THE BEHAVIOR OF THE LIFE SPAN FOR SOLUTIONS TO $u_t = \Delta u + a(x)u^p$ IN R^d

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Abstract. Let $T^*(\lambda,\phi)$ denote the life span of the positive, bounded solution $u(x,t)$ to $u_t = \Delta u + a(x)u^p$ in R^d with initial condition $u(x,0) = \lambda\phi(x)$, where $0 \lneqq a(x) \in C^\alpha(R^d)$, $0 \lneqq \phi(x) \in C_b(R^d), p > 1$, and $\lambda > 0$ is a parameter. We consider "small" initial data: $0 \lneqq \phi(x) \le \delta \exp(-\gamma|x|^2)$, where $\delta, \gamma > 0$, and "large" initial data: $c_1 \le \phi(x) \le c_2$, where $c_1, c_2 > 0$.

The life span may satisfy $T^*(\lambda,\phi) = \infty$, for λ sufficiently small, or $T^*(\lambda,\phi) < \infty$, for all $\lambda > 0$, in which case $\lim_{\lambda\to0} T^*(\lambda,\phi) = \infty$. This dichotomy depends on ϕ, a, p and d; explicit conditions are known and are stated in the paper. For all choices of ϕ, a, p and d, one has $\lim_{\lambda\to\infty} T^*(\lambda,\phi) = 0$.

In this paper, we study the asymptotic behavior of $T^*(\lambda,\phi)$ as $\lambda \to 0$ in the case that $T^*(\lambda,\phi) < \infty$, for all $\lambda > 0$, and we study the asymptotic behavior of $T^*(\lambda,\phi)$ as $\lambda \to \infty$. For two reasons, the asymptotic behavior of the life span is much more delicate in the case $\lambda \to 0$ than in the case $\lambda \to \infty$. First of all, in order to consider the asymptotics as $\lambda \to 0$, one must begin by restricting to those values of ϕ, a, p, and d for which $T^*(\lambda,\phi) < \infty$, for all $\lambda > 0$. As might be suspected, when ϕ, a, p, and d are borderline cases for the property $T^*(\lambda,\phi) < \infty$, for all $\lambda > 0$, the asymptotic rate of growth of $T^*(\lambda,\phi)$ is much faster. Second of all, since $T^*(\lambda,\phi) \to \infty$ as $\lambda \to 0$, the order of the asymptotics will depend on the global behavior of a and ϕ as well as on the dimension d. In contrast, when $\lambda \to \infty$, we have $T^*(\lambda,\phi) \to 0$, and the order of the asymptotics will not depend globally on a and ϕ; indeed, it turns out that as long as the supports of ϕ and a have a common interior point, then the asymptotic order of the life span is the same as for the ordinary differential equation $v' = v^p$. The case in which a positive distance separates the supports of ϕ and a is more interesting.

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1. **Introduction and statement of results.** In this paper, we consider bounded, positive solutions to the Cauchy problem

$$(1.1) \qquad \begin{aligned} &u_t = \Delta u + a(x)u^p, x \in R^d, t \in (0,T) \\ &u(x,0) = \lambda\phi(x), x \in R^d, \end{aligned}$$

where $0 \lneqq a(x) \in C^\alpha(R^d)$, $0 \lneqq \phi(x) \in C_b(R^d), p > 1$, and $\lambda > 0$ is a parameter. The above conditions on a and ϕ will hold throughout the paper without further mention. For most of the paper, we will consider the following two classes of initial data:

Class S. $0 \lneqq \phi(x) \leq \delta \exp(-\gamma|x|^2)$, where $\delta, \gamma > 0$.

Class L. $c_1 \leq \phi(x) \leq c_2$, where $c_1, c_2 > 0$.

Class L contains the largest admissible initial data since we are considering bounded solutions, and Class S contains all sufficiently small initial data. It can be shown that any solution with initial data from Class L becomes unbounded instantaneously if a is unbounded; thus we will always assume that a is bounded in the case of initial data from Class L, or more generally, when the initial data is unspecified. We will assume that a grows no faster than polynomially in the case of initial data from Class S. Under these conditions, it follows easily from the general theory of evolution equations [8] that there exists a unique bounded solution $u(x,t) = u(t,x;\lambda,\phi)$ to (1.1) defined on a maximal time interval $[0,T^*)$, where $T^* \equiv T^*(\lambda,\phi) \in (0,\infty]$, and such that $\lim_{t\to T^*}\sup_{x\in R^d} u(x,t) = \infty$, if $T^* < \infty$. We will call $T^*(\lambda,\phi)$ the *life span* of the solution.

The life span may satisfy $T^*(\lambda,\phi) = \infty$, for λ sufficiently small, or $T^*(\lambda,\phi) < \infty$, for all $\lambda > 0$, in which case $\lim_{\lambda\to 0} T^*(\lambda,\phi) = \infty$. This dichotomy depends on ϕ, a, p and d; explicit conditions are known and will be stated below. For all choices of ϕ, a, p and d, one has $\lim_{\lambda\to\infty} T^*(\lambda,\phi) = 0$.

In this paper, we study the asymptotic behavior of $T^*(\lambda,\phi)$ as $\lambda \to 0$ in the case that $T^*(\lambda,\phi) < \infty$, for all $\lambda > 0$, and we study the asymptotic behavior of $T^*(\lambda,\phi)$ as $\lambda \to \infty$. For two reasons, the asymptotic behavior of the life span is much more delicate in the case $\lambda \to 0$ than in the case $\lambda \to \infty$. First of all, in order to consider the asymptotics as $\lambda \to 0$, one must begin by restricting to those values of ϕ, a, p, and d for which $T^*(\lambda,\phi) < \infty$, for all $\lambda > 0$. As might be suspected, when ϕ, a, p, and d are borderline cases for the property $T^*(\lambda,\phi) < \infty$, for all $\lambda > 0$, the asymptotic rate of growth of $T^*(\lambda,\phi)$ is much faster. Second of all, since $T^*(\lambda,\phi) \to \infty$ as $\lambda \to 0$, the order of the asymptotics will depend on the global behavior of a and ϕ as well as on the dimension d. In contrast, when $\lambda \to \infty$, we have $T^*(\lambda,\phi) \to 0$, and the order of the asymptotics will not depend globally on a and ϕ; indeed, it turns out that as long as the supports of ϕ and a have a common interior point, then the asymptotic order of the life span is the same as for the ordinary differential equation $v' = v^p$. The case in which a positive distance separates the supports of ϕ and a is more interesting.

When we consider ϕ from Class S, it will be convenient to define the following possible conditions on $a(x) \gneqq 0$.

Condition $A_m, m \geq -2$.

$$c_1|x|^m \leq a(x) \leq c_2|x|^m, \text{ for } |x| \text{ sufficiently large and } c_1, c_2 > 0.$$

Condition $B_m, m < -1$.

$$a(x) \leq c|x|^m, \text{ for } |x| \text{ sufficiently large and some } c > 0.$$

Condition C_{-2}.

$$a(x) \geq c|x|^{-2}, \text{ for } |x| \text{ sufficiently large and some } c > 0.$$

We now state two results from the recent paper [9] which determine when $T^*(\lambda, \phi) < \infty$, for all $\lambda > 0$. The first result treats initial data ϕ from Class S and the second one treats initial data ϕ from Class L.

Theorem S. *Let $m \geq -2$ and assume that a satisfies Condition A_m if $m > -2$ and Condition B_{-2} if $m = -2$.*
i. Let $d \geq 2$.
a. If $1 < p \leq 1 + \frac{2+m}{d}$, then $T^*(\lambda, \phi) < \infty$, for all $\phi \gneqq 0$ and all $\lambda > 0$.
b. If $p > 1 + \frac{2+m}{d}$, then for every ϕ in Class S there exists a $\lambda_0 > 0$ such that $T^*(\lambda, \phi) = \infty$, for $\lambda < \lambda_0$.

ii. Let $d = 1$.
a. If $m > -1$ and $1 < p \leq 3 + m$, or if $-2 \leq m \leq -1$ and $1 < p \leq 2$, then $T^*(\lambda, \phi) < \infty$, for all $\phi \gneqq 0$ and all $\lambda > 0$.
b. If $m > -1$ and $p > 3 + m$, or if $-2 \leq m \leq -1$ and $p > 2$, then for every ϕ in Class S there exists a $\lambda_0 > 0$ such that $T^*(\lambda, \phi) = \infty$, for $\lambda < \lambda_0$.

Remark 1. In light of Theorem S, we define a critical exponent $p^* = p^*(d, m)$ for initial data in Class S as follows:
(1.2)

$$p^* = p^*(d,m) = \begin{cases} 1 + \frac{2+m}{d}, & \begin{array}{l}\text{if } d \geq 2 \text{ and } a \text{ satisfies Condition } A_m, m > -2, \\ \text{or if } d = 1 \text{ and } a \text{ satisfies Condition } A_m, m > -1;\end{array} \\ 2, & \begin{array}{l}\text{if } d = 1 \text{ and } a \text{ satisfies Condition } A_m, -2 < m < -1, \\ \text{or Condition } B_m, m = -2.\end{array} \end{cases}$$

If $d \geq 2$ and a satisfies Condition B_{-2}, we don't define a critical exponent since T^* can be infinite for all $p > 1$.

Remark 2. In the case $m = 0$ and $p \neq p^* = 1 + \frac{2}{d}$, the above result goes back to Fujita [3]. The case $m = 0$ and $p = p^*$ was solved by Kobayashi, Sino, and Tanaka [5] and by Aronson and Weinberger [1]. For $m > 0$ and $p \neq p^*$, the result follows from the work of Bandle and Levine [2] together with the work of Levine and Meier [7].

Theorem L. *i. Let $d = 1$ or 2, $p > 1$, and $a \gneqq 0$. Then $T^*(\lambda, \phi) < \infty$, for all ϕ in Class L and all $\lambda > 0$.*
ii. Let $d \geq 3$ and $p > 1$.
a. If a is bounded and satisfies satisfies C_{-2}, then $T^*(\lambda, \phi) < \infty$, for all ϕ in Class L and all $\lambda > 0$.
b. If a satisfies $B_{-2-\epsilon}$ for some $\epsilon > 0$, then for every ϕ in Class L there exists a $\lambda_0 > 0$ such that $T^*(\lambda, \phi) = \infty$, for $\lambda < \lambda_0$.

In light of the above results, when we study the asymptotic behavior of $T^*(\lambda,\phi)$ as $\lambda\to 0$, we will assume that the conditions of Theorem S, i-a or ii-a, or Theorem L, i or ii-a are in effect.

In the sequel, the notation $T^*(\lambda,\phi)\sim f(\lambda)$ as $\lambda\to 0$ ($\lambda\to\infty$) means that there exist positive constants $c_1,c_2>0$ such that $c_1 f(\lambda)\le T^*(\lambda,\phi)\le c_2 f(\lambda)$ for $\lambda>0$ sufficiently small (large).

We will prove the following two theorems concerning the asymptotics of $T^*(\lambda,\phi)$ as $\lambda\to 0$.

Theorem 1. *Let $T^*(\lambda,\phi)$ denote the blow-up time of the solution to (1.1). Assume that the initial data ϕ belong to Class S. If $d\ge 2$, let $m>-2$, and if $d=1$, let $m\ge -2$. Assume that a satisfies Condition A_m, if $m>-2$, and Condition B_{-2}, if $m=-2$. Let $p^*=p^*(d,m)$ be as in (1.2).*
i. Let $p\in(1,p^*)$.
a. If $(d,m)\ne(1,-1)$, then

$$T^*(\lambda,\phi)\sim\lambda^{\frac{2(1-p)}{d(p^*-p)}},\ \textit{as } \lambda\to 0.$$

b. If $(d,m)=(1,-1)$ (in which case $p^*=2$), then there exist constants $c_1,c_2>0$ such that

$$\frac{c_1}{(|\log\lambda|)^{\frac{2}{2-p}}}\lambda^{\frac{2(1-p)}{2-p}}\le T^*(\lambda,\phi)\le c_2\lambda^{\frac{2(1-p)}{2-p}},\ \textit{for small } \lambda.$$

ii. Let $p=p^*$.
a. If $(d,m)\ne(1,-1)$, then there exists a constant $c_1>0$ and for every $\epsilon>0$, a constant $c_2>0$ such that

$$c_1\lambda^{1-p^*}\le\log T^*(\lambda,\phi)\le c_2\lambda^{1-p^*-\epsilon},\ \textit{for small } \lambda.$$

b. If $(d,m)=(1,-1)$ (in which case $p^*=2$), then there exists a constant $c_1>0$ and for every $\epsilon>0$, a constant $c_2>0$ such that

$$c_1\lambda^{-\frac{1}{2}}\le\log T^*(\lambda,\phi)\le c_2\lambda^{-\frac{1}{2}-\epsilon},\ \textit{for small } \lambda.$$

Theorem 2. *Let $T^*(\lambda,\phi)$ denote the life span of the solution to (1.1) and assume that the initial data ϕ belong to Class L.*
i. Let $d=1$.
a. If a satisfies Condition $A_m, m\in(-1,0]$, then

$$T^*(\lambda,\phi)\sim\lambda^{\frac{2(1-p)}{2+m}},\ \textit{as } \lambda\to 0.$$

b. If a satisfies Condition A_{-1}, then there exists a constant $c_1>0$ and for every $\epsilon>0$ a constant $c_2>0$ such that

$$c_1\frac{\lambda^{2(1-p)}}{|\log\lambda|^2}\le T^*(\lambda,\phi)\le c_2\frac{\lambda^{2(1-p)}}{|\log\lambda|^{2-\epsilon}},\ \textit{for small } \lambda.$$

c. If a satisfies Condition $B_{-1-\epsilon}$ for some $\epsilon>0$, then

$$T^*(\lambda,\phi)\sim\lambda^{2(1-p)},\ \textit{as } \lambda\to 0.$$

ii. Let $d = 2$.
a. If a satisfies Condition $A_m, m \in (-2, 0]$, then

$$T^*(\lambda, \phi) \sim \lambda^{\frac{2(1-p)}{2+m}}, \text{ as } \lambda \to 0.$$

b. If a satisfies Condition A_{-2}, then there exists a constant $c_1 > 0$ and for every $\epsilon > 0$ a constant $c_2 > 0$ such that

$$c_1 \lambda^{\frac{1-p}{2}} \le \log T^*(\lambda, \phi) \le c_2 \lambda^{\frac{1-p}{2} - \epsilon}, \text{ for small } \lambda.$$

c. If a satisfies Condition $B_{-2-\epsilon}$, for some $\epsilon > 0$, then there exists a constant $c_1 > 0$ and for each $\epsilon > 0$ a constant $c_2 > 0$ such that

$$c_1 \lambda^{1-p} \le \log T^*(\lambda, \phi) \le c_2 \lambda^{1-p-\epsilon}, \text{ for small } \lambda.$$

iii. Let $d \ge 3$.
a. If a satisfies Condition $A_m, m \in (-2, 0]$, then

$$T^*(\lambda, \phi) \sim \lambda^{\frac{2(1-p)}{2+m}}, \text{ as } \lambda \to 0.$$

b. If a satisfies Condition A_{-2}, then there exists a constant $c_1 > 0$ and for each $\epsilon > 0$ a constant $c_2 > 0$ such that

$$c_1 \lambda^{1-p} \le \log T^*(\lambda, \phi) \le \lambda^{1-p-\epsilon}, \text{ for small } \lambda.$$

Remark. In the case that $a(x) \equiv 1$, which corresponds to $m = 0$ and to Condition A_0 in Theorems 1 and 2, Lee and Ni [6] studied $T^*(\lambda, \phi)$ as $\lambda \to 0$. Let $I(l)$, $l \ge 0$, denote the class of initial data ϕ satisfying $0 < \liminf_{|x|\to\infty} |x|^l \phi(x) \le \limsup_{|x|\to\infty} |x|^l \phi(x) < \infty$. Let $p^* = p^*(d, 0) = 1 + \frac{2}{d}$. If $1 < p \le p^*$, then as a particular case of Theorem S, or by the results cited in Remark 2 following Theorem S, it follows that $T^*(\lambda, \phi) < \infty$, for all $\lambda > 0$ and all $\phi \gneqq 0$. When $p > 1 + \frac{2}{d}$, Lee and Ni showed that if ϕ belongs to $I(l)$ for some l, then $T^*(\lambda, \phi) < \infty$, for all $\lambda > 0$ if and only if $l < \frac{2}{p-1}$. They then obtained the following asymptotic behavior for $T^*(\lambda, \phi)$ as $\lambda \to 0$: If $1 < p < p^*$ and $l > d$, then $T^*(\lambda, \phi) \sim \lambda^{\frac{2(1-p)}{d(p^*-p)}}$. If $p = p^*$ and $l > d$, then $\log T^*(\lambda, \phi) \sim \lambda^{1-p}$. When $1 < p < p^*$ and $l = d$, they obtained $T^*(\lambda, \phi) \sim (\lambda \log \frac{1}{\lambda})^{\frac{2(1-p)}{d(p^*-p)}}$; for $p = p^*$ and $l = d$, they obtained $\log T^*(\lambda, \phi) \sim \lambda^{\frac{1-p}{p}}$. When $1 < p \le p^*$ and $0 \le l < d$, and when $p > p^*$ and $l < \frac{2}{p-1}$, they obtained $T^*(\lambda, \phi) \sim \lambda^{\overline{\frac{2(1-p)}{d(p^*-p+\frac{d-l}{d}(p-1))}}}$.

Maintaining the assumption $a(x) \equiv 1$, Gui and Wang [4] improved on the asymptotics of Lee and Ni in certain particular cases as follows. Under the condition $\lim_{|x|\to\infty} \phi(x) = \phi_\infty > 0$, they showed that $\lim_{\lambda\to 0} \lambda^{p-1} T^*(\lambda, \phi) = \frac{1}{p-1} \phi_\infty^{-(p-1)}$. Under the condition that $\phi(x)$ is radially symmetric, obeys certain regularity conditions, and satisfies $\lim |x| \to \infty |x|^l \phi(x) = L > 0$, where $0 < l < \min(\frac{2}{p-1}, d)$, they showed that $\lim_{\lambda\to 0} \lambda^{\overline{\frac{2(p-1)}{d(p^*-p+\frac{d-l}{d}(p-1))}}} T^*(\lambda, \phi)$ exists and is positive.

The results we've obtained above in Theorems 1 and 2 for the asymptotics of $T^*(\lambda,\phi)$ in the case of general $a(x)$ are restricted to small initial data (Class S) and large initial data (Class L). Before one can study the asymptotics of $T^*(\lambda,\phi)$ for intermediate sized initial data of class $I(l), l>0$, one must first prove a theorem for intermediate sized initial data analogous to Theorems S and L for small and large initial data, in order to determine for which values of l (depending on m, p, and d), it will be true that $T^*(\lambda,\phi)<\infty$, for all $\lambda>0$.

We now turn to the asymptotics for $T^*(\lambda,\phi)$ as $\lambda\to\infty$.

Theorem 3. *Let $T^*(\lambda,\phi)$ denote the life span of the solution to (1.1). Assume that a is bounded and let ϕ be arbitrary bounded initial data.*
i. If there exists an $x_0\in R^d$ such that $a(x_0),\phi(x_0)>0$, then

$$T^*(\lambda,\phi)\sim\lambda^{1-p},\ \ as\ \lambda\to\infty.$$

ii. If $dist(supp(a),supp(\phi))>0$, then

$$T^*(\lambda,\phi)\sim(\log\lambda)^{-1},\ \ as\ \lambda\to\infty.$$

Remark 1. In the case $a(x)\equiv 1$, Theorem 3(i) was proved in Lee and Ni [6] and then improved upon in Gui and Wang [4] where it was shown that $\lim_{\lambda\to\infty}\lambda^{p-1}T^*(\lambda,\phi)=\frac{1}{p-1}||\phi||_\infty^{1-p}$. Note that $\lambda^{1-p}\frac{||\phi||_\infty^{1-p}}{p-1}$ is the exact life span for the ordinary differential equation $v'(t)=v^p(t)$ with $v(0)=\lambda||\phi||_\infty$.

Remark 2. One can show that if ϕ belongs to Class S, then Theorem 3 continues to hold when a is polynomially bounded. We leave this to the reader.

The proofs of Theorems 1, 2 and 3 will be given in sections 2, 3, and 4 respectively. In the sequel, we will use the notation

$$p(t,x,y)=(4\pi t)^{-\frac{d}{2}}\exp(-\frac{|y-x|^2}{4t}).$$

We conclude this section by noting the following well-known integral representation which holds for bounded solutions $u(x,t)$ to (1.1):

$$u(x,t)=\lambda\int_{R^d}p(t,x,y)\phi(y)dy+\int_0^t\int_{R^d}p(t-s,x,y)a(y)u^p(y,s)dyds. \tag{1.3}$$

2. Proof of Theorem 1. We begin with the proof of the upper bounds. For the case $1<p<p^*$, we will need the following simple lemma which follows easily from (1.3).

Lemma 1. *Let $u(x,t)$ satisfy (1.1). Then for any $t_0\in(0,T)$, there exists a $c>0$ such that*

$$u(x,t)\geq\lambda ct^{-\frac{d}{2}}\exp(-\frac{|x|^2}{2t}),\ \ for\ t\in[t_0,T),x\in R^d. \tag{2.1}$$

Proof. The proof can be found in [9]. □

In order to treat the case $p = p^*$, we will need the following important refinement of Lemma 1.

Lemma 2. *Let $u(x,t)$ satisfy (1.1) and assume that $p = p^*$, where $p^* = p^*(d,m)$ is as in (1.2). Then for any $t_0 \in (0,T)$ and any positive integer k, there exists a constant C_k such that*

$$(2.2)\qquad u(x,t) \geq C_k \lambda^{p^k} t^{-\frac{d}{2}} (\log(1+t))^{\sum_{j=0}^{k-1} p^j} \exp(-\frac{|x|^2}{t}), \ \textit{for } t \in [t_0, T), x \in R^d.$$

Furthermore, if $(d,m) = (1,-1)$, in which case $p^* = 2$, the inequality (2.2) also holds with the exponent $\sum_{j=0}^{k-1} p^j$ replaced by $2\sum_{j=0}^{k-1} p^j$.

Proof. Most of the work has already been done in [9, Proposition 1]. It was proven there that if $p = p^*$ and if

$$(2.3)\qquad u(x,t) \geq ct^{-\frac{d}{2}} \exp(-\frac{|x|^2}{t}), \quad \text{for } t \in [t_0,T), x \in R^d,$$

then (2.2) holds with $\lambda = 1$ and $k = 1$. The method of proof consisted of inserting the estimate in (2.3) into the righthand side of (1.3) and then doing some analysis. It is trivial to check the proof and verify that if c is replaced by λc in (2.3), then (2.2) holds with $k = 1$. Since (2.3) with c replaced by λc holds automatically by Lemma 1, the proof of Lemma 2 is complete in the case $k = 1$. One now proceeds by induction, taking the improved estimate (2.2) for any positive interger k, and inserting it in (1.3). Applying the same analysis in [9] noted above to get from (2.3) to (2.2) with $k = 1$, one obtains (2.2) with k replaced by $k+1$. (In order to see this more clearly, we note that if one defines $l(s) = ps + 1$, then the exponent of the logarithmic term in (2.2) is just the $(k-1)$-th iterate of $l(1)$; that is $l^{(k-1)}(1) = \sum_{j=0}^{k-1} p^j$. The point is that whenever one has an estimate of the form $u(x,t) \geq c\lambda^r t^{-\frac{d}{2}} (\log(1+t))^s \exp(-\frac{|x|^2}{t})$, for some constant c, then the analysis noted above will improve that estimate to $u(x,t) \geq C\lambda^{rp} t^{-\frac{d}{2}} (\log(1+t))^{ps+1} \exp(-\frac{|x|^2}{t})$, for some constant C.

In the special case that $(d,m) = (1,-1)$, the analysis in [9] can be refined as follows. Equation (2.35) in [9] which followed directly from Lemma 4 in that paper (the same lemma appears as Lemma 3 farther along in this paper), reads as follows in the case $(d,m) = (1,-1)$: $\int_R p(\tau,0,y)a(y)dy \geq c_1 \tau^{-\frac{1}{2}}$, for $\tau \geq 1$. However, that lemma in fact gives the stronger inequality $\int_R p(\tau,0,y)a(y)dy \geq c_1 \tau^{-\frac{1}{2}} \log(1+\tau)$, for $\tau \geq 1$. Using this stronger inequality and proceeding as above, one obtains (2.2) with the exponent $\sum_{j=0}^{k-1} p^j$ replaced by $2\sum_{j=0}^{k-1} p^j$. □

We can now give the
Proof of the upper bound. Recall that $p^*(d,m) = 1 + \frac{2+m}{d}$, if $d \geq 2$ and $m > -2$, or if $d = 1$ and $m \geq -1$. However, if $d = 1$ and $m \in [-2,-1)$, then $p^*(d,m) = 2 > 1 + \frac{2+m}{d}$. We will prove Theorem 1 under the assumption that $p^*(d,m) = 1 + \frac{2+m}{d}$, that is, under the assumption that $d \geq 2$ and $m > -2$ or that $d = 1$ and $m \geq -1$. Afterwards, we will describe how to handle the exceptional case $d = 1$ and $m \in [-2,-1)$. Thus, in what follows below, we assume that $1 < p \leq p^* = 1 + \frac{2+m}{d}$.

Let $D_n = \{x \in R^d : n < |x| < 2n\}$, if $d \geq 2$, and $D_n = \{x \in R : n < x < 2n\}$, if $d = 1$. Let $\mu_n > 0$ denote the principal eigenvalue of $-\Delta$ in D_n, and let ψ_n

denote the corresponding positive eigenfunction, normalized by $\int_{D_n} \psi_n(x)dx = 1$. Note that since D_n contains a d-dimensional cube of length kn for an appropriate constant $k \in (0,1)$, it follows that there exists a constant $c > 0$ such that

$$\mu_n \leq \frac{c}{n^2}. \tag{2.4}$$

By assumption, $a(x)$ satisfies Condition A_m for some $m > -2$; thus there exists an n_0 such that

$$a(x) \geq c_1|x|^m, \text{ for } |x| \geq n_0. \tag{2.5.}$$

From now on, we will always assume that $n \geq n_0$. Define

$$F_n(t) = \int_{D_n} u(x,t)\psi_n(x)dx, \text{ for } 0 \leq t < T^*(\lambda, \phi).$$

Let $\nu(x)$ denote the outward unit normal to D_n at $x \in \partial D_n$. Integrating by parts, using (2.4), (2.5), and the fact that $\psi_n = 0$ and $\nabla\psi_n \cdot \nu \leq 0$ on ∂D_n, and applying Jensen's inequality, we obtain

$$\begin{aligned} F_n'(t) &= \int_{D_n} u_t(x,t)\psi_n(x)dx = \int_{D_n} (\Delta u(x,t) + a(x)u^p(x,t))\psi_n(x)dx \\ &\geq -\mu_n F_n(t) + c_1 n^m \int_{D_n} u^p(x,t)\psi_n(x)dx \geq -\frac{c}{n^2}F_n(t) + c_1 n^m F_n^p(t). \end{aligned} \tag{2.6}$$

Assume for the moment that we can find a time t_n such that

$$\frac{cF_n(t_n)}{n^2} \leq \frac{1}{2}c_1 n^m F_n^p(t_n). \tag{2.7}$$

Since the expression $\frac{1}{2}c_1 n^m z^p - \frac{c}{n^2}z$ is an increasing function of z for $z \geq z_0$, where z_0 is the positive root of the aforementioned expression, it follows from (2.6) that (2.7) also holds if t_n is replaced by any $t \in (t_n, T^*(\lambda, \phi))$. Using this observation along with (2.6) and (2.7), we obtain

$$F_n'(t) \geq \frac{1}{2}c_1 n^m F_n^p(t), \text{ for } t \in [t_n, T^*(\lambda, \phi)). \tag{2.8}$$

Integrating (2.8) gives

$$\frac{F_n^{1-p}(t)}{p-1} \leq \frac{F_n^{1-p}(t_n)}{p-1} - \frac{1}{2}c_1 n^m (t - t_n), \text{ for } t \in [t_n, T^*(\lambda, \phi)). \tag{2.9}$$

Since the right hand side of (2.9) is equal to 0 when $t = t_n + \frac{2F_n^{1-p}(t_n)}{c_1(p-1)n^m}$, it follows from (2.9) that $F_n(t)$ and consequently $\sup_{D_n} u(x,t)$ must blow up by this value of t; that is,

$$T^*(\lambda, \phi) \leq t_n + \frac{2F_n^{1-p}(t_n)}{c_1(p-1)n^m}. \tag{2.10}$$

From (2.7) and (2.10), we obtain

$$T^*(\lambda,\phi) \le t_n + \frac{n^2}{c(p-1)}. \tag{2.11}$$

We now choose a t_n for which (2.7) holds. We consider the cases $1<p<p^*$ and $p=p^*$ separately. For the case $1<p<p^*$, we use Lemma 1 to conclude that there exists a $C>0$ such that $u(x,n^2)\ge C\lambda n^{-d}$, for $n<|x|<2n$; thus

$$F_n(n^2) \ge C\lambda n^{-d}. \tag{2.12}$$

Choosing $t_n = n^2$ and using (2.12), we see that (2.7) will be satisfied if $n=n(\lambda)$ is chosen such that

$$\lambda^{p-1} = \alpha n^{d(p-1)-m-2} = \alpha n^{d(p-p^*)}, \text{ for } \alpha \text{ sufficiently large},$$

or equivalently, if

$$n^2 = \alpha\lambda^{\frac{2(1-p)}{d(p^*-p)}}. \tag{2.13}$$

We conclude then that (2.11) holds if $t_n=n^2$ and n^2 satisfies (2.13). Substituting this in (2.11) gives

$$T^*(\lambda,\phi) \le const.\lambda^{\frac{2(1-p)}{d(p^*-p)}},$$

which completes the proof of the upper bound in the case $1<p<p^*$.

We now turn to the case $p=p^*$. We use Lemma 2 to conclude that for any positive integer k, there exists a constant $C_k>0$ such that $u(x,n^2)\ge C_k\lambda^{p^k}n^{-d}(\log(1+n^2))^{\sum_{j=0}^{k-1}p^j}$, for $n<|x|<2n$; thus

$$F_n(n^2) \ge C_k\lambda^{p^k}n^{-d}(\log(1+n^2))^{\sum_{j=0}^{k-1}p^j}. \tag{2.14}$$

Choosing $t_n=n^2$, using (2.14), and recalling that $p=p^*=1+\frac{2+m}{d}$, we see that (2.7) will be satisfied if $n=n(\lambda)$ is chosen such that

$$\lambda^{p^k}(\log(1+n^2))^{\sum_{j=0}^{k-1}p^j} \ge \alpha_k, \text{ for } \alpha_k \text{ sufficiently large}. \tag{2.15}$$

It follows that (2.15) will be satisfied if

$$n^2 \ge \exp(\beta\lambda^{q_k}), \text{ for } \beta>0 \text{ sufficiently large, where } q_k = -\frac{p^k}{\sum_{j=0}^{k-1}p^j}. \tag{2.16}$$

Thus, (2.11) holds if $t_n=n^2$ and n^2 satisfies (2.16) for some positive integer k. Substituting this into (2.11) and using the fact that $\lim_{k\to\infty}q_k = 1-p$, it follows that for every $\epsilon>0$, there exists a constant $\beta>0$ such that

$$T^*(\lambda,\phi) \le const.\exp(\beta\lambda^{1-p-\epsilon}), \tag{2.17}$$

which completes the proof of the upper bound in the case $p=p^*$, except when $(d,m)=(1,-1)$. When $(d,m)=(1,-1)$, it follows from Lemma 2 that the exponent $\sum_{j=0}^{k-1}p^j$ appearing in (2.14) may be replaced by $2\sum_{j=0}^{k-1}p^j$, and thus, by the same analysis as above, we obtain the estimate $T^*(\lambda,\phi) \le const.\exp(\beta\lambda^{\frac{1-p}{2}-\epsilon}) = const.\exp(\beta\lambda^{-\frac{1}{2}-\epsilon})$, where the last equality follows since $p = p^*(1,-1) = 2$.

We now discuss how to handle the exceptional case $d = 1$ and $m \in [-2,-1)$, in which case $p^*(d,m) = 2$. We define $D_n = (-n,n)$ and maintain all the other definitions. If (2.6) were to hold with m replaced by -1, then continuing the proof as before, we would obtain the desired result; namely that $T^*(\lambda,\phi) \le c\lambda^{\frac{2(1-p)}{d(2-p)}}$, if $1 < p < 2$, and that $T^*(\lambda,\phi)$ satisfies (2.17), if $p = 2$. In order to replace m by -1 in (2.6), we must show that

$$(2.18) \quad \int_{D_n} a(x)u^p(x,t)\psi_n(x)dx \ge c_1 n^{-1}\int_{D_n} u^p(x,t)\psi_n(x)dx, \text{ for some } c_1 > 0.$$

We have $\psi_n(x) = \frac{\pi}{4n}\cos(\frac{\pi x}{2n})$. Using this, it is not hard to show that (2.18) holds for all a satisfying $0 \lneqq a(x) \le c|x|^{-1}$, as long as for each t, $u(x,t)$ is nonincreasing on $(0,\infty)$ and nondecreasing on $(-\infty,0)$. Now it was shown in [9] that $u(x,t)$ is an even function which decreases on $(0,\infty)$ if a and ψ are both even and nonincreasing on $(0,\infty)$. This completes the proof for these special choices of a and ψ. For the general case, we may assume without loss of generality that there exists a point $x_0 \in R$ such that $a(x_0), \psi(x_0) \ne 0$. (Indeed if not, then we just consider the function $v(x,t) \equiv u(x,t+1)$, which satisfies the same differential equation and has strictly positive initial data.) Then one can choose $\hat{a}$ and $\hat{\psi}$ which satisfy $\hat{a} \le a$ and $\hat{\psi} \le \psi$ and such that $\hat{a}(x_0+x)$ and $\hat{\psi}(x_0+x)$ are even functions of x, nonincreasing for $x \in (0,\infty)$. Let $\hat{u}(x,t)$ denote the solution corresponding to $\hat{a}$ and $\hat{\phi}$ and let $\hat{T}^*(\hat{\phi},\lambda)$ denote its life span. From the special case above, we conclude that $\hat{T}^*(\hat{\phi},\lambda)$ satisfies the inequality in (2.17), if $p = 2$, and that $\hat{T}^*(\hat{\phi},\lambda) \le c\lambda^{\frac{2(1-p)}{d(2-p)}}$, if $1 < p < 2$. By the maximum principle, $\hat{u}(x,t) \le u(x,t)$; thus $T^*(\lambda,\phi) \le \hat{T}^*(\phi,\lambda)$, and the upper bound obtained for $\hat{T}^*(\hat{\phi},\lambda)$ holds also for $T^*(\lambda,\phi)$. □

We now turn to the
Proof of the lower bound. For the proof, we will need the following three lemmas from advanced calculus which appear as Lemmas 4,5, and 6 in [9].

Lemma 3. *(i) Let $m > -d$. If $a(x) \ge 0$ satisfies $\tilde{c}_1|x|^m \le a(x) \le \tilde{c}_2|x|^m$ for large $|x|$ and for constants $\tilde{c}_1, \tilde{c}_2 > 0$, then for any $t_0 > 0$, there exist constants $c_1, c_2 > 0$ such that*

$$c_1 t^{\frac{m}{2}} \le \int_{R^d} p(t,0,y)a(y)dy \le c_2 t^{\frac{m}{2}}, \text{ for } t \ge t_0.$$

(ii) If $a(x) \ge 0$ satisfies $\tilde{c}_1|x|^{-d} \le a(x) \le \tilde{c}_2|x|^{-d}$ for large $|x|$ and for constants $\tilde{c}_1, \tilde{c}_2 > 0$, then for any $t_0 > 0$, there exist constants $c_1, c_2 > 0$ such that

$$c_1 t^{-\frac{d}{2}} log(1+t) \le \int_{R^d} p(t,0,y)a(y)dy \le c_2 t^{-\frac{d}{2}} log(1+t), \text{ for } t \ge t_0.$$

(iii) Let $m < -d$. If $a(x) \gneqq 0$ satisfies $a(x) \le C(1+|x|)^m$, for some constant $C > 0$, then for any $t_0 > 0$, there exist constants $c_1, c_2 > 0$ such that

$$c_1 t^{-\frac{d}{2}} \le \int_{R^d} p(t,0,y)a(y)dy \le c_2 t^{-\frac{d}{2}}, \text{ for } t \ge t_0.$$

Lemma 4. *For each $m > 0$, there exists a constant $c > 0$ such that*

$$\int_{R^d} p(t,x,y)(1+|y|)^m dy \le c(1+t^{\frac{m}{2}}+|x|^m), \text{ for } x \in R^d, t > 0.$$

Lemma 5. *For $m \le 0$ and $t > 0$, the function $H(x) \equiv \int_{R^d} p(t,x,y)(1+|y|)^m dy$ attains its maximum at $x = 0$.*

We can now give the

Proof of the lower bound. To prove that a given number T provides a lower bound for $T^*(\lambda,\phi)$, we will make the following argument. Define $u_0(x,t) = \lambda \int_{R^d} p(t,x,y)\phi(y)dy$, where ϕ belongs to Class S, and

$$u_{n+1}(x,t) = u_0(x,t) + \int_0^t \int_{R^d} p(t-s,x,y)a(y)u_n^p(y,s)dyds, n \ge 0. \tag{2.19}$$

By induction, $u_{n+1}(x,t) \ge u_n(x,t)$. If

$$u(x,t) \equiv \lim_{n\to\infty} u_n(x,t) < \infty, \quad \text{for } x \in R^d \text{ and } t \in [0,T),$$

then it follows from the monotone convergence theorem and (2.19) that u satisfies (1.3) for $x \in R^d$ and $t \in (0,T)$; hence $T^*(\lambda,\phi) \ge T$. Thus, to obtain an estimate of the form $T^*(\lambda,\phi) \ge T$, it is enough to show that if

$$\phi(y) \le \delta p(k,0,y), \tag{2.20}$$

for $k, \delta > 0$, then

$$\sup_n u_n(x,t) < \infty, \text{ for } x \in R^d, t \in [0,T). \tag{2.21}$$

To obtain (2.21), we consider the inductive hypothesis

$$u_n(x,t) \le cp(t+k,0,x), \text{ for } x \in R^d, t \in [0,T), \tag{2.22}$$

where $c = c(\lambda) > 0$. Note that from (2.20), it follows that (2.22) holds for $n = 0$ with $c = \lambda\delta$ and $T = \infty$. To complete the proof of the lower bound, we will verify the inductive step above for an appropriate choice of $c = c(\lambda)$ and for $T = T(\lambda)$ satisfying the requirements of the theorem.

In the sequel C will denote a positive constant whose value will change from term to term. Using (2.19), (2.20), and (2.22), we obtain

$$\begin{aligned} &u_{n+1}(x,t) \le \lambda\delta p(t+k,0,x) \\ &+ c^p C \int_0^t \int_{R^d} p(t-s,x,y)a(y)(k+s)^{-\frac{d}{2}p} \exp\left(-\frac{p|y|^2}{4(k+s)}\right) dyds. \end{aligned} \tag{2.23}$$

Using the equality

$$\begin{aligned} &\exp\left(-\frac{|y-x|^2}{4(t-s)} - \frac{p|y|^2}{4(k+s)}\right) \\ &= \exp\left(-\frac{1}{4(t-s)R(s,t)}|y-R(s,t)x|^2\right) \exp\left(-\frac{pR(s,t)|x|^2}{4(k+s)}\right), \end{aligned}$$

where $R(s,t) = \frac{k+s}{k+s+p(t-s)}$, (2.23) can be rewritten as

$$
\begin{aligned}
&u_{n+1}(x,t) \le \lambda\delta p(t+k,0,x) \\
(2.24) \qquad &+ c^pC\int_0^t\int_{R^d} p(R(s,t)(t-s), R(s,t)x, y)a(y)(k+s)^{-\frac{d}{2}p}(R(s,t))^{\frac{d}{2}} \\
&\times \exp\left(-\frac{pR(s,t)|x|^2}{4(k+s)}\right)dyds.
\end{aligned}
$$

At this stage in the proof, we must consider two cases separately. The first case is when $m > 0$, that is when a satisfies Condition A_m with $m > 0$, and the second case is when $m \le 0$, that is when a satisfies either Condition A_m with $m \in (-2,0]$, or Condition B_m with $m = -2$. We treat the case $m > 0$ first. By assumption, $a(x) \le C(1+|x|)^m$, for some $m > 0$. We may assume in fact that $a(x) = C(1+|x|)^m$; indeed, it follows by induction that replacing $a(x)$ by $C(1+|x|)^m$ just increases u_{n+1}. Carrying out the integration over R^d in (2.24) with this choice of a, and using Lemma 4 with t and x replaced by $R(s,t)(t-s)$ and $R(s,t)x$, the final term on the right hand side of (2.24) reduces to

$$
\begin{aligned}
&c^pC\int_0^t (k+s)^{-\frac{d}{2}p}(R(s,t))^{\frac{d}{2}}\left[1+(R(s,t))^{\frac{m}{2}}(t-s)^{\frac{m}{2}}+(R(s,t))^m|x|^m\right] \\
(2.25) \qquad &\times \exp\left(-\frac{pR(s,t)|x|^2}{4(k+s)}\right)ds.
\end{aligned}
$$

Multiplying outside the integral in (2.25) by the factor $\exp\left(-\frac{|x|^2}{4(t+k)}\right)$, multiplying inside the integral by its reciprocal, and simplifying the argument in the exponential term, (2.25) may be rewritten as

(2.26)

$$
\begin{aligned}
&c^pC\exp\left(-\frac{|x|^2}{4(t+k)}\right)\int_0^t (k+s)^{-\frac{d}{2}p}(R(s,t))^{\frac{d}{2}}[1+(R(s,t))^{\frac{m}{2}}(t-s)^{\frac{m}{2}}+(R(s,t))^m|x|^m] \\
&\times \exp\left(-\frac{(p-1)R(s,t)|x|^2}{4(t+k)}\right)ds.
\end{aligned}
$$

We now write $(R(s,t))^m|x|^m\exp\left(-\frac{(p-1)R(s,t)|x|^2}{4(t+k)}\right) = (R(s,t))^{\frac{m}{2}}z^{\frac{m}{2}}\exp\left(-\frac{(p-1)z}{4(t+k)}\right)$, where

$z = R(s,t)|x|^2$. Differentiating and using the fact that $p > 1$, it is easy to check that as a function of $z > 0$, the expression $z^{\frac{m}{2}}\exp\left(-\frac{(p-1)z}{4(t+k)}\right)$ attains its maximum at $z = \frac{2(t+k)m}{(p-1)}$. The maximum value then is $(\frac{2(t+k)m}{(p-1)})^{\frac{m}{2}}\exp(-\frac{m}{2})$. From this it follows that

$$
\begin{aligned}
(2.27) \qquad &(R(s,t))^m|x|^m\exp\left(-\frac{(p-1)R(s,t)|x|^2}{4(t+k)}\right) \le C(R(s,t))^{\frac{m}{2}}(t+k)^{\frac{m}{2}}, \\
&\text{for all } x \in R^d, t > 0, \text{ and } 0 < s < t.
\end{aligned}
$$

From (2.27) and the fact that $p > 1$, it follows that the expression in (2.26) is smaller than

(2.28)

$$
\begin{aligned}
&c^pC\exp(-\frac{|x|^2}{4(t+k)}) \\
&\times\left[\int_0^t (k+s)^{-\frac{d}{2}p}(R(s,t))^{\frac{d}{2}}ds + \int_0^t (k+s)^{-\frac{d}{2}p}(R(s,t))^{\frac{d}{2}+\frac{m}{2}}[(t-s)^{\frac{m}{2}}+(t+k)^{\frac{m}{2}}]ds\right].
\end{aligned}
$$

We now carry out the integration in (2.28), making the change of variables $u = \frac{s}{t}$. Recalling that $p \leq p^* = 1 + \frac{2+m}{d}$, recalling that $R(s,t) = \frac{k+s}{k+s+p(t-s)}$, and noting that $k + t \leq k + tu + pt(1-u) < p(k+t)$, for $u \in [0,1]$, we have

(2.29)

$$\int_0^t (k+s)^{-\frac{d}{2}p}(R(s,t))^{\frac{d}{2}}ds = \int_0^1 (k+tu)^{-\frac{d}{2}p}\left(\frac{k+tu}{k+tu+pt(1-u)}\right)^{\frac{d}{2}} tdu$$

$$= (t+k)^{-\frac{d}{2}} \int_0^1 (k+tu)^{\frac{d}{2}(1-p)}\left(\frac{k+t}{k+tu+pt(1-u)}\right)^{\frac{d}{2}} tdu$$

$$\leq C(t+k)^{-\frac{d}{2}} \int_0^1 (k+tu)^{\frac{d}{2}(1-p)}du \leq \begin{cases} C(t+k)^{-\frac{d}{2}}, & \text{if } p > 1+\frac{2}{d} \\ C(t+k)^{-\frac{d}{2}}\log(t+k), & \text{if } p = 1+\frac{2}{d} \\ C(t+k)^{1-p\frac{d}{2}}, & \text{if } p < 1+\frac{2}{d} \end{cases}$$

and

(2.30)

$$\int_0^t (k+s)^{-\frac{d}{2}p}(R(s,t))^{\frac{d}{2}+\frac{m}{2}}\left[(t-s)^{\frac{m}{2}} + (t+k)^{\frac{m}{2}}\right]ds$$

$$= \int_0^1 (k+tu)^{-\frac{d}{2}p}\left(\frac{k+tu}{k+tu+pt(1-u)}\right)^{\frac{d}{2}+\frac{m}{2}}\left[t^{\frac{m}{2}}(1-u)^{\frac{m}{2}} + (t+k)^{\frac{m}{2}}\right] tdu$$

$$\leq C(t+k)^{-\frac{d}{2}} \int_0^1 (k+tu)^{(1-p)\frac{d}{2}+\frac{m}{2}} tdu \leq \begin{cases} C(t+k)^{-\frac{d}{2}+\frac{d}{2}(p^*-p)}, & \text{if } p < p^* \\ C(t+k)^{-\frac{d}{2}}\log(t+k), & \text{if } p = p^*. \end{cases}$$

From (2.25), (2.26), (2.28), (2.29), and (2.30), we conclude now that if $p < p^*$, then the final term on the right hand side of (2.24) is smaller than $c^pC(t+k)^{-\frac{d}{2}+\frac{d}{2}(p^*-p)}\exp\left(-\frac{|x|^2}{4(t+k)}\right)$, and if $p = p^*$, then the final term on the right hand side of (2.24) is smaller than $c^pC(t+k)^{-\frac{d}{2}}\log(t+k)\exp\left(-\frac{|x|^2}{4(t+k)}\right)$. Substituting this in (2.24), we obtain

$$\begin{aligned} u_{n+1}(x,t) &\leq \lambda\delta p(t+k,0,x) + c^pC(t+k)^{-\frac{d}{2}+\frac{d}{2}(p^*-p)}\exp\left(-\frac{|x|^2}{4(t+k)}\right) \\ &= \left(\lambda\delta + c^pC(t+k)^{\frac{d}{2}(p^*-p)}\right)p(t+k,0,x), \text{ for } x \in R^d, t \geq 0, \text{ if } p < p^*, \end{aligned} \tag{2.31}$$

and

$$\begin{aligned} u_{n+1}(x,t) &\leq \lambda\delta p(t+k,0,x) + c^pC(t+k)^{-\frac{d}{2}}\log(t+k)\exp\left(-\frac{|x|^2}{4(t+k)}\right) \\ &= \left(\lambda\delta + c^pC\log(t+k)\right)p(t+k,0,x), \text{ for } x \in R^d, t \geq 0, \text{ if } p = p^*. \end{aligned} \tag{2.32}$$

Choosing $c = c(\lambda) = 2\lambda\delta$, we find that the inequality $\lambda\delta + c^pC(t+k)^{\frac{d}{2}(p^*-p)} \leq c$ will hold for small λ as long as

$$t \leq T = T(\lambda) \equiv \begin{cases} C\lambda^{\frac{2(1-p)}{d(p^*-p)}}, & \text{if } p < p^* \\ \exp(-C\lambda^{1-p}), & \text{if } p = p^*. \end{cases}$$

It then follows from (2.31) and (2.32) that

$$u_{n+1}(x,t) \le cp(t+k,0,x), \text{ for } x \in R^d, t \in [0,T).$$

This verifies the inductive hypothesis (2.22) and proves that $T^*(\lambda,\phi) \ge T(\lambda)$.

We now turn to the case $m \in [-2,0]$, that is, the case in which a satisfies Condition A_m with $m \in (-2,0]$ or Condition B_m with $m=-2$. (Recall that the case in which a satisfies Condition B_{-2}, is allowed only if $d=1$.) By assumption, $a(x) \le C(1+|x|)^m$, for some $m \in [-2,0]$. As before, we may assume in fact that $a(x) = C(1+|x|)^m$. With this choice of a, it follows from Lemma 5 that the inside integral, $\int_{R^d} p\left(R(s,t)(t-s), R(s,t)x, y\right)(1+|y|)^m dy$, appearing on the right hand side of (2.24), attains its maximum as a function of x when $x=0$. Thus, the final term on the right hand side of (2.24) is less that or equal to

$$\begin{aligned}(2.33)\qquad & c^pC\int_0^t\int_{R^d} p\left(R(s,t)(t-s),0,y\right)(1+|y|)^m(k+s)^{-\frac{d}{2}p}(R(s,t))^{\frac{d}{2}} \\ & \times \exp\left(-\frac{pR(s,t)|x|^2}{4(k+s)}\right)dyds.\end{aligned}$$

We now appeal to Lemma 3 to carry out the integration over y in (2.33). Using Lemma 3, whose inequalities hold for, say, $t \ge 1$, along with the fact that $\int_{R^d} p(t,0,y)\,(1+|y|)^m\,dy \le 1$, for $t \in [0,1]$ and $m \le 0$, it follows that

(2.34)

$$\int_{R^d} p(t,0,y)(1+|y|)^m dy \le \begin{cases} Ct^{\frac{m}{2}}, \quad \text{for } t>0, & \text{if } m \in (-2,0] \text{ and } d \ge 2 \text{ or } m \in (-1,0] \text{ and } d=1 \\ Ct^{-\frac12} \text{ for } t>0, & \text{if } m \in [-2,-1) \text{ and } d=1 \\ Ct^{-\frac12}\log(2+t), \text{ for } t>0, & \text{if } m=-1 \text{ and } d=1 \end{cases}.$$

We will complete the proof under the assumption that $m \in (-2,1]$ and $d \ge 2$ or that $m \in (-1,0]$ and $d=1$, and leave it to the reader to complete the proof in the two other cases spelled out in (2.34), using the very same argument. Applying (2.34) with t replaced by $R(s,t)(t-s)$, it follows that the expression in (2.33) is less than or equal to

$$(2.35)\qquad c^pC\int_0^t \left(R(s,t)(t-s)\right)^{\frac{m}{2}}(k+s)^{-\frac{d}{2}p}(R(s,t))^{\frac{d}{2}}\exp\left(-\frac{pR(s,t)|x|^2}{4(k+s)}\right)ds.$$

Since $p>1$ and since $\frac{R(s,t)}{k+s} = \frac{1}{k+s+p(t-s)} \ge \frac{1}{k+pt}$, for $s \in [0,t]$, it follows that $\exp\left(-\frac{pR(s,t)|x|^2}{4(k+s)}\right) \le \exp\left(-\frac{|x|^2}{4(t+k)}\right)$. Therefore, the expression in (2.35) is less than or equal to

$$(2.36)\qquad c^pC\exp\left(-\frac{|x|^2}{4(t+k)}\right)\int_0^t \left(R(s,t)(t-s)\right)^{\frac{m}{2}}(k+s)^{-\frac{d}{2}p}(R(s,t))^{\frac{d}{2}}ds.$$

We now carry out the integration in (2.36), making the substitution $u=\frac{s}{t}$. Recalling that $1<p\le p^* = 1+\frac{2+m}{d}$, that $m \in (-2,0]$, and that $R(s,t) =$

$\frac{k+s}{k+s+p(t-s)}$, and using the fact that $k+t \leq k+tu+pt(1-u) < p(k+t)$, for $u \in [0,1]$, we have

$$
\begin{aligned}
&\int_0^t (R(s,t)(t-s))^{\frac{m}{2}} (k+s)^{-\frac{d}{2}p}(R(s,t))^{\frac{d}{2}} ds \\
&\leq \frac{Ct^{\frac{m}{2}}}{(t+k)^{\frac{m}{2}+\frac{d}{2}}} \int_0^1 (k+tu)^{\frac{m}{2}+(1-p)\frac{d}{2}}(1-u)^{\frac{m}{2}} tdu \\
&\leq C(t+k)^{-\frac{d}{2}} \int_0^1 (k+tu)^{\frac{m}{2}+(1-p)\frac{d}{2}}(1-u)^{\frac{m}{2}} tdu \\
&\leq C(t+k)^{-\frac{d}{2}+\frac{d}{2}(p^*-p)}.
\end{aligned} \tag{2.37}
$$

From (2.24), (2.33), (2.35), (2.36), and (2.37), we conclude that

$$
u_{n+1}(x,t) \leq \left(\lambda\delta + c^p C(t+k)^{\frac{d}{2}(p^*-p)}\right) p(t+k,0,x). \tag{2.38}
$$

The rest of the proof is now the same as in the case $m > 0$, starting after (2.32).□

3. Proof of Theorem 2. We begin with the

Proof of the upper bound. We will prove the upper bound under the assumption that a satisfies Condition A_m for some $m \geq -1$, if $d = 1$, and that a satisfies Condition A_m for some $m \geq -2$, if $d \geq 2$. After the completion of the proof we will describe how to prove the exceptional cases in which $d = 1$ and a satisfies Condition $B_{-1-\epsilon}$ for some $\epsilon > 0$, or $d = 2$ and a satisfies Condition $B_{-2-\epsilon}$, for some $\epsilon > 0$.

As in the proof of the upper bound in Theorem 1, we define $D_n = \{x \in R^d : n < |x| < 2n\}$, if $d \geq 2$, and $D_n = \{x \in R : n < x < 2n\}$, if $d = 1$, we let $\mu_n > 0$ denote the principal eigenvalue of $-\Delta$ in D_n, we let ψ_n denote the corresponding positive eigenfunction, normalized by $\int_{D_n} \psi_n(x)dx = 1$, and we define $F_n(t) = \int_{D_n} u(x,t)\psi_n(x)dx$, for $0 \leq t < T^*(\lambda,\phi)$. The analysis made between (2.4) and (2.11) shows that (2.11) holds as long as (2.7) holds.

We break the rest of the proof up into two cases. The first case is when $d = 1$ and a satisfies Condition A_m for some $m > -1$, or $d \geq 2$ and a satisfies Condition A_m for some $m > -2$. In this case, we choose $t_n = 0$. Since $u(x,0) \geq k\lambda$ for some constant $k > 0$, we have $F_n(0) \geq k\lambda$. It then follows that (2.7) will hold with $t_n = 0$ as long as $(k\lambda)^{p-1} \geq \frac{2c}{c_1} n^{-m-2}$. Thus, we may choose $n^2 = k_1 \lambda^{\frac{2(1-p)}{m+2}}$, for sufficiently large k_1. Choosing n^2 as above, substituting it in (2.11), and setting $t_n = 0$ completes the proof of the upper bound.

For the case in which either $d = 1$ and a satisfies Condition A_{-1}, $d = 2$ and a satisfies Condition A_{-2}, or $d \geq 3$ and a satisfies Condition A_{-2}, as well as for the exceptional case $d = 2$ and a satisfies Condition $B_{-2-\epsilon}$, we need the following lemma.

Lemma 6. *Let $u(x,t)$ satisfy (1.1) and assume that the initial data ϕ belong to Class L.*
i. Let $d = 1$ and let a satisfy Condition A_{-1}. Then for any positive integer k, there exists a constant $c > 0$ such that

$$
u(x,t) \geq c\lambda^{p^k} \exp(-\frac{x^2}{ct}) \left((t+1)^{\frac{1}{2}} \log(1+t)\right)^{(\sum_{j=0}^{k-1} p^j)}.
$$

ii. Let $d = 2$ and a satisfy Condition A_{-2}. Then for any positive integer k, there exists a constant $c > 0$ such that

$$u(x,t) \geq c\lambda^{p^k} \exp(-\frac{|x|^2}{ct})(\log(1+t))^{2(\sum_{j=0}^{k-1} p^j)}.$$

iii. Let either $d \geq 3$ and a satisfy Condition A_{-2}, or $d = 2$ and a satisfy Condition $B_{-2-\epsilon}$, for some $\epsilon > 0$. Then for any positive integer k, there exists a constant $c > 0$ such that

$$u(x,t) \geq c\lambda^{p^k} \exp(-\frac{|x|^2}{ct})(\log(1+t))^{(\sum_{j=0}^{k-1} p^j)}.$$

Proof. The proofs, which rely on Lemma 3, are similar; thus we will only prove (i). In the sequel, c will denote a positive constant whose value will change from term to term. By Lemma 3 and the assumption on a,

$$\int_R p(s,x,y)a(y)dy \geq c(s+1)^{-\frac{1}{2}} \log(1+s), \text{ for } s > 0. \tag{3.1}$$

Since ϕ belongs to Class L, the first term on the right hand side of (1.3) is larger or equal to $c\lambda$ and thus $u(x,t) \geq c\lambda$. Substituting this into the second term on the right hand side of (1.3), and using (3.1), we obtain

$$\begin{aligned} u(x,t) &\geq c\lambda^p \int_0^t \int_R p(t-s,x,y)a(y)dyds \\ &\geq c\lambda^p \int_{t_0}^t (s+1)^{-\frac{1}{2}} \log(1+s)ds \geq c\lambda^p (t+1)^{\frac{1}{2}} \log(1+t), \end{aligned}$$

which proves the lemma in the case $k = 1$. We now proceed by induction, assuming the estimate to hold for a positive integer k. Taking this estimate, and substituting again into the second term on the right hand side of (1.3), using (3.1), and recalling that c changes from line to line, we obtain

$$\begin{aligned} &u(x,t) \geq c\lambda^{p^{k+1}} \int_0^t \int_R p(t-s,x,y)a(y)\exp(-\frac{px^2}{cs}) \Big((s+1)^{\frac{1}{2}} \log(1+s)\Big)^{\sum_{j=1}^k p^j} dyds \\ &\geq c\lambda^{p^{k+1}} \int_0^t \int_R (4\pi(t-s))^{-\frac{1}{2}} \exp(-\frac{y^2}{2(t-s)}) \exp(-\frac{x^2}{2(t-s)} - \frac{px^2}{cs}) \\ &\times a(y) \Big((s+1)^{\frac{1}{2}} \log(1+s)\Big)^{\sum_{j=1}^k p^j} dyds \\ &\geq c\lambda^{p^{k+1}} \int_0^t (1+t-s)^{-\frac{1}{2}} \log(1+t-s) \Big((s+1)^{\frac{1}{2}} \log(1+s)\Big)^{\sum_{j=1}^k p^j} \\ &\times \exp(-\frac{x^2}{2(t-s)} - \frac{px^2}{cs})ds \\ &\geq c\lambda^{p^{k+1}} \exp(-\frac{x^2}{ct}) \int_{\frac{t}{4}}^{\frac{t}{2}} (1+s)^{\frac{1}{2}(\sum_{j=1}^k p^j - 1)} (\log(1+s))^{\sum_{j=0}^k p^j} ds \\ &\geq c\lambda^{p^{k+1}} \exp(-\frac{x^2}{ct}) \Big((1+t)^{\frac{1}{2}} \log(1+t)\Big)^{\sum_{j=0}^k p^j}. \end{aligned}$$

$\square$

We now prove the upper bound when $d = 1$ and a satisfies Condition A_{-1}. As above, c will denote a positive constant whose value changes from term to term. By Lemma 6-i, it follows that $F_n(t) \geq c\lambda^{p^k}((1+t)^{\frac{1}{2}}\log(1+t))^{\sum_{j=0}^{k-1} p^j}$, for some $c > 0$. Recalling that $m = -1$ in the present case, it follows then that (2.7) will hold with $t_n = n^2$, if we choose n to satisfy

$$\left[((1+n^2)^{\frac{1}{2}}\log(1+n^2))^{\sum_{j=0}^{k-1} p^j}\lambda^{p^k}\right]^{p-1} \geq \frac{c}{n}. \tag{3.2}$$

Using the fact that $\sum_{j=0}^{k-1} p^j = \frac{p^k-1}{p-1}$, we find that (3.2) will hold if $n(\log(1+n))^{1-p^{-k}} \geq c\lambda^{1-p}$. Thus we may pick $n = c\frac{\lambda^{1-p}}{|\log\lambda|^{1-p^{-k}}}$. Substituting this along with $t_n = n^2$ in (2.11), and using the fact that k can be chosen arbitrarily large, it follows that for any $\epsilon > 0$, there exists a $c > 0$ such that $T^*(\lambda,\phi) \leq c\frac{\lambda^{2(1-p)}}{|\log\lambda|^{2-\epsilon}}$. This proves the upper bound.

The proof for the case in which $d = 2$ and a satisfies Condition A_{-2} is identical to the proof for the case in which $d \geq 3$ and a satisfies Condition A_{-2}, except for the fact that in the former case one invokes Lemma 6-ii and in the latter case Lemma 6-iii. Thus we will only prove the latter case. As before, c denotes a positive constant whose value changes from term to term. Since $m = -2$, it follows that (2.7) will hold as long as

$$F_n(t_n) \geq c, \quad \text{for sufficiently large } c, \text{ independent of } n. \tag{3.3}$$

Thus, choose $n = n_0$. (Recall from the line following (2.5) that we are always assuming that $n \geq n_0$.) By Lemma 6-iii, $F_{n_0}(t) \geq c\lambda^{p^k}(\log(1+t))^{\sum_{j=0}^{k-1} p^j}$. Thus (3.3) will hold with $n = n_0$ and $t_{n_0} = \exp(c\lambda^{-q_k})$, where $q_k = \frac{p^k}{\sum_{j=0}^{k-1} p^j}$. Substituting $n = n_0$ and t_{n_0} as above in (2.11) , we obtain $\log T^*(\lambda,\phi) \leq c\lambda^{-q_k}$. Since k is arbitrary and $\lim_{k\to\infty} q_k = p-1$, it follows that for any $\epsilon > 0$, there exists a $c > 0$ such that $\log T^*(\lambda,\phi) \leq c\lambda^{1-p-\epsilon}$. This proves the upper bound.

We now discuss how to handle the two exceptional cases. The case in which $d = 1$ and a satisfies Condition $B_{-1-\epsilon}$, for some $\epsilon > 0$, is treated exactly as we treated the exceptional case $d = 1$ and $m \in [-2,-1)$ in Theorem 2. Namely, in the special case that a and ϕ are even functions, nonincreasing on $(0,\infty)$, the solution $u(x,t)$ is also even and nonincreasing on $(0,\infty)$ for each $t \geq 0$. From this, it follows that (2.18) holds, and thus (2.7) holds with $m = -1$. The rest of the proof now continues like the proof above in the case $m \in (-1,0]$, except that one sets $m = -1$. This gives the upper bound for these special a and ϕ, and the general case then follows by the application of the maximum principle used to complete the proof of the exceptional case in Theorem 2.

The case in which $d = 2$ and a satisfies Condition $B_{-2-\epsilon}$ for some $\epsilon > 0$ is treated similarly. We consider the special case that a and ϕ are radially symmetric and nonincreasing in $|x|$ and let $D_n = \{x \in R^2 : |x| < n\}$. In this case it can by shown that $u(x,t)$ is radially symmetric and decreasing in $|x|$, from which it is easy to show that (2.18) holds with n^{-1} replaced by n^{-2}. Thus, (2.7) holds with $m = -2$. Noting that Lemma 6-(iii) holds for the case $d \geq 3$ and a satisfying Condition A_{-2} as well as for the case at hand, the rest of the proof goes through just as the proof above in the case $d \geq 3$ and a satisfying Condition A_{-2}. This gives the upper bound for these special a and ϕ. The general case follows by the application of the maximum principle used to complete the proof of the exceptional case in Theorem 2.

We now turn to the

Proof of the lower bound. We will use the same type of idea used to prove the lower bound in Theorem 1; however, the calculations are much simpler here. Define $u_0(x,t) = \lambda \int_{R^d} p(t,x,y)\phi(y)dy$, where ϕ belongs to Class L, and

$$u_{n+1}(x,t) = u_0(x,t) + \int_0^t \int_{R^d} p(t-s,x,y)a(y)u_n^p(y,s)dyds, n \geq 0. \tag{3.4}$$

By induction, $u_{n+1}(x,t) \geq u_n(x,t)$. If

$$u(x,t) \equiv \lim_{n\to\infty} u_n(x,t) < \infty, \quad \text{for } x \in R^d \text{ and } t \in [0,T),$$

then it follows from the monotone convergence theorem and (3.4) that u satisfies (1.3) for $x \in R^d$ and $t \in (0,T)$; hence $T^*(\lambda,\phi) \geq T$. Thus, to obtain an estimate of the form $T^*(\lambda,\phi) \geq T$, it is enough to show that if

$$\phi(y) \leq \delta, \tag{3.5}$$

for $\delta > 0$, then

$$\sup_n u_n(x,t) < \infty, \text{ for } x \in R^d, t \in [0,T). \tag{3.6}$$

To obtain (3.6), we consider the inductive hypothesis

$$u_n(x,t) \leq c \text{ for } x \in R^d, t \in [0,T), \tag{3.7}$$

where $c = c(\lambda) > 0$. Note that (3.7) holds for $n = 0$ with $c = \lambda\delta$ and $T = \infty$. To complete the proof of the lower bound, we will verify the inductive step above for an appropriate choice of $c = c(\lambda)$ and for $T = T(\lambda)$ satisfying the requirements of the theorem. In the sequel, C will denote a positive constant whose value will change from term to term. By (3.4), (3.5) and (3.7), we have

$$u_{n+1}(x,t) \leq \lambda\delta + c^p \int_0^t \int_{R^d} p(t,x,y)a(y)dyds. \tag{3.8}$$

At this point, we apply Lemma 3. Since the proofs are very similar in all the cases, we will content ourselves with proving the "typical case"—namely, the case in which either $d = 1$ and a satisfies Condition $A_m, m \in (-1,0]$, or $d \geq 2$ and a satisfies Condition $A_m, m \in (-2,0]$—and one of the exceptional cases. First consider the "typical case" defined above. Then by Lemma 3 and (3.8), we obtain

$$u_{n+1}(x,t) \leq \lambda\delta + Cc^p \int_0^t s^{\frac{m}{2}} ds \leq \lambda\delta + Cc^p t^{\frac{m}{2}+1}. \tag{3.9}$$

Choosing $c = c(\lambda) = 2\lambda\delta$, it follows that the inequality $\lambda\delta + Cc^p t^{\frac{m}{2}+1} \le c$ will hold as long as $t \le T = T(\lambda) = C\lambda^{\frac{2(1-p)}{m+2}}$. It then follows from (3.9) that $u_{n+1}(x,t) \le c$, for $x \in R^d, t \in [0,T)$. This verifies the inductive hypothesis (3.7) and proves that $T^*(\lambda,\phi) \ge T(\lambda)$.

We now prove the lower bound in one of the exceptional cases—the case that $d = 1$ and that a satisfies Condition A_{-1}. Applying Lemma 3 to (3.8), we obtain

$$u_{n+1}(x,t) \le \lambda\delta + Cc^p \int_0^t s^{-\frac{1}{2}} \log(2+s)ds \le \lambda\delta + Cc^p t^{\frac{1}{2}} \log(2+t). \tag{3.10}$$

Choosing $c(\lambda) = 2\lambda\delta$, it follows that the inequality $\lambda\delta + Cc^p t^{\frac{1}{2}} \log(2+t) \le c$ will hold as long as $t \le T = T(\lambda) = C\frac{\lambda^{2(1-p)}}{|\log\lambda|^2}$. It then follows from (3.10) that $u_{n+1}(x,t) \le c$, for $x \in R^d, t \in [0,T)$. This verifies the inductive hypothesis (3.7) and proves that $T^*(\lambda,\phi) \ge T(\lambda)$. □

4. Proof of Theorem 3. *Proof of part (i)—upper bound.* Choose a smooth bounded domain $D \subset R^d$ for which

$$c_1 \equiv \inf_{x\in D} a(x) > 0, c_2 \equiv \inf_{x\in D} \phi(x) > 0. \tag{4.1}$$

Let $\mu > 0$ denote the principal eigenvalue of $-\Delta$ in D and let ψ denote the corresponding positive eigenfunction, normalized by $\int_D \psi(x)dx = 1$. Define

$$F(t) = \int_D u(x,t)\psi(x)dx, \text{ for } 0 \le t < T^*(\lambda,\phi).$$

Let $\nu(x)$ denote the outward unit normal to D at $x \in \partial D$. Integrating by parts, using (4.1) and the fact that $\psi = 0$ and $\nabla\psi \cdot \nu \le 0$ on ∂D, and applying Jensen's inequality, we obtain

$$\begin{aligned} F'(t) &= \int_D u_t(x,t)\psi(x)dx = \int_D (\Delta u(x,t) + a(x)u^p(x,t))\psi(x)dx \\ &\ge -\mu F(t) + c_1 \int_D u^p(x,t)\psi(x)dx \ge -\mu F(t) + c_1 F^p(t). \end{aligned} \tag{4.2}$$

By (4.1), it follows that $F(0) \ge c_2\lambda$. Thus there exists a λ_0 such that $\mu F(0) \le \frac{1}{2}c_1 F^p(0)$, for $\lambda \ge \lambda_0$. From now on, we will always assume that $\lambda \ge \lambda_0$. Since the expression $-\mu z + \frac{1}{2}c_1 z^p$ is an increasing function of z for $z \ge z_0$, where z_0 is the positive root of the aforementioned expression, it then follows from (4.2) that $\mu F(t) \le \frac{1}{2}c_1 F^p(t)$, for $t \in [0, T^*(\lambda,\phi))$. Thus we obtain from (4.2) that $F'(t) \ge \frac{1}{2}c_1 F^p(t)$, for $t \in [0, T^*(\lambda,\phi))$. Integrating gives

$$\frac{F^{1-p}(t)}{p-1} \le \frac{F^{1-p}(0)}{p-1} - \frac{1}{2}c_1 t \le \frac{(c_2\lambda)^{1-p}}{p-1} - \frac{1}{2}c_1 t. \tag{4.3}$$

Since the righthand side of (4.3) equals 0, when $t = \frac{2(c_2\lambda)^{1-p}}{c_1(p-1)}$, it follows that $F(t)$ must blow up by this value of t. This gives the upper bound $T^*(\lambda,\phi) \le C\lambda^{1-p}$, for some constant $C > 0$.

Proof of part (i)—lower bound. We argue exactly as we did from (3.4) until (3.8) for the lower bound in Theorem 2. From (3.8), it follows that $u_{n+1} \leq \lambda\delta + Cc^p t$, for $t \in [0,1]$ and some $C > 0$. Choosing $c = c(\lambda) = 2\lambda\delta$, it follows that the inequality $\lambda\delta + Cc^p t \leq c$ will hold as long as $t \leq \frac{(\delta\lambda)^{1-p}}{C}$ and $t \leq 1$. Thus, $T^*(\lambda,\phi) \geq const.\lambda^{1-p}$, for λ sufficiently large.

Proof of part (ii)—upper bound. Let $D \subset R^d$ be a smooth bounded domain for which

$$c_1 \equiv \inf_{x\in D} a(x) > 0. \tag{4.4}$$

Choose l such that $l > dist(x, supp(\phi))$, for all $x \in D$. It then follows from (1.3) that

(4.5)
$$u(x,t) \geq \lambda \int_{R^d} \frac{1}{(4\pi t)^{-\frac{d}{2}}} \exp(-\frac{|y-x|^2}{4t})\phi(y)dy \geq \lambda C t^{-\frac{d}{2}} \exp(-\frac{l^2}{4t}), \text{ for } x \in D, t > 0.$$

Now define $F(t)$ exactly as in the proof of part(i) above. By (4.5),

$$F(t_0) \geq \lambda^\gamma, \text{ for some } \gamma > 0 \text{ and } t_0 = \frac{C}{\log\lambda}, \text{ with } C \text{ sufficiently large.} \tag{4.6}$$

In particular then, there exists a λ_0 such that the inequality $\mu F(t) \leq \frac{1}{2}c_1 F^p(t)$ holds for $t = t_0$ and $\lambda \geq \lambda_0$ From now on, we assume that $\lambda \geq \lambda_0$. From (4.2), it follows that the above inequality in fact holds for all $t \geq t_0$; thus $F'(t) \geq \frac{1}{2}c_1 F^p(t)$, for $t \geq t_0$, and integrating gives

$$\frac{F^{1-p}(t)}{p-1} \leq \frac{F^{1-p}(t_0)}{p-1} - \frac{1}{2}c_1(t-t_0). \tag{4.7}$$

As in the proof of part (i), the value of t for which the righthand side of (4.7) equals zero constitutes an upper bound for $T^*(\lambda,\phi)$. Thus, from (4.6) and (4.7), we obtain

$$T^*(\lambda,\phi) \leq t_0 + \frac{2F^{1-p}(t_0)}{c_1(p-1)} \leq \frac{C}{\log\lambda} + \frac{2\lambda^{\gamma(1-p)}}{c_1(p-1)} \leq \frac{C_1}{\log\lambda},$$

which completes the proof of the upper bound.

Proof of part (ii)—lower bound. Let $D_1 = supp(\phi)$ and $D_2 = supp(a)$. Without loss of generality, assume that $a(x) \leq 1$ and $\phi(x) \leq 1$. Define $u_0(x,t) = \lambda \int_{R^d} p(t,x,y)\phi(y)dy$ and

$$u_{n+1}(x,t) = u_0(x,t) + \int_0^t \int_{R^d} p(t-s,x,y)a(y)u_n^p(y,s)dyds, n \geq 0. \tag{4.8}$$

By induction, $u_{n+1}(x,t) \geq u_n(x,t)$. If

$$u(x,t) \equiv \lim_{n\to\infty} u_n(x,t) < \infty, \quad \text{for } x \in D_2 \text{ and } t \in [0,T),$$

then since a is supported on D_2, it follows from (4.8) that $u(x,t) \equiv \lim_{n\to\infty} u_n(x,t) < \infty$, for $x \in R^d$ and $t \in [0,T)$, and then it follows from the monotone convergence theorem that u satisfies (1.3) for $x \in R^d$ and $t \in (0,T)$; hence $T^*(\lambda,\phi) \geq T$. Thus, to obtain an estimate of the form $T^*(\lambda,\phi) \geq T$, it is enough to show that

$$\sup_n u_n(x,t) < \infty, \text{ for } x \in D_2, t \in [0,T). \tag{4.9}$$

To obtain (4.9), we consider the inductive hypothesis

$$u_n(x,t) \leq c \int_{D_1} p(2t,x,z)dz, x \in D_2, t \in [0,T). \tag{4.10}$$

Note that

$$\int_{D_1} p(t,x,z)dz \leq 2^{\frac{d}{2}} \int_{D_1} p(2t,x,z)dz; \tag{4.11}$$

thus (4.10) holds for $n = 0$ with $c = 2^{\frac{d}{2}}\lambda$ and $T = \infty$. To complete the proof of the lower bound, we will verify the inductive step above for an appropriate choice of $c = c(\lambda)$ and for $T = \frac{k}{\log \lambda}$, where $k > 0$.

From the assumption on a and ϕ, we have

$$\begin{aligned} u_{n+1}(x,t) &= u_0(x,t) + \int_0^t \int_{R^d} p(t-s,x,y)a(y)u_n^p(y,s)dyds \\ &\leq \lambda \int_{D_1} p(t,x,z)dz + \int_0^t \int_{D_2} p(t-s,x,y)u_n^p(y,s)dyds. \end{aligned} \tag{4.12}$$

We now use (4.10) and Jensen's inequality to estimate $u_n^p(y,s)$ appearing in (4.12). It turns out that in order for the rest of the proof to work, the application of Jensen's inequality must be done with a little care as follows:

(4.13)

$$\begin{aligned} &u_n^p(y,s) \leq (c\int_{D_1} p(2s,y,z)dz)^p = (2^{\frac{d}{2}} c \int_{D_1} p(4s,y,z)\exp(-\frac{|y-z|^2}{16s})dz)^p \\ &\leq 2^{\frac{pd}{2}} c^p \int_{D_1} p(4s,y,z)\exp(-p\frac{|y-z|^2}{16s})dz = 2^{\frac{pd}{2}} c^p \int_{D_1} (16\pi s)^{-\frac{d}{2}} \exp(-(p+1)\frac{|y-z|^2}{16s})dz. \end{aligned}$$

Using (4.13) and the fact that

$$\begin{aligned} &\frac{|y-x|^2}{4(t-s)} + (p+1)\frac{|y-z|^2}{16s} \\ &= \frac{(p+1)(t-s)+4s}{(p+1)(t-s)s}\left|y - \frac{4xs + z(p+1)(t-s)}{(p+1)(t-s)+4s}\right|^2 + \frac{p+1}{4(p+1)(t-s)+16s}|x-z|^2, \end{aligned}$$

and integrating out over y, we obtain

$$\begin{aligned} &\int_0^t \int_{D_2} p(t-s,x,y)u_n^p(y,s)dyds \\ &\leq Cc^p \int_0^t \int_{D_1} (8\pi t)^{-\frac{d}{2}} \exp(-\frac{p+1}{4(p+1)(t-s)+16s}|x-z|^2)dzds, \end{aligned} \tag{4.14}$$

for some $C > 0$. Since $p > 1$, we have

$$\frac{p+1}{4(p+1)(t-s)+16s} = \frac{1}{8t} + \frac{(p+1)t+(p-3)s}{2t\,(4(p+1)(t-s)+16s)} \geq \frac{1}{8t} + \frac{\gamma}{t}, \tag{4.15}$$

for some $\gamma > 0$. Letting $l = dist(D_1, D_2) > 0$, it follows from (4.14) and (4.15) that

(4.16)

$$\int_0^t \int_{D_2} p(t-s,x,y)u_n^p(y,s)dyds \leq Cc^p t\exp(-\frac{l\gamma}{t}) \int_{D_1} p(2t,x,z)dz, \text{ for } x \in D_2, t > 0.$$

From (4.11), (4.12), and (4.16), we conclude that

$$u_{n+1}(x,t) \leq \left(2^{\frac{d}{2}}\lambda + Cc^p t\exp(-\frac{l\gamma}{t})\right) \int_{D_1} p(2t,x,z)dz. \tag{4.16}$$

Choosing $c = c(\lambda) = 2^{\frac{d+1}{2}}\lambda$, we find that the inequality $2^{\frac{d}{2}}\lambda + Cc^p t\exp(-\frac{l\gamma}{t}) \leq c$ will hold if $t \leq T = T(\lambda) \equiv \frac{k}{\log\lambda}$, for $k > 0$ sufficiently small. □

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