. In light of these observations, it follows that (3.4-a) will hold if

$$(M_0+1)W_1 - \frac{1}{5}(K - C_0)(\frac{1+|x|^2}{R^2-|x|^2})^l \leq 0, \text{ whenever } \frac{1+|x|^2}{R^2-|x|^2} \leq L_0.$$

Or equivalently, if

$$K \geq C_0 + 5(\frac{R^2-|x|^2}{1+|x|^2})^l (M_0+1)W_1, \text{ whenever } \frac{1+|x|^2}{R^2-|x|^2} \leq L_0. \tag{3.7}$$

Thus, we must show that the right hand side of (3.7) is bounded in R and x under the constraint $\frac{1+|x|^2}{R^2-|x|^2} \leq L_0$. Substituting for W_1 in the right hand side of (3.7) and using the assumption that $\sum_{i,j=1}^n a_{i,j}(x)x_ix_j \leq C(1+|x|^2)|x|^2$, one finds that it is enough to show that $(\frac{1+|x|^2}{R^2-|x|^2})^{l-1}\frac{(R^2+1)^2|x|^2}{(R^2-|x|^2)^3}$ is bounded in R and x under the above constraint. Since $(\frac{1+|x|^2}{R^2-|x|^2})^{l-1}$ is trivially bounded under the constraint, it remains only to consider $\frac{(R^2+1)^2|x|^2}{(R^2-|x|^2)^3}$. The constraint above is equivalent to the constraint $|x|^2 \leq \frac{L_0R^2-1}{L_0+1}$. From this it is clear that under the constraint, $\frac{(R^2+1)^2|x|^2}{(R^2-|x|^2)^3}$ is bounded in R and x.

We now turn to (3.4-b). The constraint $\psi_R(x,t) \geq M_0$ along with the condition $t \leq T$ guarantee the existence of a $c_0 \in (0,1)$ such that $\phi_R(x) - 1 \geq c_0\phi_R(x)$. Note that c_0 depends on T, but not on R. Thus, under the constraint, we have $\psi(x,t) = (\phi_R(x)-1)\exp(K(t+1)) \geq c_0\phi_R(x)\exp(K(t+1))$. Therefore (3.4-b) will hold if we show that K can be picked independent of R and such that $W_1 - \frac{1}{5}c_0K - \frac{1}{5}c_0\left(\log\phi_R + \log c_0 + K(t+1)\right)^{2+\epsilon} \leq 0$ holds under the constraint. We will always assume that $K \geq -\log c_0$. Thus it suffices to show that $W_1 - \frac{1}{5}c_0(\log\phi_R)^{2+\epsilon}$ is bounded from above under the constraint, independent of R. Substituting for ϕ_R and W_1 and using the assumption $\sum_{i,j=1}^n a_{i,j}(x)x_ix_j \leq C(1+|x|^2)|x|^2$, it is sufficient to show that $4l^2C(1+|x|^2)^{2l-1}(R^2-|x|^2)^{-2l-2}(R^2+1)^2|x|^2 - \frac{1}{5}c_0\left(\frac{1+|x|^2}{R^2-|x|^2}\right)^{(2+\epsilon)l}$

is bounded from above under the constraint, or equivalently, that

$$\left(\frac{1+|x|^2}{R^2-|x|^2}\right)^{(2+\epsilon)l}\left(4l^2C\frac{(R^2+1)^2|x|^2}{(1+|x|^2)^3}\left(\frac{R^2-|x|^2}{1+|x|^2}\right)^{l\epsilon-2}-\frac{c_0}{5}\right) \tag{3.8}$$

is bounded from above under the constraint.

We may assume that $4l^2C\frac{(R^2+1)^2|x|^2}{(1+|x|^2)^3}\left(\frac{R^2-|x|^2}{1+|x|^2}\right)^{l\epsilon-2} \geq \frac{c_0}{5}$, since otherwise it is clear that (3.8) holds. From this inequality and the assumption that $l\epsilon > 2$, it follows that

$$\frac{1+|x|^2}{R^2-|x|^2} \leq \left(\frac{20l^2C}{c_0}\frac{(R^2+1)^2|x|^2}{(1+|x|^2)^3}\right)^{\frac{1}{l\epsilon-2}}. \tag{3.9}$$

Furthermore, the constraint $\psi_R \geq M_0$ guarantees that

$$\frac{1+|x|^2}{R^2-|x|^2} \geq (\log(1+M_0\exp(-K(T+1))))^{\frac{1}{l}} \equiv \gamma_0 > 0, \tag{3.10}$$

which can be written in the form

$$|x|^2 \geq \frac{\gamma_0 R^2-1}{\gamma_0+1}. \tag{3.11}$$

If $|x|$ satisfies (3.11), then the right hand side of (3.9) is bounded. Therefore, in (3.8), the terms $(\frac{1+|x|^2}{R^2-|x|^2})^{(2+\epsilon)l}$ and $\frac{(R^2+1)^2|x|^2}{(1+|x|^2)^3}$ are bounded. And by (3.10), the term $(\frac{R^2-|x|^2}{1+|x|^2})^{l\epsilon-2}$ is also bounded. This completes the proof of (3.8).

We now turn to the case that the initial condition g is not equal to 0. We assume now in addition that condition (F-4b) is in effect. Fix $R > 1$ and $T \in (0,\infty)$. Let $\psi_R(x,t)$ be as in the proof above for the case $g = 0$, but corresponding to the function H appearing in condition (F-4b), rather than corresponding to the function F. (That is, the parameter K appearing in the definition of ψ_R is chosen sufficiently large so that (3.5) holds with $f(x,\psi_R)$ replaced by $H(\psi_R)$.)

In [6], for the case $f(x,u) = V(x)u - \gamma(x)u^p$, we showed that there exists a minimal solution u_g to (1.1); that is, a solution u_g with the property that $u_g(x,t) \leq u(x,t)$, for any solution u to (1.1) with initial data g. In fact, the proofs there go through for general locally Lipschitz continuous f as long as a universal a priori upper bound exists. Thus, in light of Theorem 1, there exists a minimal solution u_g. (In fact, u_g is obtained by taking the solution of (4.1) below and letting $m \to \infty$.)

Now define $\hat{\psi}_R(x,t) = \psi_R(x,t) + u_g$. Then

$$
\begin{aligned}
L\hat{\psi}_R + f(x,\hat{\psi}_R) - \frac{\partial \hat{\psi}_R}{\partial t} &= (L\psi_R + H(\psi_R) - \frac{\partial \psi_R}{\partial t}) \\
+ (Lu_g + f(x,u_g) - \frac{\partial u_g}{\partial t}) &+ (f(x,\psi_R + u_g) - f(x,u_g) - H(\psi_R)).
\end{aligned} \tag{3.12}
$$

The first of the three terms on the right hand side of (3.12) is non-positive by (3.5), the second term is non-positive because u_g is a solution to (1.1), and the third term is non-positive by the definition of H in (F-4b). The argument used above in the paragraph in which (3.5) appears then shows that any solution u to (1.1) must satisfy $u(x,t) \le u_g(x,t) + \psi_R(x,t)$, for $|x| < R$ and $t \in (0,T]$. Letting $R \to \infty$ and then $T \to \infty$ as before shows that $u = u_g$. □

4. Proof of Theorem 2. *i.* We need to utilize certain constructions that were carried out in [6, section 2] for the case that $f(x,u) = V(x) - \gamma(x)u^p$. These constructions are based on results in [10], and hold with the same proofs for general locally Lipschitz continuous f as long as a universal a priori upper bound on solutions exists. Thus, in light of Theorem 1, they hold for f satisfying (F-1) and (F-2).

Let $B_m \subset R^n$ denote the open ball of radius m centered at the origin. There exists a solution $u \in C^{2,1}(B_m \times (0,\infty)) \cap C(B_m \times [0,\infty)) \cap C(\bar{B}_m \times (0,\infty))$ to the equation

$$
\begin{aligned}
&u_t = Lu + f(x,u), \ (x,t) \in B_m \times (0,\infty); \\
&u(x,0) = g(x), \ x \in B_m; \\
&u(x,t) = 0, \ (x,t) \in \partial B_m \times (0,\infty),
\end{aligned} \tag{4.1}
$$

for any $0 \le g \in C(\bar{B}_m)$. (See the beginning of the proof of Theorem 1 in [6], where the above construction is first made in the case that g is compactly supported in B_m, and then extended to the case that $g \in C(\bar{B}_m)$.)

Now let W be an arbitrary subsolution to (1.5). For $m > 0$ and a positive integer k, let $\psi_{m,k} \in C^\infty(R^n)$ satisfy

$$\psi_{m,k}(x) = 0, \ |x| \le m \text{ and } |x| > 2m+1$$
$$\psi_{m,k}(x) = k, \ m+1 \le |x| \le 2m$$
$$0 \le \psi_{m,k} \le k.$$

There exists a nonnegative solution $U_{m,k} \in C^{2,1}(B_{2m} \times (0,\infty)) \cap C(\bar{B}_{2m} \times (0,\infty))$ to the equation

$$\begin{aligned} &u_t = Lu + f(x,u) + \psi_{m,k}, \ (x,t) \in B_{2m} \times (0,\infty); \\ &u(x,0) = g_m, \ x \in B_{2m}; \\ &u(x,t) = 0, \ (x,t) \in \partial B_{2m} \times (0,\infty), \end{aligned} \tag{4.2}$$

where $g_m \ge 0$ is continuous and satisfies

$$g_m(x) = \begin{cases} 0, \text{ for } \ x \in B_m \\ m^2 W, \text{ for } x \in B_{2m} - B_{m+1} \end{cases}.$$

(This construction is similar to the one in [6, equation (2.5)].) Also,
(4.3)

$$U(x,t) \equiv \lim_{m\to\infty} \lim_{k\to\infty} U_{m,k}(x,t) \text{ is a solution to (1.1) with initial condition } g=0.$$

(See the two full paragraphs after equation (2.5) in [6], ignoring equation (2.6) and the concept of a maximal solution that appears there.)

Consider (4.1) with m replaced by $2m$, with the nonlinearity f replaced by G as in condition (F-4a), and with $g = W$. Denote the solution to this equation by u_m. We will show below that

$$W - u_m \le U_{m,k}, \ \text{in } B_{\frac{3m}{2}} \times [0,\infty), \text{for } k \text{ sufficiently large, depending on } \ m. \tag{4.4}$$

Let v_∞ denote the solution to $v' = G(v)$ with $v_\infty(0) = \infty$, as in Lemma 1-i (note that by condition (F-4a), G satisfies the requirement in Lemma 1-i). Since

$f(x,u) \leq G(u)$, we have $f(x,v_\infty) - v_\infty' \leq G(v_\infty) - v_\infty' = 0$. Also, since G is Lipschitz, it follows from the uniqueness theorem for ordinary differential equations that $v_\infty(t) > 0$, for all $t \geq 0$. Using these facts along with the fact that $u_m = 0$ on ∂B_{2m} and the fact that $v_\infty(0) = \infty$, it follows from the maximum principle in Proposition 1 that

$$u_m(x,t) \leq v_\infty(t) \text{ in } B_{2m} \times (0,\infty). \tag{4.5}$$

Letting $k \to \infty$ and then letting $m \to \infty$, it follows from (4.3) that the right hand side of (4.4) converges to a solution U of (1.1) with initial data $g = 0$. By the uniqueness assumption, $U = 0$. Using this with (4.5) then gives

$$W(x) \leq v_\infty(t) \text{ in } R^n \times (0,\infty). \tag{4.6}$$

We now show that

$$\lim_{t\to\infty} v_\infty(t) = c_0, \text{ where } c_0 \text{ is the largest root of } G(u) = 0. \tag{4.7}$$

To see this, let v_c be as in Lemma 1. Integrating, changing variables and letting $c \to \infty$, we obtain

$$\int_{v_\infty(t)}^{\infty} \frac{1}{-G(u)} du = t. \tag{4.8}$$

Letting $t \to \infty$ in (4.8) and using the fact that G is locally Lipschitz proves (4.7). The theorem now follows from (4.6) and (4.7).

It remains to prove (4.4). Let $D_{2m} = \{(x,t) \in B_{2m} \times (0,\infty) : u_m(x,t) \leq W(x)\}$. Since $U_{m,k} \geq 0$, it suffices to prove (4.4) in $\left(B_{\frac{3m}{2}} \times [0,\infty)\right) \cap D_{2m}$. Let $V = W - u_m$. We have

$$\begin{aligned} &LV + f(x,V) - V_t = (LW + f(x,W)) - (Lu_m + G(u_m) - (u_m)_t) + \\ &f(x, W - u_m) - f(x,W) + G(u_m) \geq \\ &f(x, W - u_m) - f(x,W) + G(u_m) \geq 0 \text{ in } (B_{2m} \times (0,\infty)) \cap D_{2m}, \end{aligned} \tag{4.9}$$

where the first inequality follows from the definitions of W and u_m, and the second inequality follows from the definition of G. On the other hand, we have

$$LU_{m,k} + f(x, U_{m,k}) - (U_{m,k})_t = -\psi_{m,k} \leq 0 \text{ in } B_{2m} \times (0, \infty). \tag{4.10}$$

We now show that for sufficiently large k, depending on m,

$$V(x,t) \leq U_{m,k}(x,t), \text{ on } \left(\partial B_{\frac{3m}{2}} \times [0,\infty)\right) \cap D_{2m}. \tag{4.11}$$

Define $Q(x) = \left(l^2 - (m+1+l-|x|)^2\right) W(x)$, where $l = \frac{1}{2}(m-1)$. Note that $Q > 0$ in the annulus $A_{m+1,2m} \equiv \{m+1 < |x| < 2m\}$ and vanishes on $\partial A_{m+1,2m}$. Clearly $LQ + f(x,Q)$ is bounded in $A_{m+1,2m} \times [0,\infty)$. Thus for k sufficiently large, we have $LQ + f(x,Q) \geq -\psi_{m,k}$ in $A_{m+1,2m} \times [0,T]$. Since $Q(x) \leq l^2 W(x) < m^2 W(x) = g_m(x) = U_{m,k}(x,0)$ on $A_{m+1,2m}$, and since Q vanishes on $\partial A_{m+1,2m}$, it follows by the maximum principle in Proposition 1 that $U_{m,k} \geq Q$ in $A_{m+1,2m} \times [0,\infty)$, for k sufficiently large. Substituting $|x| = \frac{3m}{2}$ in Q, we conclude that for $m \geq 4$ and sufficiently large k, $U_{m,k}(x,t) \geq Q(x) = (l^2 - \frac{1}{4})W(x) > W(x)$ on $\partial B_{\frac{3m}{2}} \times [0,\infty)$. This proves (4.11) since $V \leq W$ on D_{2m}. In light of (4.9)-(4.11), the fact that $0 = V \leq U_{m,k}$ on ∂D_{2m} and the fact that $V(x,0) = 0$, (4.4) now follows from the maximum principle in Proposition 1.

ii. Since the largest root c_0 of the equation $G(u) = 0$ is $c_0 = 0$, one direction follows from part (i). The proof of the other direction is essentially the same as the proof of [6, Theorem 4-ii].

□

5. Proof of Theorem 5-ii. For technical reasons, it will be necessary to treat the cases $d > \frac{2p}{p-1}$ and $d = \frac{2p}{p-1}$ separately.

We first consider the case $d > \frac{2p}{p-1}$. For ϵ and R satisfying $0 < \epsilon < 1$ and $R > 1$, and some $l \in (0,1]$, define

$$\phi_{R,\epsilon}(x) = ((|x| - \epsilon)(R - |x|))^{-\frac{2}{p-1}}(1 + |x|)^{\frac{2}{p-1}}(1 + \frac{\epsilon^l}{|x|^l} R^{\frac{2}{p-1}}). \tag{5.1}$$

Also, for R and ϵ as above, and some $\gamma > 0$, define

$$\psi_{R,\epsilon}(x,t) = \phi_{R,\epsilon}(x)\exp(\gamma(t+1)). \tag{5.2}$$

Note that $\psi_{R,\epsilon}(x,0) > 0$, for $|x| \in (\epsilon, R)$, and $\psi_{R,\epsilon}(x,t) = \infty$, for $|x| = \epsilon$ or $|x| = R$. We will show that for all sufficiently large R and all sufficiently small ϵ, and for γ sufficiently large and l sufficiently small, independent of those R and ϵ, one has

$$\Delta\psi_{R,\epsilon} - \psi_{R,\epsilon}^p - (\psi_{R,\epsilon})_t \le 0, \quad \text{for } \epsilon < |x| < R \ \text{ and } t > 0. \tag{5.3}$$

It then follows from the maximum principle in Proposition 1 that every solution $u(x,t)$ to (1.6) satisfies

$$u(x,t) \le \psi_{R,\epsilon}(x,t), \quad \text{for } \epsilon < |x| < R \ \text{ and } t \in [0,\infty). \tag{5.4}$$

Substituting (5.1) and (5.2) in (5.4), letting $\epsilon \to 0$, and then letting $R \to \infty$, one concludes that $u(x,t) \equiv 0$. Thus, it remains to show (5.3).

From now on we will use radial coordinates, writing $\phi(r)$ for $\phi(x)$ with $|x| = r$, and similarly for ψ. We have

$$\begin{aligned}
&\exp(-\gamma(t+1))(\psi_{R,\epsilon})_r = \\
&-(\frac{2}{p-1})((r-\epsilon)(R-r))^{-\frac{2}{p-1}-1}(R+\epsilon-2r)(1+r)^{\frac{2}{p-1}}(1+\frac{\epsilon^l}{r^l}R^{\frac{2}{p-1}}) \\
&+(\frac{2}{p-1})((r-\epsilon)(R-r))^{-\frac{2}{p-1}}(1+r)^{\frac{2}{p-1}-1}(1+\frac{\epsilon^l}{r^l}R^{\frac{2}{p-1}}) \\
&-l((r-\epsilon)(R-r))^{-\frac{2}{p-1}}(1+r)^{\frac{2}{p-1}}\frac{\epsilon^l}{r^{l+1}}R^{\frac{2}{p-1}},
\end{aligned} \tag{5.5}$$

and

$$\exp(-\gamma(t+1))\big((r-\epsilon)(R-r)\big)^{-\frac{2}{p-1}-2}(\psi_{R,\epsilon})_{rr} =$$

$$(5.6)\quad \begin{aligned}
&(\frac{2}{p-1})(\frac{2}{p-1}+1)(R+\epsilon-2r)^2(1+r)^{\frac{2}{p-1}}(1+\frac{\epsilon^l}{r^l}R^{\frac{2}{p-1}})\\
&+2(\frac{2}{p-1})(r-\epsilon)(R-r)(1+r)^{\frac{2}{p-1}}(1+\frac{\epsilon^l}{r^l}R^{\frac{2}{p-1}})\\
&-2(\frac{2}{p-1})^2(r-\epsilon)(R-r)(R+\epsilon-2r)(1+r)^{\frac{2}{p-1}-1}(1+\frac{\epsilon^l}{r^l}R^{\frac{2}{p-1}})\\
&+2l(\frac{2}{p-1})(r-\epsilon)(R-r)(R+\epsilon-2r)(1+r)^{\frac{2}{p-1}}\frac{\epsilon^l}{r^{l+1}}R^{\frac{2}{p-1}}\\
&+(\frac{2}{p-1})(\frac{2}{p-1}-1)((r-\epsilon)(R-r))^2(1+r)^{\frac{2}{p-1}-2}(1+\frac{\epsilon^l}{r^l}R^{\frac{2}{p-1}})\\
&-2l(\frac{2}{p-1})((r-\epsilon)(R-r))^2(1+r)^{\frac{2}{p-1}-1}\frac{\epsilon^l}{r^{l+1}}R^{\frac{2}{p-1}}\\
&+l(l+1)((r-\epsilon)(R-r))^2(1+r)^{\frac{2}{p-1}}\frac{\epsilon^l}{r^{l+2}}R^{\frac{2}{p-1}}.
\end{aligned}$$

Using (5.1), (5.2), (5.5) and the fact that $\frac{2}{p-1}+2=\frac{2p}{p-1}$, we have

$$\exp(-\gamma(t+1))\big((r-\epsilon)(R-r)\big)^{-\frac{2}{p-1}-2}\Big(\frac{d-1}{r}(\psi_{R,\epsilon})_r-\psi^p_{R,\epsilon}-(\psi_{R,\epsilon})_t\Big) =$$

$$(5.7)\quad \begin{aligned}
&-(\frac{2}{p-1})(\frac{d-1}{r})(r-\epsilon)(R-r)(R+\epsilon-2r)(1+r)^{\frac{2}{p-1}}(1+\frac{\epsilon^l}{r^l}R^{\frac{2}{p-1}})\\
&+(\frac{2}{p-1})(\frac{d-1}{r})((r-\epsilon)(R-r))^2(1+r)^{\frac{2}{p-1}-1}(1+\frac{\epsilon^l}{r^l}R^{\frac{2}{p-1}})\\
&-l(\frac{d-1}{r})((r-\epsilon)(R-r))^2(1+r)^{\frac{2}{p-1}}\frac{\epsilon^l}{r^{l+1}}R^{\frac{2}{p-1}}\\
&-\gamma((r-\epsilon)(R-r))^2(1+r)^{\frac{2}{p-1}}(1+\frac{\epsilon^l}{r^l}R^{\frac{2}{p-1}})\\
&-(1+r)^{\frac{2p}{p-1}}(1+\frac{\epsilon^l}{r^l}R^{\frac{2}{p-1}})^p\exp((p-1)\gamma(t+1)).
\end{aligned}$$

We will show that for all sufficiently large R and sufficiently small ϵ, and for γ sufficiently large and l sufficiently small, independent of those R and ϵ, the sum of the right hand sides of (5.6) and (5.7) is non-positive. This will prove (5.3).

We denote the seven terms on the right hand side of (5.6) by J_1-J_7, and the five terms on the right hand side of (5.7) by I_1-I_5. Note that the terms that are positive are J_1, J_2, J_4, J_5, J_7 and I_2. In what follows, M will denote a positive number that can be made as large as one desires by choosing γ sufficiently large. Consider first those r satisfying $r \geq cR$, where c is a fixed positive number. For r in this range, we have $|I_5| \geq MR^{\frac{2}{p-1}+2}(1+\epsilon^l R^{\frac{2}{p-1}-l})$. It is easy to see that for M sufficiently large, $|I_5|$ dominates each of the positive terms, uniformly over large R and small ϵ, and thus (since M can be made arbitrarily large) also the sum of all of the positive terms. Now consider those r for which $\delta_0 \leq r \leq C$, for some constants $0 < \delta_0 < C$. For r in this range and ϵ sufficiently small, we have $|I_4| \geq MR^2(1+\epsilon^l R^{\frac{2}{p-1}})$, and it is easy to see that for M sufficiently large, $|I_4|$ dominates each of the positive terms, uniformly over large R and small ϵ, and thus, also the sum of all of the positive terms. One can also show that the transition from r of order unity to r of order R causes no problem. Thus, we conclude that for any fixed $\delta_0 > 0$ and γ sufficiently large, the sum of the right hand sides of (5.6) and (5.7) is negative for all large R and small ϵ. Note that all this holds uniformly over $l \in (0,1]$. The parameter l has not been needed yet.

We now turn to the delicate situation—when $\epsilon \leq r \leq \delta_0$. For later use, we remind the reader that δ_0 may be chosen as small as one likes. (Note that at $r=\epsilon$, all the terms vanish except J_1 and I_5. Using the fact that $\frac{2}{p-1}+2=\frac{2p}{p-1}$, it is easy to see that for sufficiently large γ, $|I_5(\epsilon)|$ dominates $J_1(\epsilon)$, uniformly over all large R and small ϵ. However, when r is small, but on an order larger than ϵ, the analysis becomes a lot more involved.) In the sequel, whenever we say that a condition holds for γ or M sufficiently large, or for l sufficiently small, we mean independent of R and ϵ.

We first take care of the easy terms. Clearly, $J_5 \leq |I_4|$ if γ is sufficiently large. Also $J_7 = \frac{l+1}{d-1}|I_3| \leq |I_3|$, if l is chosen sufficiently small. (This last inequality holds since by assumption, $d > \frac{2p}{p-1}$; thus, $d > 2$ for all choices of p.)

We now show that for γ sufficiently large, $J_2 \leq |I_4| + |I_5|$, for $\epsilon \leq r \leq \delta_0$. (We are reusing $|I_4|$ here. Later we will reuse $|I_5|$. This is permissible because γ can be chosen as large as we like.) To show this inequality, it suffices to show that for M sufficiently large,

$$(r-\epsilon)R \leq M(r-\epsilon)^2R^2 + M(1+\frac{\epsilon^l}{r^l}R^{\frac{2}{p-1}})^{p-1}, \text{ for } r \in [\epsilon, \delta_0]. \tag{5.8}$$

A trivial calculation shows that the left hand side of (5.8) is less than the first term on the right hand side if $r \geq \epsilon + \frac{1}{RM}$. If $r \in [\epsilon, \epsilon + \frac{1}{RM}]$, then the left hand side of (5.8) is less than or equal to $\frac{1}{M}$ while the second term on the right hand side is greater than M. We conclude that (5.8) holds with $M \geq 1$.

It remains to consider J_1, J_4 and I_2. We will show that for γ sufficiently large,

$$J_1 + J_4 + I_2 + I_1 + I_5 \leq 0, \text{ for } r \in [\epsilon, \delta_0]. \tag{5.9}$$

Since I_2 has the factor $(r-\epsilon)^2$, while I_1 has the factor $(r-\epsilon)$, and since $\frac{R-r}{R+\epsilon-2r}$ can be made arbitrarily close to 1 by choosing R sufficiently large, it follows that for any $\eta > 0$, we have $I_2 \leq \eta |I_1|$, for $r \in [\epsilon, \delta_0]$, if we choose δ_0 sufficiently small and R sufficiently large. Note that $J_4 \leq \frac{2l}{d-1}|I_1|$. Thus,

$$J_1 + J_4 + I_2 + I_1 \leq J_1 + (1 - \frac{2l}{d-1} - \eta) I_1 = J_1 + (1-\kappa) I_1, \tag{5.10}$$

where $\kappa = \frac{2l}{d-1} + \eta$. Also note that since we are free to choose l and η as small as we like, the same holds for κ. We have

$$\begin{aligned} &J_1 + (1-\kappa) I_1 = (1+r)^{\frac{2}{p-1}} (R+\epsilon-2r)(1+\frac{\epsilon^l}{r^l} R^{\frac{2}{p-1}}) \times \\ &\Big(\frac{2(p+1)}{(p-1)^2}(R+\epsilon-2r) - (1-\kappa)\frac{2(d-1)}{p-1}(\frac{r-\epsilon}{r})(R-r)\Big). \end{aligned} \tag{5.11}$$

From the assumption that $d > \frac{2p}{p-1}$, it follows that for κ sufficiently small and R sufficiently large,

$$\Big(\frac{2(p+1)}{(p-1)^2}(R+\epsilon-2r) - (1-\kappa)\frac{2(d-1)}{p-1}(\frac{r-\epsilon}{r})(R-r)\Big) \leq C\frac{\epsilon}{r} R, \ r \in [\epsilon, \delta_0], \tag{5.12}$$

for some $C > 0$. From (5.10)-(5.12), we obtain

$$\begin{aligned} &J_1 + J_4 + I_2 + I_1 \leq C\frac{\epsilon}{r} R (1+r)^{\frac{2}{p-1}} (R+\epsilon-2r)(1+\frac{\epsilon^l}{r^l} R^{\frac{2}{p-1}}), \\ &\text{for } r \in [\epsilon, \delta_0]. \end{aligned} \tag{5.13}$$

In light of (5.13), in order to prove (5.9), it suffices to show that

$$\frac{\epsilon}{r} R^2 \leq M (1 + \frac{\epsilon^l}{r^l} R^{\frac{2}{p-1}})^{p-1}, \ r \in [\epsilon, \delta_0], \tag{5.14}$$

for sufficiently large M. Choose l sufficiently small so that $l(p-1) \leq 1$. Then the right hand side of (5.14) is greater or equal to $M\frac{\epsilon}{r}R^2$.

We now turn to the case $d = \frac{2p}{p-1}$. For ϵ and R satisfying $0 < \epsilon < 1$ and $R > 1$, and some $c \geq 2$, define

$$\phi_{R,\epsilon}(x) = ((|x|-\epsilon)(R-|x|))^{-\frac{2}{p-1}}(1+|x|)^{\frac{2}{p-1}}\big(1+(\frac{R^2}{\log\frac{c|x|}{\epsilon}})^{\frac{1}{p-1}}\big). \tag{5.15}$$

Note that the only difference between $\phi_{R,\epsilon}$ here and $\phi_{R,\epsilon}$ in the previous case is that the term $\frac{\epsilon^l}{|x|^l}$ has been changed to $(\frac{1}{\log\frac{c|x|}{\epsilon}})^{\frac{1}{p-1}}$. As before, we define

$$\psi_{R,\epsilon}(x,t) = \phi_{R,\epsilon}(x)\exp(\gamma(t+1)),$$

and convert to radial coordinates, with $|x| = r$. Note that $\frac{1}{p-1}+1 = \frac{p}{p-1}$ and $\frac{1}{p-1}+2 = \frac{2p-1}{p-1}$. In place of (5.6) and (5.7), we have

(5.16)

$$\begin{aligned}
&\exp(-\gamma(t+1))\big((r-\epsilon)(R-r)\big)^{-\frac{2}{p-1}-2}(\psi_{R,\epsilon})_{rr} = \\
&(\frac{2}{p-1})(\frac{2}{p-1}+1)(R+\epsilon-2r)^2(1+r)^{\frac{2}{p-1}}\big(1+(\frac{R^2}{\log\frac{cr}{\epsilon}})^{\frac{1}{p-1}}\big) \\
&+2(\frac{2}{p-1})(r-\epsilon)(R-r)(1+r)^{\frac{2}{p-1}}\big(1+(\frac{R^2}{\log\frac{cr}{\epsilon}})^{\frac{1}{p-1}}\big) \\
&-2(\frac{2}{p-1})^2(r-\epsilon)(R-r)(R+\epsilon-2r)(1+r)^{\frac{2}{p-1}-1}\big(1+(\frac{R^2}{\log\frac{cr}{\epsilon}})^{\frac{1}{p-1}}\big) \\
&+(\frac{2}{p-1})^2(r-\epsilon)(R-r)(R+\epsilon-2r)(1+r)^{\frac{2}{p-1}}\frac{1}{r(\log\frac{cr}{\epsilon})^{\frac{p}{p-1}}}R^{\frac{2}{p-1}} \\
&+(\frac{2}{p-1})(\frac{2}{p-1}-1)((r-\epsilon)(R-r))^2(1+r)^{\frac{2}{p-1}-2}\big(1+(\frac{R^2}{\log\frac{cr}{\epsilon}})^{\frac{1}{p-1}}\big) \\
&-(\frac{2}{p-1})^2((r-\epsilon)(R-r))^2(1+r)^{\frac{2}{p-1}-1}\frac{1}{r(\log\frac{cr}{\epsilon})^{\frac{p}{p-1}}}R^{\frac{2}{p-1}} \\
&+(\frac{1}{p-1})((r-\epsilon)(R-r))^2(1+r)^{\frac{2}{p-1}}\big(\frac{1}{r^2(\log\frac{cr}{\epsilon})^{\frac{p}{p-1}}}+\frac{p}{(p-1)r^2(\log\frac{cr}{\epsilon})^{\frac{2p-1}{p-1}}}\big)R^{\frac{2}{p-1}}
\end{aligned}$$

and

$$
\begin{aligned}
&\exp(-\gamma(t+1))\big((r-\epsilon)(R-r)\big)^{-\frac{2}{p-1}-2}\Big(\frac{d-1}{r}(\psi_{R,\epsilon})_r-\psi_{R,\epsilon}^p-(\psi_{R,\epsilon})_t\Big)=\\
&-(\frac{2}{p-1})(\frac{d-1}{r})(r-\epsilon)(R-r)(R+\epsilon-2r)(1+r)^{\frac{2}{p-1}}\big(1+(\frac{R^2}{\log\frac{cr}{\epsilon}})^{\frac{1}{p-1}}\big)\\
&+(\frac{2}{p-1})(\frac{d-1}{r})((r-\epsilon)(R-r))^2(1+r)^{\frac{2}{p-1}-1}\big(1+(\frac{R^2}{\log\frac{cr}{\epsilon}})^{\frac{1}{p-1}}\big)\\
&-(\frac{1}{p-1})(\frac{d-1}{r})((r-\epsilon)(R-r))^2(1+r)^{\frac{2}{p-1}}\frac{1}{r(\log\frac{cr}{\epsilon})^{\frac{p}{p-1}}}R^{\frac{2}{p-1}}\\
&-\gamma((r-\epsilon)(R-r))^2(1+r)^{\frac{2}{p-1}}\big(1+(\frac{R^2}{\log\frac{cr}{\epsilon}})^{\frac{1}{p-1}}\big)\\
&-(1+r)^{\frac{2p}{p-1}}\big(1+(\frac{R^2}{\log\frac{cr}{\epsilon}})^{\frac{1}{p-1}}\big)^p\exp((p-1)\gamma(t+1)).
\end{aligned}
\tag{5.17}
$$

As before, we denote the terms in (5.16) and (5.17) by $J_1 - J_7$ and $I_1 - I_5$ respectively. It's easy to see that the analysis in the previous case carries over to the present case when r satisfies $r \geq \delta_0$, where, as above, δ_0 is an arbitrary positive constant. It remains to consider $r \in [\epsilon, \delta_0]$.

Exactly as in the previous case, we have $J_5 \leq |I_4|$ and $J_2 \leq |I_4| + |I_5|$, and similar to the previous case, it is easy to see that if c is chosen sufficiently large, then $J_7 \leq |I_3|$. (For this last inequality, we use the fact that the condition $d = \frac{2p}{p-1}$ guarantees that $d > 2$.) We now consider the term I_2. Using the fact that $d - 1 = \frac{p+1}{p-1}$, and replacing $\frac{r-\epsilon}{r}$ by 1, we have

$$
I_2 \leq \frac{2(p+1)}{(p-1)^2}(r-\epsilon)(R-r)^2(1+r)^{\frac{2}{p-1}-1}\big(1+(\frac{R^2}{\log\frac{cr}{\epsilon}})^{\frac{1}{p-1}}\big),
$$

whereas

$$
|J_3| = 2(\frac{2}{p-1})^2\frac{R+\epsilon-2r}{R-r}(r-\epsilon)(R-r)^2(1+r)^{\frac{2}{p-1}-1}\big(1+(\frac{R^2}{\log\frac{cr}{\epsilon}})^{\frac{1}{p-1}}\big).
$$

Since $\frac{R+\epsilon-2r}{R-r}$ can be made arbitrarily close to 1 by choosing R sufficiently large, we have $I_2 \leq |J_3|$. (Notice that this argument does not work in the case that $d > \frac{2p}{p-1}$ if d is chosen sufficiently large. On the other hand, the method of dealing with I_2 that was used above in the case $d > \frac{2p}{p-1}$—namely, treating it together with

I_1—does not work in the present case that $d = \frac{2p}{p-1}$. It is because of this that it has been necessary to split the proof into two cases.)

Now consider the term J_4. In the case that $d > \frac{2p}{p-1}$, J_4 was treated together with I_1; in the present borderline case, this will not work. It is here that the amended form of $\phi_{R,\epsilon}$ is needed. We have

$$J_4 \leq CR^{\frac{2p}{p-1}}(\log \frac{cr}{\epsilon})^{-\frac{p}{p-1}}, \text{ for } r \in [\epsilon, \delta_0],$$

for some $C > 0$. On the other hand,

$$|I_5| \geq MR^{\frac{2p}{p-1}}(\log \frac{cr}{\epsilon})^{-\frac{p}{p-1}}, \text{ for } r \in [\epsilon, \delta_0],$$

where M can be chosen as large as one wants by choosing γ sufficiently large. Thus, by choosing γ sufficiently large, we have $J_4 \leq |I_5|$.

Finally, the term J_1 is treated as it was in the previous case, but without the addition of J_4 and I_2. Using the fact that $d = \frac{2p}{p-1}$, the analysis in (5.11)-(5.13) gives

$$J_1 + I_1 \leq C\frac{\epsilon}{r}R(1+r)^{\frac{2}{p-1}}(R+\epsilon-2r)\big(1+(\frac{R^2}{\log \frac{cr}{\epsilon}})^{\frac{1}{p-1}}\big). \tag{5.18}$$

Comparing the right hand side of (5.18) with $|I_5|$, one sees that the inequality

$$J_1 + I_1 + I_5 \leq 0, \text{ for } r \in [\epsilon, \delta_0],$$

will hold with γ chosen sufficiently large if

$$\frac{\epsilon}{r}R^2 \leq M\big(1+(\frac{R^2}{\log \frac{cr}{\epsilon}})^{\frac{1}{p-1}}\big)^{p-1}, \text{ for } r \in [\epsilon, \delta_0], \tag{5.19}$$

holds with M chosen sufficiently large. The right hand side of (5.19) is larger than $MR^2(\log \frac{cr}{\epsilon})^{-1}$; thus, (5.19) holds since $\frac{\epsilon}{r}(\log \frac{cr}{\epsilon})$ is bounded for $r \in [\epsilon, \delta_0]$, uniformly over small ϵ. This completes the proof of Theorem 6-ii. □

Acknowledgement. I thank Henri Berestycki for sending me the preprint [3]. The results there concerning the existence/nonexistence of stationary solutions in the case of periodic operators proved to be a catalyst for this paper. I also thank Haim Brezis for several useful comments on an earlier draft.

References

1. Bandle, C. and Marcus, M., *Asymptotic behaviour of solutions and their derivatives, for semilinear elliptic problems with blowup on the boundary*, Ann. Inst. H. Poincare' Anal. Non Line'aire **12** (1995), 155–171.
2. Baras, P. and Pierre, M., *Problèmes paraboliques semi-linéaires avec données mesures*, Applicable Anal. **18** (1984), 111–149.
3. Berestycki, H., Hamel, F. and Roques, L., *Analysis of the periodically fragmented environment model: I-Influence of periodic heterogeneous environment on species persistence*, prepint (2004).
4. Brezis, H., *Semilinear equations in R^N without condition at infinity*, Appl. Math. Optim. **12** (1984), 271-282.
5. Englander, J., *Criteria for the existence of positive solutions to the equation $\rho(x)\Delta u = u^2$ in R^d for all $d \geq 1$—a new probabilistic approach*, Positivity **4** (2000), 327–337.
6. Englander, J. and Pinsky, R., *Uniqueness/nonuniqueness for nonnegative solutions of second-order parabolic equations of the form $u_t = Lu + Vu - \gamma u^p$ in $\mathbf{R}^n$*, J. Differential Equations **192** (2003), 396–428.
7. Ishige, K. and Murata, M., *An intrinsic metric approach to uniqueness of the positive Cauchy problem for parabolic equations*, Math. Z. **227** (1998), 313-335.
8. Keller, J. B., *On solutions of $\Delta u = f(u)$*, Comm. Pure Appl. Math. **10** (1957), 503–510.
9. Kenig, C. and Ni, W. M., *in exterior Dirichlet problem with applications to some nonlinear equations arising in geometry*, Amer. J. Math. **106** (1984), 689-702.
10. Lieberman, G. M., *Second Order Parabolic Differential Operators*, World Scientific Publishing Co. Singapore, 1996.
11. Lin, F., *On the elliptic equation $D_i[a_{i,j}(x)D_jU] - k(x)U + K(x)U^p = 0$*, Proc. Amer. Math. Soc. (1985), 219-226.
12. Osserman, Robert, *On the inequality $\Delta u \geq f(u)$*, Pacific J. Math. **7** (1957), 1641–1647.
13. Pazy, A., *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.