THE SPEED OF A GENERAL RANDOM WALK REINFORCED BY ITS RECENT HISTORY

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ABSTRACT. We consider a class of random walks whose increment distributions depend on the average value of the process over its most recent N steps. We investigate the speed of the process, and in particular, the limiting speed as the "history window" $N \to \infty$.

1. Introduction and Statement of Results

Over the past couple of decades, many papers have been devoted to the study of edge or vertex reinforced random walks and excited (also known as "cookie") random walks on $\mathbb{Z}$. These processes have a simple underlying transition mechanism—such as simple symmetric random walk—but this mechanism is "reinforced" or "excited" depending on the location of the random walk and its complete history at that location. For survey papers which include many references, see [4] and [3].

In this paper, we consider random walks on $\mathbb{R}$ with a simpler and very natural mechanism for reinforcement; namely, the reinforcement is catalyzed by the behavior of the random walk path over a bounded interval of its history, irrespective of its present location. In fact, we will define two versions of such a process. To define these processes, let $N, l \in \mathbb{N}$, let $\{P_i^{(\text{inc})}\}_{i=0}^l$ be probability measures on $\mathbb{R}$ with finite expectations $\mu_i = \int_{-\infty}^{\infty} x P_i^{(\text{inc})}(dx)$, and let $\{r_i\}_{i=1}^l$ be a sequence. We make the following assumption.

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Assumption A. The sequence $\{\mu_i\}_{i=0}^l$ of expectations corresponding to the measures $\{P_i^{(\mathrm{inc})}\}_{i=0}^l$ is strictly increasing, and the sequence $\{r_i\}_{i=1}^l$ satisfies

$$r_i < \mu_i < r_{i+1},\ i = 1, \cdots, l-1;$$

$$\mu_0 < r_1 \ \text{ and } r_l < \mu_l.$$

In our notation for the processes, we suppress the dependence on all the above parameters with the exception of N, which is the only parameter that will vary. One version of the process, the *instantaneous* version, will be denoted by $\{X_n^{N;I}\}_{n=0}^\infty$, while the other version, the *delayed* version, will be denoted by $\{X_n^{N;D}\}_{n=0}^\infty$. Most of this paper will concern the delayed version, but we define the instantaneous version first, because this will make it easier to describe the delayed version. For convenience, we define $r_0 = -\infty$ and $r_{l+1} = +\infty$ for the following definition.

The instantaneous version $\{X_n^{N;I}\}_{n=0}^\infty$ is defined as follows. Let $X_0^{N;I} = 0$ and let $\{X_n^{N;I}\}_{n=1}^N$ be distributed like a random walk with increment distribution $P_{i_0}^{(\mathrm{inc})}$, for some i_0. The continuation of the process is defined inductively as follows. Let $n \ge N+1$ and let i be such that the process used the distribution $P_i^{(\mathrm{inc})}$ at time $n-1$. The process looks back at its most recent N steps. If the average value, $\frac{X_{n-1}^{N;I} - X_{n-1-N}^{N;I}}{N}$, of those steps fell in the range $[r_i, r_{i+1})$, then at time n the process jumps with increment distribution $P_i^{(\mathrm{inc})}$. However, if the average value of those steps was strictly less than r_i, then at time n the process jumps with increment distribution $P_{i-1}^{(\mathrm{inc})}$, while if the average value of those steps was larger or equal to r_{i+1}, then at time n the process jumps with increment distribution $P_{i+1}^{(\mathrm{inc})}$.

The delayed version $\{X_n^{N;D}\}_{n=0}^\infty$ is defined similarly, the only difference being that this process is required to use any particular jump distribution at least N consecutive times, thereby insuring that the reinforcement that causes the process to switch from one increment distribution, say i, to another increment distribution is due to the behavior of the process while in the i regime. Thus, $\{X_n^{N;D}\}_{n=0}^N$ is defined identically to $\{X_n^{N;I}\}_{n=0}^N$, and for each time $n \ge N+1$, if the jump distribution used at time $n-1$ was not used at time $n-N$, then the jump distribution used at time $n-1$ is automatically used again at time n, while otherwise the jump distribution at time n is determined as it was for the instantaneous version.

We call each version of the process a random walk reinforced by its recent history. Both versions are natural models for the fortunes of various economic commodities, such as stocks, or for the popularity of various social trends, which respond positively to recent success and negatively to recent failure.

We call N the *history window* and $\{r_i\}_{i=1}^{l}$ the *threshold* levels. In Assumption B below, we specify a simple condition to ensure that the processes will almost surely jump an infinite number of times according to each of the $l+1$ increment distributions.

In this paper, we investigate the speeds of these processes. For the delayed version, it's rather easy to show that the speed exists almost surely and is almost surely constant.

Proposition 1. *Let Assumptions A and Assumption B (given below) hold. Then the speed*

$$s^D(N, r_1, \cdots, r_l) := \lim_{n\to\infty} \frac{X_n^{N;D}}{n}$$

exists almost surely, is almost surely constant and is independent of the initial state.

The proof of the proposition is embedded in the proof of the main result of this paper, Theorem 1, and is noted where it occurs. The main result concerns the limiting speed of the delayed version as the history window $N \to \infty$. Here is the condition we impose to ensure that the processes will almost surely jump an infinite number of times according to each of the $l+1$ increment distributions.

Assumption B.

$$P_i^{(\text{inc})}\big((-\infty, r_i)\big) > 0 \text{ and } P_i^{(\text{inc})}\big([r_{i+1}, \infty)\big) > 0, \text{ for } i = 1, \cdots, l-1;$$
$$P_0^{(\text{inc})}([r_1, \infty)) > 0, \quad P_l^{(\text{inc})}((-\infty, r_l)) > 0.$$

(Assumption B is a bit stronger than necessary to ensure that the process will almost surely jump an infinite number of times according to each of the $l+1$ increment distributions, but we use it so as to simplify the exposition.)

A key technical tool that will be used is Cramér's large deviations theorem for the empirical mean of an IID sequence. In order to have this at our disposal, we need to make a two-sided exponent moment assumption on the increment distributions $\{P_i^{(\mathrm{inc})}\}_{i=0}^l$. Let

$$M_{P_i^{(\mathrm{inc})}}(t)=\int_{-\infty}^{\infty}e^{tx}P_i^{(\mathrm{inc})}(dx)$$

denote the moment generating function of the distribution $P_i^{(\mathrm{inc})}$.

Assumption C. There exists a $t_0>0$ such that $M_{P_i^{(\mathrm{inc})}}(\pm t_0)<\infty$, for $i=0,1,\cdots,l$.

Let $I_i(r)$ denote the Legendre-Fenchel transformation for the distribution $P_i^{(\mathrm{inc})}$, defined by

$$I_i(r)=\sup_{\lambda\in\mathbb{R}}\big(\lambda r-\log M_{P_i^{(\mathrm{inc})}}(\lambda)\big),\ r\in\mathbb{R}. \tag{1.1}$$

We recall several facts about I_i that we will need and that hold under Assumption C [1].

$$\begin{aligned}&I_i(r)<\infty \text{ if and only if either } r\le\mu_i \text{ and } P_i^{(\mathrm{inc})}(-\infty,r])>0,\ \text{ or}\\ &r>\mu_i \text{ and } P_i^{(\mathrm{inc})}([r,\infty))>0.\end{aligned} \tag{1.2}$$

Let $x_i^+=\sup\{x\in\mathbb{R}:I_i(x)<\infty\}$ and $x_i^-=\inf\{x\in\mathbb{R}:I_i(x)<\infty\}$. Then

$$\begin{aligned}&I_i(\mu_i)=0;\\ &I_i:[\mu_i,x_i^+)\to[0,\infty) \text{ is continuous and strictly increasing;}\\ &I_i:(x_i^-,\mu_i]\to[0,\infty) \text{ is continuous and strictly decreasing.}\end{aligned} \tag{1.3}$$

And we recall an elementary large deviations result, a version of Cramér's theorem [1]: if $S_n^{(i)}$ is the sum of n IID random variables distributed as

$P_i^{(\mathrm{inc})}$, and $P_i^{(\mathrm{inc})}$ satisfies Assumption C, then

(1.4)

$$\lim_{n\to\infty}\frac{1}{n}\log P(\frac{S_n^{(i)}}{n}\ge r)=\lim_{n\to\infty}\frac{1}{n}\log P(\frac{S_n^{(i)}}{n}>r)=-I_i(r),\ \mu_i\le r<x_i^+;$$
$$\lim_{n\to\infty}\frac{1}{n}\log P(\frac{S_n^{(i)}}{n}\le r)=\lim_{n\to\infty}\frac{1}{n}\log P(\frac{S_n^{(i)}}{n}<r)=-I_i(r),\ x_i^-<r\le\mu_i.$$

We can now state the main result.

Theorem 1. *Let Assumptions A,B, and C hold. Define*

$$\Lambda_0=I_0(r_1);$$
$$\Lambda_i=I_i(r_{i+1})+\sum_{k=1}^{i}\big(I_k(r_k)-I_k(r_{k+1})\big),\ 1\le i\le l-1;$$
$$\Lambda_l=I_l(r_l)+\sum_{k=1}^{l-1}\big(I_k(r_k)-I_k(r_{k+1})\big).$$

If $\max_{0\le i\le l}\Lambda_i$ *occurs uniquely at* $i=i_0$*, then the speed* $s^D(N,r_1,\cdots,r_l)$ *of the delayed process* $\{X_n^{N;D}\}_{n=0}^{\infty}$ *satisfies*

$$\lim_{N\to\infty}s^D(N,r_1,\cdots,r_l)=\mu_{i_0}.$$

Remark. It is sometimes convenient to have one formula that holds for all $\{\Lambda_i\}_{i=0}^{l}$. Using the convention that a summation of the form $\sum_{i=1}^{0}$ is equal to 0, and defining $r_{l+1}=r_l$ (for convenience in defining the process, we had defined $r_{l+1}=\infty$), we can write

$$\Lambda_i=I_i(r_{i+1})+\sum_{k=1}^{i}(I_k(r_k)-I_k(r_{k+1}),\ 0\le i\le l;\quad r_{l+1}:=r_l.$$

Example. The Legendre-Fenchel transformation of the Gaussian distribution $N(\mu,\sigma^2)$ is given by $I(r)=\frac{(r-\mu)^2}{2\sigma^2}$. Let $P_i^{(\mathrm{inc})}\sim N(\mu_i,\sigma_i^2),\ 0\le i\le l$. Define $r_{l+1}=r_l$. If

$$\arg\max_{i\in\{0,\cdots,l\}}\Big[\frac{(r_{i+1}-\mu_i)^2}{\sigma_i^2}+\sum_{k=1}^{i}\frac{(r_k-\mu_k)^2-(r_{k+1}-\mu_k)^2}{\sigma_k^2}\Big]$$

occurs uniquely at i_0, then the limiting speed for the one-step delayed version is μ_{i_0}.

In the instantaneous version, the passage from one regime, say i, to a neighboring regime, say $i+1$, will frequently be accompanied by a number of short time oscillations between the two regimes before the process securely ensconces itself in the new regime $i+1$. Because of technical difficulties related to these oscillations, we can only prove a theorem for the limiting speed of the instantaneous version in the case $l=1$.

Theorem 2. *Let $l=1$ and let Assumptions A,B, and C hold. The speed of the instantaneous process $\{X_n^{N;I}\}_{n=0}^\infty$ almost surely satisfies*

(1.5)
$$\lim_{N\to\infty}\limsup_{n\to\infty}\frac{X_n^{N;I}}{n}=\lim_{N\to\infty}\liminf_{n\to\infty}\frac{X_n^{N;I}}{n}=\begin{cases}\mu_0, & \textit{if } I_0(r_1)>I_1(r_1);\\ \mu_1, & \textit{if } I_1(r_1)>I_0(r_1).\end{cases}$$

In the instantaneous version, define the N-dimensional differences process $\{Z_n^{N;I}\}_{n=0}^\infty$ by

$$Z_n^{N;I}=(X_{n+1}^{N;I}-X_n^{N;I},X_{n+2}^{N;I}-X_{n+1}^{N;I},\cdots,X_{n+N}^{N;I}-X_{n+N-1}^{N;I}).$$

It is easy to see that this is a Markov process. In [5] we studied the speed of the instantaneous version $\{X_n^{N;I}\}_{n=0}^\infty$ under the assumption that the increment distributions $\{P_i^{(\mathrm{inc})}\}_{i=0}^l$ are all Bernoulli distributions on $\{-1,1\}$; $P_i^{(\mathrm{inc})}\sim\mathrm{Ber}(p_i)$, so $\mu_i=2p_i-1$. Thus, those processes lived on $\mathbb{Z}$ and made only nearest-neighbor jumps. In that version, we were able to calculate explicitly the invariant measure π^N (defined on $\{-1,1\}^N$) of the differences process $\{Z_n^{N;I}\}_{n=0}^\infty$, and this allowed us to obtain an explicit formula for the speed $s^I(N,r_1,\cdots,r_l)$. What made the explicit calculation of the invariant distribution possible was the fact that π^N turned out to be constant on the level sets $\{z\in\{-1,1\}^N:\sum_{i=1}^N z_i=M\}$, for any M. Even in the case that the increment distributions $\{P_i^{(\mathrm{inc})}\}_{i=0}^l$ are all supported on a fixed set of size three, the explicit calculation of the invariant measure of the differences process does not seem possible in general. Exploiting this explicit formula for the speed $s^I(N,r_1^{(N)},\cdots,r_l^{(N)})$ in the case of Bernoulli increment distributions, in [5] we proved the equivalent of Theorem 1 for the instantaneous version. The expressions $\{\Lambda\}_{i=0}^{l}$ in the case of these Bernoulli distributions appear there in explicit form, but their connection to the Legendre-Fenchel transformation is not mentioned. The delicate borderline cases, when $\max_{0\le i\le l}\Delta_i$ does not occur uniquely were also resolved, in each case of which the limiting speed was a certain linear combination of the speeds $\{\mu_i\}_{i=0}^{l}$. In this paper, we work on exponential scale, via (1.4), so we cannot handle the borderline cases.

We now turn to the organization of the rest of the paper. Theorem 1 is proved very quickly in section 3, but this is only after a number of technical propositions are proved in the rather long section 2. As already noted, the proof of Proposition 1 is embedded in the proof of Theorem 1. The proof of Theorem 2 is given in section 4.

Here is a rough outline of the idea of the proof of Theorem 1. Let $\{Y_m^{N;D}\}_{m=0}^{\infty}$ denote the Markov chain (more specifically, birth and death chain) on $\{0,\cdots,l\}$ that follows the changes of the increment distribution utilized by the delayed version $\{X_n^{N;D}\}_{n=0}^{\infty}$ of the random walk reinforced by its recent history. Thus, $Y_0^{N;D}=i_0$, since the process $\{X_n^{N;D}\}_{n=0}^{\infty}$ starts out using the increment distribution $P_{i_0}^{(\mathrm{inc})}$. If the first time the process $\{X_n^{N;D}\}_{n=0}^{\infty}$ changes its increment distribution, it switches from distribution $P_{i_0}^{(\mathrm{inc})}$ to distribution $P_j^{(\mathrm{inc})}$ ($j=i_0+1$ or $j=i_0-1$), then $Y_1^{N;D}=j$. In general, $Y_m^{N;D}=k$, if after switching increment distribution m times, the process $\{X_n^{N;D}\}_{n=0}^{\infty}$ is using the increment distribution $P_k^{(\mathrm{inc})}$. Propositions 2 and 3, the first two propositions of section 2, are the key technical results that are used to prove Proposition 4, which gives tight exponential estimates as $N\to\infty$ on the transition probabilities of the Markov chain $\{Y_m^{N;D}\}_{m=0}^{\infty}$. Since $\{Y_m^{N;D}\}_{m=0}^{\infty}$ is a birth and death chain, its invariant distribution can be written down explicitly in terms of its transition probabilities; thus we obtain tight exponential estimates on the behavior of this invariant measure as $N\to\infty$. Proposition 5 calculates the exponential order as $N\to\infty$ of the expected number of steps made by $\{X_n^{N;D}\}_{n=0}^{\infty}$ between the time it enters a particular increment distribution regime and the time it switches to another increment distribution regime, while Proposition 6 calculates the expected distance between it position upon entering a particular increment distribution regime and its position upon switching to another increment distribution regime. The proof of Theorem 1 in section 3 follows easily from Propositions 5 and 6 along with the asymptotic behavior of the invariant measure for the Markov chain $\{Y_m^{N;D}\}_{m=0}^{\infty}$.

2. A series of propositions

We will use the following notation throughout the paper.

$$\begin{aligned} &a_N \approx b_N \text{ means } \lim_{N\to\infty} \frac{1}{N}\log a_N = \lim_{N\to\infty} \frac{1}{N}\log b_N;\\ &a_N \lessapprox b_N \text{ means } \limsup_{N\to\infty} \frac{1}{N}\big(\log a_N - \log b_N\big) \le 0. \end{aligned}$$

The random walk with increment distribution $P_i^{(\mathrm{inc})}$ will be denoted by $\{S_n^{(i)}\}_{n=0}^{\infty}$. Also, we will use the notation

$$S_{j,k}^{(i)} = S_k^{(i)} - S_j^{(i)}, \text{ for } 0 \le j < k.$$

In order to reduce the cumbersome notation, we define as follows $Z_n^{N,i}$, for $n \geq 1$ and $1 \leq i \leq l-1$:

(2.1)

$$Z_n^{N,i} = 1, \text{ if}$$
$$\max\Big(\frac{S^{(i)}_{(n-1)N,nN}}{N}, \frac{S^{(i)}_{(n-1)N+1,nN+1}}{N}, \cdots, \frac{S^{(i)}_{(n-1)N+N-1,nN+N-1}}{N}\Big) \geq r_{i+1} \text{ and}$$
$$\min\Big(\frac{S^{(i)}_{(n-1)N,nN}}{N}, \frac{S^{(i)}_{(n-1)N+1,nN+1}}{N}, \cdots, \frac{S^{(i)}_{(n-1)N+N-1,nN+N-1}}{N}\Big) \geq r_i;$$
$$Z_n^{N,i} = -1, \text{ if}$$
$$\max\Big(\frac{S^{(i)}_{(n-1)N,nN}}{N}, \frac{S^{(i)}_{(n-1)N+1,nN+1}}{N}, \cdots, \frac{S^{(i)}_{(n-1)N+N-1,nN+N-1}}{N}\Big) < r_{i+1} \text{ and}$$
$$\min\Big(\frac{S^{(i)}_{(n-1)N,nN}}{N}, \frac{S^{(i)}_{(n-1)N+1,nN+1}}{N}, \cdots, \frac{S^{(i)}_{(n-1)N+N-1,nN+N-1}}{N}\Big) < r_i;$$
$$Z_n^{N,i} = -11, \text{ if}$$
$$\max\Big(\frac{S^{(i)}_{(n-1)N,nN}}{N}, \frac{S^{(i)}_{(n-1)N+1,nN+1}}{N}, \cdots, \frac{S^{(i)}_{(n-1)N+N-1,nN+N-1}}{N}\Big) \geq r_{i+1} \text{ and}$$
$$\min\Big(\frac{S^{(i)}_{(n-1)N,nN}}{N}, \frac{S^{(i)}_{(n-1)N+1,nN+1}}{N}, \cdots, \frac{S^{(i)}_{(n-1)N+N-1,nN+N-1}}{N}\Big) < r_i;$$
$$Z_n^{N,i} = 0, \text{ otherwise.}$$

Note that $\{Z_n^{N,i}\}_{n=1}^{\infty}$ are identically distributed, and that each of $\{Z_{2n}^{N,i}\}_{n=1}^{\infty}$ and $\{Z_{2n-1}^{N,i}\}_{n=1}^{\infty}$ is an independent sequence.

We begin with two key propositions. These propositions serve as a basis for the rest of the results in this section. For both of them, we will need the FKG correlation inequality [2] in the following form. Let $\{W_i\}_{i=1}^{M}$ be independent real-valued random variables and define $W = (W_1, \cdots, W_M)$.

Let $f, g : \mathbb{R}^M \to \mathbb{R}$. Then

(2.2)

$Ef(W)g(W) \ge Ef(W)Eg(W)$, if f and g are either both increasing or both decreasing in each of their M variables;

$Ef(W)h(W) \le Ef(W)Eh(W)$, if one of f and g is increasing and the other one is decreasing in each of its M variables.

Proposition 2. *Let* $1 \le i \le l-1$. *Then*

$$\begin{aligned} &P(Z_1^{N,i} = 1) \approx e^{-NI_i(r_{i+1})};\\ &P(Z_1^{N,i} = -1) \approx e^{-NI_i(r_i)};\\ &P(Z_1^{N,i} = -11) \lessapprox e^{-N\big(I_i(r_i)+I_i(r_{i+1})\big)}. \end{aligned} \tag{2.3}$$

Proof. We will prove the first and third formulas in (2.3); the second one is proved analogously to the first. By (1.4), we have

$$\begin{aligned} &P(Z_1^{N,i} = 1) \le P\big(\max\big(\frac{S^{(i)}_{0,N}}{N}, \frac{S^{(i)}_{1,N+1}}{N}, \cdots, \frac{S^{(i)}_{N-1,2N-1}}{N}\big) \ge r_{i+1}\big) \le \\ &\sum_{j=0}^{N-1} P(\frac{S^{(i)}_{j,N+j}}{N} \ge r_{i+1}) \approx Ne^{-NI_i(r_{i+1})} \approx e^{-NI_i(r_{i+1})}. \end{aligned} \tag{2.4}$$

Also

(2.5)

$$P(Z_1^{N,i} = 1) =$$

$$P\big(\max\big(\frac{S^{(i)}_{0,N}}{N}, \frac{S^{(i)}_{1,N+1}}{N}, \cdots, \frac{S^{(i)}_{N-1,2N-1}}{N}\big) \ge r_{i+1}\big) \times$$

$$P\big(\min\big(\frac{S^{(i)}_{0,N}}{N}, \frac{S^{(i)}_{1,N+1}}{N}, \cdots, \frac{S^{(i)}_{N-1,2N-1}}{N}\big) \ge r_i | \max\big(\frac{S^{(i)}_{0,N}}{N}, \frac{S^{(i)}_{1,N+1}}{N}, \cdots, \frac{S^{(i)}_{N-1,2N-1}}{N}\big) \ge r_{i+1}\big).$$

By (1.4),

(2.6)

$$P\big(\max\big(\frac{S^{(i)}_{0,N}}{N}, \frac{S^{(i)}_{1,N+1}}{N}, \cdots, \frac{S^{(i)}_{N-1,2N-1}}{N}\big) \ge r_{i+1}\big) \ge P(\frac{S^{(i)}_{0,N}}{N} \ge r_{i+1}) \approx e^{-NI_i(r_{i+1})}.$$

The following inequality follows from the FKG correlation inequality (2.2).

(2.7)

$$P\big(\min\big(\frac{S^{(i)}_{0,N}}{N},\frac{S^{(i)}_{1,N+1}}{N},\cdots,\frac{S^{(i)}_{N-1,2N-1}}{N}\big)\ge r_i|\max\big(\frac{S^{(i)}_{0,N}}{N},\frac{S^{(i)}_{1,N+1}}{N},\cdots,\frac{S^{(i)}_{N-1,2N-1}}{N}\big)\ge r_{i+1}\big)\ge P\big(\min\big(\frac{S^{(i)}_{0,N}}{N},\frac{S^{(i)}_{1,N+1}}{N},\cdots,\frac{S^{(i)}_{N-1,2N-1}}{N}\big)\ge r_i\big).$$

To see that (2.7) follows from (2.2), let $x=(x_1,\cdots,x_{2N-1})\in\mathbb{R}^{2N-1}$, let $s_{i,j}=\sum_{k=i+1}^{j}x_k$, for $0\le i<j\le 2N-1$, and define

$$f(x)=1_{\min\big(\frac{s_{0,N}}{N},\frac{s_{1,N+1}}{N},\cdots,\frac{s_{N-1,2N-1}}{N}\big)\ge r_i};$$

$$g(x)=1_{\max\big(\frac{s_{0,N}}{N},\frac{s_{1,N+1}}{N},\cdots,\frac{s_{N-1,2N-1}}{N}\big)\ge r_{i+1}}.$$

Denote the increments of the random walk $\{S^{(i)}_n\}_{n=0}^{\infty}$ by $\{W^{(i)}_n\}_{n=1}^{\infty}$; that is, $S^{(i)}_n=\sum_{k=1}^{n}W^{(i)}_k$. Let $W^{(i)}=(W^{(i)}_1,\cdots,W^{(i)}_{2N-1})$. Then (2.7) is equivalent to $Ef(W^{(i)})g(W^{(i)})\ge Ef(W^{(i)})Eg(W^{(i)})$, and this latter inequality follows from (2.2).

From (2.7) and (1.4) we have

(2.8)

$$P\big(\min\big(\frac{S^{(i)}_{0,N}}{N},\frac{S^{(i)}_{1,N+1}}{N},\cdots,\frac{S^{(i)}_{N-1,2N-1}}{N}\big)\ge r_i|\max\big(\frac{S^{(i)}_{0,N}}{N},\frac{S^{(i)}_{1,N+1}}{N},\cdots,\frac{S^{(i)}_{N-1,2N-1}}{N}\big)\ge r_{i+1}\big)\ge 1-P\big(\min\big(\frac{S^{(i)}_{0,N}}{N},\frac{S^{(i)}_{1,N+1}}{N},\cdots,\frac{S^{(i)}_{N-1,2N-1}}{N}\big)<r_i\big)\ge 1-NP\big(\frac{S^{(i)}_{0,N}}{N}<r_i\big)\approx 1,\ \text{as } N\to\infty.$$

The first formula in (2.3) now follows from (2.4)-(2.8).

We now turn to the third formula in (2.3). We have

(2.9)

$$P(Z^{N,i}_1=-11)=P\big(\max\big(\frac{S^{(i)}_{0,N}}{N},\cdots,\frac{S^{(i)}_{N-1,2N-1}}{N}\big)\ge r_{i+1}\big)\times P\big(\min\big(\frac{S^{(i)}_{0,N}}{N},\frac{S^{(i)}_{1,N+1}}{N},\cdots,\frac{S^{(i)}_{N-1,2N-1}}{N}\big)<r_i|\max\big(\frac{S^{(i)}_{0,N}}{N},\frac{S^{(i)}_{1,N+1}}{N},\cdots,\frac{S^{(i)}_{N-1,2N-1}}{N}\big)\ge r_{i+1}\big).$$

By (1.4),

(2.10)

$$P\big(\max\big(\frac{S^{(i)}_{0,N}}{N},\cdots,\frac{S^{(i)}_{N-1,2N-1}}{N}\big)\ge r_{i+1}\big)\le NP\big(\frac{S^{(i)}_{0,N}}{N}\ge r_{i+1}\big)\approx e^{-NI_i(r_{i+1})}.$$

The first inequality below follows from the FKG inequality (2.2) similarly to the way (2.7) followed from (2.2). Using this and (1.4), we have

(2.11)
$$P\big(\min\big(\frac{S^{(i)}_{0,N}}{N},\frac{S^{(i)}_{1,N+1}}{N},\cdots,\frac{S^{(i)}_{N-1,2N-1}}{N}\big)<r_i|\max\big(\frac{S^{(i)}_{0,N}}{N},\frac{S^{(i)}_{1,N+1}}{N},\cdots,\frac{S^{(i)}_{N-1,2N-1}}{N}\big)\ge r_{i+1}\big)\le$$
$$P\big(\min\big(\frac{S^{(i)}_{0,N}}{N},\frac{S^{(i)}_{1,N+1}}{N},\cdots,\frac{S^{(i)}_{N-1,2N-1}}{N}\big)<r_i\big)\le NP(\frac{S^{(i)}_{0,N}}{N}\le r_i)\approx e^{-NI_i(r_i)}.$$

The third formula in (2.3) follows from (2.9)-(2.11). □

Proposition 3. *Let* $1\le i\le l-1$. *Define*

$$\tau^{N,i}=\inf\big\{n\ge 0:\frac{S^{(i)}_{n,N+n}}{N}\notin[r_i,r_{i+1})\big\}. \tag{2.12}$$

Then

$$\begin{aligned}&P(\frac{S^{(i)}_{\tau^{N,i},N+\tau^{N,i}}}{N}<r_i)\approx e^{-N\big(I_i(r_i)-I_i(r_{i+1})\big)^+};\\&P(\frac{S^{(i)}_{\tau^{N,i},N+\tau^{N,i}}}{N}\ge r_{i+1})\approx e^{-N\big(I_i(r_{i+1})-I_i(r_i)\big)^+}.\end{aligned} \tag{2.13}$$

Proof. Assume without loss of generality that $I_i(r_i)\ge I_i(r_{i+1})$. If $I_i(r_i)>I_i(r_{i+1})$, then it suffices to prove the first formula in (2.13) since the two terms on the left hand side of (2.13) add up to one. If $I_i(r_i)=I_i(r_{i+1})$, then the proofs of the two formulas in (2.13) are almost identical. Thus, in this case too we will prove only the first formula. Suppressing the dependence on N, let

(2.14)
$$\sigma^{(e)}_i=\inf\big\{2n\ge 2: Z^{N,i}_{2n}\ne 0\big\},\ \ \sigma^{(o)}_i=\inf\big\{2n-1\ge 1, Z^{N,i}_{2n-1}\ne 0\big\}.$$

Using Proposition 2 and the fact that each of $\{Z^{N,i}_{2n}\}_{n=1}^\infty$ and $\{Z^{N,i}_{2n-1}\}_{n=1}^\infty$ is an IID sequence, it follows that

$$\begin{aligned}&P(Z^{N,i}_{\sigma^{(*)}_i}=-1)\approx e^{-N\big(I_i(r_i)-I_i(r_{i+1})\big)},\\&P(Z^{N,i}_{\sigma^{(*)}_i}=-11)\lessapprox e^{-NI_i(r_i)},\\&\text{both when } \sigma^{(*)}_i=\sigma^{(e)}_i \text{ and when } \sigma^{(*)}_i=\sigma^{(o)}_i.\end{aligned} \tag{2.15}$$

Now

$$\Big\{\frac{S^{(i)}_{\tau^{N,i},N+\tau^{N,i}}}{N}<r_i\Big\}\subset\big\{Z^{N,i}_{\sigma^{(e)}_i}\in\{-1,-11\}\big\}\cup\big\{Z^{N,i}_{\sigma^{(0)}_i}\in\{-1,-11\}\big\};$$

thus, it follows from (2.15) that

$$P\big(\frac{S^{(i)}_{\tau^{N,i},N+\tau^{N,i}}}{N}<r_i\big)\lessapprox e^{-N\big(I_i(r_i)-I_i(r_{i+1})\big)}. \tag{2.16}$$

To prove an inequality in the other direction, let $a_N=P(Z^{(N,i)}_1=-1)$ and $b_N=P(Z^{(N,i)}_1\in\{1,-11\})$, where we have suppressed the dependence on i. From Proposition 2,

$$a_N\approx e^{-NI_i(r_i)},\quad b_N\approx e^{-NI_i(r_{i+1})}. \tag{2.17}$$

We have for any positive integer M,

(2.18)

$$\Big\{\frac{S^{(i)}_{\tau^{N,i},N+\tau^{N,i}}}{N}<r_i\Big\}\supset\big(\cap_{n=1}^{2M}\{Z^{N,i}_n\in\{0,-1\}\}\big)\cap\big(\cup_{n=1}^{M}\{Z^{N,i}_{2n}=-1\}\big).$$

Thus,

$$\begin{aligned}P\big(\frac{S^{(i)}_{\tau^{N,i},N+\tau^{N,i}}}{N}<r_i\big)&\ge P(\cup_{n=1}^{M}\{Z^{N,i}_{2n}=-1\})\times\\ &P\big(\cap_{n=1}^{2M}\{Z^{N,i}_n\in\{0,-1\}\}\big|\cup_{n=1}^{M}\{Z^{N,i}_{2n}=-1\}\big).\end{aligned} \tag{2.19}$$

Since $\{Z^{N,i}_{2n}\}_{n=1}^{M}$ are IID, it follows that

$$P(\cup_{n=1}^{M}\{Z^{N,i}_{2n}=-1\})=1-(1-a_N)^M. \tag{2.20}$$

From the definitions, it follows that

$$\{Z^{N,i}_n\in\{0,-1\}\}=\cap_{m=(n-1)N}^{nN-1}\{\frac{S^{(i)}_{m,N+m}}{N}<r_{i+1}\} \tag{2.21}$$

and

(2.22)

$$\{Z^{N,i}_{2n}=-1\}=\big(\cap_{m=(2n-1)N}^{2nN-1}\{\frac{S^{(i)}_{m,N+m}}{N}<r_{i+1}\}\big)\cap\big(\cup_{m=(2n-1)N}^{2nN-1}\{\frac{S^{(i)}_{m,N+m}}{N}<r_i\}\big).$$

From (2.21) and (2.22), along with the FKG inequality (2.2), we have

(2.23)

$$P\big(\cap_{n=1}^{2M}\{Z^{N,i}_n\in\{0,-1\}\}\big|\cup_{n=1}^{M}\{Z^{N,i}_{2n}=-1\}\big)\ge P\big(\cap_{n=1}^{2M}\{Z^{N,i}_n\in\{0,-1\}\}\big).$$

To see this, let $x=(x_1,\cdots,x_{(2M+1)N-1})\in\mathbb{R}^{(2M+1)N-1}$, let $s_{i,j}=\sum_{k=i+1}^j x_k$, for $0\le i<j\le (2M+1)N-1$, and define

$$f(x)=\max(1,\big[\sum_{n=1}^M 1_{\max\big(\frac{s_{m,N+m}}{N}:\ m\in\{(2n-1)N,\cdots,2nN-1\}\big)<r_{i+1}}\times$$
$$1_{\min\big(\frac{s_{m,N+m}}{N}:\ m\in\{(2n-1)N,\cdots,2nN-1\}\big)<r_i}\big];$$
$$g(x)=1_{\max\big(\frac{s_{m,N+m}}{N}:\ m\in\{1,\cdots,2MN-1\}\big)<r_{i+1}}.$$

Denote the increments of the random walk $\{S_n^{(i)}\}_{n=0}^\infty$ by $\{W_n^{(i)}\}_{n=1}^\infty$, and let $W^{(i)}=(W_1^{(i)},\cdots,W^{(i)}_{(2M+1)N-1})$. Then (2.23) is equivalent to $Ef(W^{(i)})g(W^{(i)})\ge Ef(W^{(i)})Eg(W^{(i)})$, and this latter inequality follows from (2.2).

Similarly, the FKG inequality (2.2) gives

$$P\big(\cap_{n=1}^{2M}\{Z_n^{N,i}\in\{0,-1\}\}\big)\ge\big(P(Z_n^{N,i}\in\{0,-1\})\big)^{2M}=(1-b_N)^{2M}.$$

Thus,

$$P\big(\cap_{n=1}^{2M}\{Z_n^{N,i}\in\{0,-1\}\}\big|\cup_{n=1}^M\{Z_{2n}^{N,i}=-1\}\big)\ge(1-b_N)^{2M}. \tag{2.24}$$

Now choose $M=[\frac1{b_N}]$. We consider the two cases $I_i(r_i)>I_i(r_{i+1})$ and $I_i(r_i)=I_i(r_{i+1})$ separately. We first consider the former case. Note that $\lim_{N\to\infty}\frac{a_N}{b_N}=0$. Since $1-a_N\le e^{-a_N}$, from (2.20),

$$P(\cup_{n=1}^{[\frac1{b_N}]}\{Z_{2n}^{N,i}=-1\})\ge 1-e^{-a_N[\frac1{b_N}]}\ge\frac{a_N}{2b_N},\ \text{for large } N. \tag{2.25}$$

From (2.24),

$$\liminf_{N\to\infty}P\big(\cap_{n=1}^{2[\frac1{b_N}]}\{Z_n^{N,i}\in\{0,-1\}\}\big|\cup_{n=1}^{[\frac1{b_N}]}\{Z_{2n}^{N,i}=-1\}\big)\ge e^{-2}. \tag{2.26}$$

From (2.17), (2.19), (2.25) and (2.26), we conclude that

$$P\big(\frac{S^{(i)}_{\tau^{N,i},N+\tau^{N,i}}}{N}<r_i\big)\gtrapprox e^{-N\big(I_i(r_i)-I_i(r_{i+1})\big)}. \tag{2.27}$$

Now consider the case $I_i(r_i)=I_i(r_{i+1})$. Then similar to (2.25), we have

$$P(\cup_{n=1}^{[\frac1{b_N}]}\{Z_{2n}^{N,i}=-1\})\ge 1-e^{-a_N[\frac1{b_N}]}\ge\min(c,\frac{a_N}{2b_N}),\ \text{for some } c>0. \tag{2.28}$$

Then from (2.17), (2.19), (2.28) (2.26) and the fact that $a_N \approx b_N$, we obtain (2.27). The first formula in (2.13) follows from (2.16) and (2.27). □

Recall the process $\{Y_m^{N;D}\}_{m=0}^{\infty}$ defined at the end of section 1; it denotes the Markov processes that follows the changes of the increment distribution utilized by the delayed version $\{X_n^{N;D}\}_{n=0}^{\infty}$ of the random walk reinforced by its recent history. We denote the transitions for $\{Y_m^{N;D}\}_{m=0}^{\infty}$ by

$$p_{i,j}^{N;D} = P(Y_{m+1}^{N;D} = j | Y_m^{N;D} = i),\ i, j \in \{0, \cdots, l\},\ j = i \pm 1.$$

Using Proposition 3, the following estimates on these transition probabilities are almost immediate.

Proposition 4.

$$\begin{aligned} &p_{i,i+1}^{N;D} \approx e^{-N\left(I_i(r_{i+1}) - I_i(r_i)\right)^+};\ i \in \{1, \cdots, l-1\}; \\ &p_{i,i-1}^{N;D} \approx e^{-N\left(I_i(r_i) - I_i(r_{i+1})\right)^+};\ i \in \{1, \cdots, l-1\}; \\ &p_{0,1}^{N;D} = p_{l,l-1}^{N;D} = 1. \end{aligned} \tag{2.29}$$

Proof. The third line in (2.29) follows by definition. Noting that

$$p_{i,i+1}^{N;D} = P(\frac{S^{(i)}_{\tau^{N,i}, N+\tau^{N,i}}}{N} \geq r_{i+1}),\quad p_{i,i-1}^{N;D} = P(\frac{S^{(i)}_{\tau^{N,i}, N+\tau^{N,i}}}{N} < r_i),$$

the first two lines of (2.29) follow from Proposition 3. □

Denote the invariant distribution of the Markov chain $\{Y_m^{N;D}\}_{m=0}^{\infty}$ on $\{0, \cdots, l\}$ by $\nu^{N;D}$. The Markov chain $\{Y_m^{N;D}\}_{m=0}^{\infty}$ is a birth and death process, thus reversible, so its invariant distribution can be calculated explicitly, via the detailed balance equations: $\nu^{N;D}(i)p_{i,i+1}^{N;D} = \nu^{N;D}(i+1)p_{i+1,i}^{N;D}$, $i = 0, \cdots, l-1$. As is well-known, one has

$$\begin{aligned} &\Pi_N \nu^{N,D}(0) = 1; \\ &\Pi_N \nu^{N,D}(k) = \prod_{i=1}^{k} \frac{p_{i-1,i}^{N;D}}{p_{i,i-1}^{N;D}},\ k = 1, \cdots, l, \\ &\text{where } \Pi_N = 1 + \sum_{k=1}^{l} \prod_{i=1}^{k} \frac{p_{i-1,i}^{N;D}}{p_{i,i-1}^{N;D}}. \end{aligned} \tag{2.30}$$

Recall the definition of $\tau^{N,i},\ 1\le i\le l-1$, from (2.12). Define

$$\tau^{N,0}=\inf\big\{n\ge 0:\frac{S^{(0)}_{n,N+n}}{N}\ge r_1\big\};\quad \tau^{N,l}=\inf\big\{n\ge 0:\frac{S^{(l)}_{n,N+n}}{N}<r_l\big\}.$$

Anytime the delayed version of the random walk reinforced by its recent history switches to regime i, the number of steps during which it will operate in this regime before switching to a different regime is distributed as $\tau^{N,i}+N$, and the distance between its position upon entering regime i and its position upon switching to another regime is distributed as $S^{(i)}_{\tau^{N,i}+N}$. The next two propositions calculate the expected values of these two distributions.

Proposition 5.

$$\begin{aligned}&E\tau^{N,i}\approx e^{N\min\big(I_i(r_i),\,I_i(r_{i+1})\big)},\ 1\le i\le l-1;\\ &E\tau^{N,l,l}\approx e^{NI_l(r_l)};\quad E\tau^{N,0}\approx e^{NI_0(r_1)}.\end{aligned}\tag{2.31}$$

Proof. Let $1\le i\le l-1$. Using the notation from the proof of Proposition 3, for any positive integer L, we have

$$\begin{aligned}&\{\tau^{N,i}\ge 2LN\}=\{Z^{N,i}_n=0,\ \text{for all } n=1,\cdots,2L\}=\\ &\{\sigma^{(e)}_i>2L,\sigma^{(o)}_i>2L-1\}.\end{aligned}\tag{2.32}$$

Since $\sigma^{(e)}_i$ and $\sigma^{(o)}_i+1$ have the same distribution, it follows that

$$P(\tau^{N,i}\ge 2LN)\le P(\sigma^{(e)}_i>2L),\ L\ge 1.\tag{2.33}$$

We have

$$\sum_{L=0}^{\infty}P(\tau^{N,i}\ge 2LN)\ge\sum_{m=0}^{\infty}(\frac{m}{2N}+1)P(\tau^{N,i}=m)=1+\frac{1}{2N}E\tau^{N,i}.\tag{2.34}$$

From the definition of $\sigma^{(e)}_i$ along with Proposition 2, $\sigma^{(e)}_i$ is distributed according to a geometric distribution with parameter $p\approx e^{-N\min\big(I_i(r_i),\,I_i(r_{i+1})\big)}$; thus, $E\sigma^{(e)}_i\approx e^{N\min\big(I_i(r_i),\,I_i(r_{i+1})\big)}$. Consequently,

$$\sum_{L=0}^{\infty}P(\sigma^{(e)}_i>2L)\le\sum_{L=1}^{\infty}P(\sigma^{(e)}_i\ge L)=E\sigma^{(e)}_i\approx e^{N\min\big(I_i(r_i),\,I_i(r_{i+1})\big)}.\tag{2.35}$$

From (2.33)-(2.35), we obtain

$$E\tau^{N,i} \lessapprox e^{N\min\left(I_i(r_i),\, I_i(r_{i+1})\right)}. \tag{2.36}$$

From Proposition 2 and the definition of $\sigma_i^{(e)}$ and $\sigma_i^{(o)}$, we have for any $\epsilon > 0$ and sufficiently large N,

$$\begin{aligned} &P(\sigma_i^{(e)} > 2L, \sigma_i^{(o)} > 2L-1) = P(Z_n^{N,i} = 0,\ \text{for } n = 1, \cdots, 2L) \ge \\ &1 - 2LP(Z_1^{N,i} \neq 0) \ge 1 - 2Le^{\epsilon N - N\min\left(I_i(r_i),\, I_i(r_{i+1})\right)}. \end{aligned} \tag{2.37}$$

Letting $L_{N,\epsilon} = [e^{-2\epsilon N + N\min\left(I_i(r_i),\, I_i(r_{i+1})\right)}]$, it follows from (2.32) and (2.37) that $\lim_{N\to\infty} P(\tau^{N,i} \ge 2L_{N,\epsilon}N) = 1$. Since $\epsilon > 0$ is arbitrary, it follows that

$$E\tau^{N,i} \gtrapprox e^{N\min\left(I_i(r_i),\, I_i(r_{i+1})\right)}. \tag{2.38}$$

The first formula in (2.31) follows from (2.36) and (2.38).

The statements of Proposition 2 and Proposition 3 involve certain two-sided hitting times related to a random walk with increment distribution $P_i^{(\text{inc})}$, with $1 \le i \le l-1$. Similar one-sided results could have been written down for $i = 0$ and $i = l$. We refrained from including them in order not to incur the necessity of additional notation and an additional analogous proof. The second formula in (2.31) is proved similarly to the first formula using the corresponding one-sided hitting times. □

Proposition 6.

$$ES^{(i)}_{\tau^{N,i}+N} = \mu_i(E\tau^{N,i} + N),\ 0 \le i \le l. \tag{2.39}$$

Proof. Let $\{W_n^{(i)}\}_{n=1}^{\infty}$ be IID random variables distributed according to $P_i^{(\text{inc})}$ and consider the filtration $\mathcal{F}_n = \sigma\Big(W_1^{(i)}, \cdots, W_n^{(i)}\Big)$, $n \ge 1$. We can write $S_n^{(i)} = \sum_{j=1}^n W_j^{(i)}$. Now $M_{n+N} := S_{n+N}^{(i)} - (n+N)\mu_i$, $n \ge 0$, is a martingale with respect to $\{\mathcal{F}_{n+N}\}_{n=0}^{\infty}$. Note that $N + \tau^{N,i}$ is a stopping time with respect to $\{\mathcal{F}_{n+N}\}_{n=0}^{\infty}$. So by Doob's optional sampling theorem,

$$ES_{(\tau^{N,i}+N)\wedge L} - \mu_i E((\tau^{N,i} + N) \wedge L) = 0,\ L \ge 0.$$

Letting $L \to \infty$ and using (2.31), we obtain (2.39). □

3. Proof of Theorem 1

Recall that $\nu^{N,D}$ denotes the invariant distribution of the process $\{Y_m^{N;D}\}_{m=0}^{\infty}$. By the ergodic theorem, as $m \to \infty$ the asymptotic proportion of switches of the process $\{Y_m^{N;D}\}_{m=0}^{\infty}$ for the delayed process to the regime i is $\nu^{N,D}(i)$. As noted before Proposition 5, anytime the delayed version of the random walk reinforced by its recent history switches to regime i, the number of steps during which it will operate in this regime before switching to a different regime is distributed as $\tau^{N,i} + N$, and the distance between its position upon entering regime i and its position upon switching to another regime is distributed as $S^{(i)}_{\tau^{N,i}+N}$. Also, this random number of steps and this random distance are independent of the random number of steps the process spent and the random distance it attained in any regime in the past before the present entrance into regime i. From these observations, it is standard to deduce that the speed $s^D(N, r_1, \cdots, r_l)$, defined in Proposition 1, exists almost surely and is almost surely given by the constant

$$s^D(N, r_1, \cdots, r_l) = \frac{\sum_{i=0}^{l} \nu^{N,D}(i) E S^{(i)}_{\tau^{N,i}+N}}{\sum_{i=0}^{l} \nu^{N,D}(i)\big(E\tau^{N,i} + N\big)}. \tag{3.1}$$

This proves Proposition 1.

By Propositions 5 and 6,

$$\begin{aligned} &E S^{(i)}_{\tau^{N,i}+N} \approx \mu_i e^{N \min\big(I_i(r_i), I_i(r_{i+1})\big)}, \text{ for } 1 \le i \le l-1; \\ &E S^{(0)}_{\tau^{N,0}+N} \approx \mu_0 e^{N I_0(r_1)}, \ E S^{(l)}_{\tau^{N,l}+N} \approx \mu_l e^{N I_l(r_l)}. \end{aligned} \tag{3.2}$$

From (2.30) and Proposition 4, we have

(3.3)

$$\nu^{N;D}(0) \approx \frac{1}{\Pi_N};$$

$$\nu^{N;D}(1) \approx \frac{1}{\Pi_N} e^{N\left(I_1(r_1)-I_1(r_2)\right)^+};$$

$$\nu^{N;D}(i) \approx \frac{1}{\Pi_N} e^{N\left(I_1(r_1)-I_1(r_2)\right)^+} \prod_{k=2}^{i} e^{N\left(\left(I_k(r_k)-I_k(r_{k+1})\right)^+ - \left(I_{k-1}(r_k)-I_{k-1}(r_{k-1})\right)^+\right)},$$

$1 \le i \le l-1;$

$$\nu^{N;D}(l) \approx \frac{1}{\Pi_N} e^{N\left(I_1(r_1)-I_1(r_2)\right)^+} \prod_{k=2}^{l-1} e^{N\left(\left(I_k(r_k)-I_k(r_{k+1})\right)^+ - \left(I_{k-1}(r_k)-I_{k-1}(r_{k-1})\right)^+\right)} \times$$

$$e^{-N\left(I_{l-1}(r_l)-I_{l-1}(r_{l-1})\right)^+}.$$

Noting that $\left(I_k(r_k) - I_k(r_{k+1})\right)^+ - (I_k(r_{k+1}) - I_k(r_k))^+ = I_k(r_k) - I_k(r_{k+1})$ and recalling the definition of $\{\Lambda_i\}_{i=0}^l$ in the statement of Theorem 1, it follows from (3.2) and (3.3) that

$$\nu^{N,D}(i) ES^{(i)}_{\tau^{N,i}+N} \approx \frac{1}{\Pi_N} \mu_i e^{N\Lambda_i}, \ 0 \le i \le l. \tag{3.4}$$

Substituting (3.4) into (3.1), recalling from Proposition 6 that $\frac{ES^{(i)}_{\tau^{N,i}+N}}{E\tau^{N,i}+N} \approx \mu_i$, and letting $N \to \infty$ proves the theorem.

□

4. Proof of Theorem 2

For the proof of Theorem 2, we need the following lemma.

Lemma 1. *Let $\{Z_n\}_{n=1}^\infty$ be IID random variables satisfying $EZ_1 = \mu$ and let $S_n = \sum_{i=1}^n Z_i$. Then for every $r < \mu$,*

$$P(\frac{S_n}{n} \ge r, \ n = 1, 2, \cdots) > 0.$$

Proof. By the strong law of large numbers, $\lim_{n\to\infty} \frac{S_n}{n} = \mu$ a.s. Thus, for every $r < \mu$, there exists an N_r such that $P(\frac{S_n}{n} \ge r, \ n > N_r) > 0$. Clearly,

$P(\frac{S_n}{n} \geq r,\ n = 1, \cdots, N_r) > 0$. By the FKG inequality (2.2), we have

$$P(\frac{S_n}{n} \geq r,\ n > N_r | \frac{S_n}{n} \geq r,\ n = 1, \cdots, N_r) \geq P(\frac{S_n}{n} \geq r,\ n > N_r).$$

Thus,

$$P(\frac{S_n}{n} \geq r,\ n = 1, 2, \cdots) =$$
$$P(\frac{S_n}{n} \geq r,\ n = 1, \cdots, N_r) P(\frac{S_n}{n} \geq r,\ n > N_r | \frac{S_n}{n} \geq r,\ n = 1, \cdots, N_r) > 0.$$

□

We now turn to the proof of the theorem.

Proof of Theorem 2. Without loss of generality, assume that $I_0(r_1) < I_1(r_1)$. Since clearly $\limsup_{n\to\infty} \frac{X_n^{N;I}}{n} \leq \mu_1$ a.s., what we need to prove is that

$$\lim_{N\to\infty} \liminf_{n\to\infty} \frac{X_n^{N;I}}{n} = \mu_1. \tag{4.1}$$

Define

$$c := P(\frac{S_n^{(1)}}{n} \geq r_1,\ n = 1, 2, \cdots) > 0, \tag{4.2}$$

where the positivity of c follows from Lemma 1. Without loss of generality, we will start the instantaneous process $\{X_n^{N;I}\}_{n=0}^{\infty}$ in the $P_0^{(inc)}$-mode. The process will eventually switch to the $P_1^{(inc)}$-mode, then switch back to the $P_0^{(inc)}$, etc.

Let $T_m^{N,1}, m \geq 1$, denote the number of steps the instantaneous process spends in the $P_1^{(inc)}$-mode during its mth session in that mode, and let $T_m^{N,0}, m \geq 1$, denote the number of steps the instantaneous process spends in the $P_0^{(\mathrm{inc})}$-mode during its mth session in that mode.

Clearly $T_m^{N,0}$, for any $m \geq 1$, is stochastically dominated by $\tau^{N,0}$, where $\tau^{N,0}$ is as in (2.12). (There is equality in distribution for $m = 1$.)

The event that for all $j = 1, \cdots, N$, the average value of the first j steps of a $P_1^{(inc)}$-random walk is greater than or equal to r_1 has probability greater than or equal to c. Thus, with probability greater than or equal to c, the instantaneous process will spend at least N steps in the $P_1^{(inc)}$-mode during any session in that mode. It follows then that $T_m^{N,1}$, for any $m \geq 1$, stochastically dominates $(N+\tau^{N,1})\mathrm{Ber}(c)$, where $\tau^{N,1}$ is as in (2.12), $\mathrm{Ber}(c)$ denotes a Bernoulli random variable with probability c of being equal to 1 and probability $1-c$ of being equal to 0, and $\tau^{N,1}$ and $\mathrm{Ber}(c)$ are independent. We note that there are two reasons that $T_m^{N,1}$ stochastically dominates $(N+\tau^{N,1})\mathrm{Ber}(c)$. One is that the probability of the event described above is greater than c. The other is that $\tau^{N,1}$, the number of steps the delayed process remains in the $P_1^{(inc)}$-mode after its first N steps in that mode, is stochastically dominated by the random variable $T_m^{N,1}-N$ when this latter random variable is conditioned on the event described above. The reason for this latter domination is that whereas the first N steps of the delayed process have the distribution $\{S_j^{(i)}\}_{j=1}^N$, the first N steps of the instantaneous process conditioned on the event described above has the distribution $\{S_j^{(i)}\}_{j=1}^N$, conditioned on $\{\frac{S_j^{(1)}}{j}\ge r_1,\ j=1,2,\cdots,N\}$, and by the FKG inequality (2.2), the distribution $\{S_j^{(i)}\}_{j=1}^N$, conditioned on $\{\frac{S_j^{(1)}}{j}\ge r_1,\ j=1,2,\cdots,N\}$ dominates the distribution $\{S_j^{(i)}\}_{j=1}^N$.

The fraction of steps that the instantaneous process spends in the $P_1^{(inc)}$-mode after m sessions in each mode is given by

$$\frac{\sum_{k=1}^m T_k^{N,1}}{\sum_{k=1}^m (T_k^{N,1}+T_k^{N,0})}. \tag{4.3}$$

By the above noted stochastic domination, we can define on one and the same space $\{T_k^{N,1}\}_{k=1}^\infty$ and $\{T_k^{N,0}\}_{k=1}^\infty$ along with $\{\tau_k^{N,i}\}_{k=1}^\infty$, $i=0,1$, and $\{\mathrm{Ber}(c)_k\}_{k=1}^\infty$, where these last three sequences are mutually independent IID sequences distributed respectively as $\tau^{N,i}, i=0,1$, and $\mathrm{Ber}(c)$, such that

$$\frac{\sum_{k=1}^m T_k^{N,1}}{\sum_{k=1}^m (T_k^{N,1}+T_k^{N,0})}\ge\frac{\sum_{k=1}^m (N+\tau_k^{N,1})\mathrm{Ber}(c)_k}{\sum_{k=1}^m (N+\tau_k^{N,1})\mathrm{Ber}(c)_k+\tau_k^{N,0}}\quad\text{a.s.} \tag{4.4}$$

By the strong law of large numbers,

$$\lim_{m\to\infty}\frac{\sum_{k=1}^m (N+\tau_k^{N,1})\mathrm{Ber}(c)_k}{\sum_{k=1}^m (N+\tau_k^{N,1})\mathrm{Ber}(c)_k+\tau_k^{N,0}}=\frac{c(N+E\tau^{N,1})}{c(N+E\tau^{N,1})+E\tau^{N,0}}\quad\text{a.s.} \tag{4.5}$$

By Proposition 5 (with $l=1$) and the assumption that $I_0(r_1)<I_1(r_1)$, it follows that $\lim_{N\to\infty}\frac{E\tau^{N,1}}{E\tau^{N,0}}=\infty$. Using this with (4.4) and (4.5), we

conclude that the asymptotic fraction of steps that the instantaneous process spends in the $P_1^{(inc)}$-mode satisfies

$$\lim_{N\to\infty}\liminf_{m\to\infty}\frac{\sum_{k=1}^m T_k^{N,1}}{\sum_{k=1}^m (T_k^{N,1}+T_k^{N,0})}=1 \text{ a.s.}$$

From this we conclude that (4.1) holds. □

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References

[1] Dembo, A. and Zeitouni, O., *Large Deviations Techniques and Applications* 2nd edition, Springer-Verlag, New York, (1998).

[2] Fortuin, C. and Kasteleyn, P., *Correlation Inequalities on Some Partially Ordered Sets*, Commun. Math. Phys. **22** (1971), 89-103.

[3] Kosygina, E. and Zerner M. *Excited random walks: results, methods, open problems*, Bull. Inst. Math. Acad. Sin. (N.S.) **8** (2013), 105-157.

[4] Pemantle, R. *A survey of random processes with reinforcement*, Probab. Surv. **4** (2007), 1-79.

[5] Pinsky, R. *The speed of a random walk excited by its recent history*, Adv. in Appl. Probab. 48 (2016), 215-234.

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