Lecture 1

1 Examples of symmetries in ODE

$$y'' = w(x, y, y')$$

A solution is invariant under $y-$shift $y(x) \Longrightarrow y(x) + c$ iff $w = w(x, y')$. In that case we can reduce it to a first order ODE by $z = y'$, so

$$z' = w(x, z)$$

A solution is invariant under $x-$shift $y(x) \Longrightarrow y(x + c)$ iff $w = w(y, y')$. We can transform to $x = x(y)$ so $x' = 1/y'$, $x''(y)(y')^2 + x'y'' = 0$ so $y'' = -x''/(x')^3$ so

$$\frac{d^2x}{dy^2} = -\left(\frac{dx}{dy}\right)^3 w\left(y, \left(\frac{dx}{dy}\right)^{-1}\right)$$

and can be reduced to first order $z = x'$,

$$z' = -z^3 w\left(y, z^{-1}\right) \ .$$

In some cases we cannot see any symmetry directly:

$$y'' = (x - y)(y')^3$$

but a systematic study show a rich structure of symmetries. This hints that this is a special equation. Indeed $x = x(y)$ implies

$$x'' = y - x$$

is a *linear* equation!

The object is to study a systematic way to discover the symmetries of ODE's and PDE's, and how to exploit them to establish explicit solutions, whenever possible. This way we may sometime uncover hidden structures (such as linearity), ect.

Another application: If a system of symmetries is known (from, say, geometrical or physical reasons), then we may discover the general form of equation subjected to this symmetry.

2 Pre-requisite

Elementary courses in ODE and PDE, basic notions in algebra (axioms of a group). We shall try to avoid the notion of a differentiable manifold, since a symmetry is, in our case, a *local* property.

3 A Lie group is a connected subset

$G \subset \mathbb{R}^m$ and a function $G \times G \to G$ which is continuous (in the induced topology) and satisfies the group axioms.

Examples:

- $\mathbb{R}^m$ $(+)$
- The general linear group $\mathbb{GL}(n, \mathbb{R})$- invertible $n \times n$ matrices under multiplications (open subset of $\mathbb{R}^{n^2}$).
- $\mathbb{SL}(n)$ given by $\det(A) = 1$.
- Special Orthogonal group $\mathbb{SO}(n) \subset \mathbb{SL}(n)$ given by $AA^T = I$. Example: $SO(2)$ is the rotation in the plane:

$$G = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \ , \quad 0 \le \theta \le 2\pi \right\}$$

  identified with the unit circle $\mathbb{S}^1 \backsimeq \{e^{i\theta}\}$.

- The affine group

$$\mathbf{A} \in \mathbb{GL}(n+1, \mathbb{R}) \ ; \mathbf{A} := \begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix}, \quad A \in \mathbb{GL}(n, \mathbb{R}), \quad a \in \mathbb{R}^n$$

  also satisfies $(A, a) \cdot (B, b) = (AB, a + Ab)$.

- The Euclidian group is a subgroup of the affine group where $\mathbf{A} \in \mathbb{SO}(n)$. Example, $n = 2$. We can identify (x, y) with $z = x + iy \in \mathbb{C}$ and set $G = \{(e^{i\theta}, z) \ ; \ \theta \in [0, 2\pi), \ z \in \mathbb{C}$ with the multiplication $(e^{i\theta_1}, z_1) \cdot (e^{i\theta_2}, z_2) = \left(e^{i(\theta_1+\theta_2)}, z_1 + e^{i\theta_1} z_2\right)$.
- If $n = 2r$ even and

$$J := \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$$

  then

$$Sp(2r) := \{A \in \mathbb{GL}(2r, \mathbb{R}); \ A^T J A = J$$

  is the *Symplectic group*.

4 Action of Lie groups

G on $M \subset \mathbb{R}^d$ is given by a function $\Psi(G \times M) \to M$ such that

i) If $(h, x) \in G \times M$ then $\Psi(g, \Psi(h, x)) = \Psi(g \cdot h, x)$.

ii) For all $x \in M$ $\Psi(e, x) = x$.

iii) $\Psi(g^{-1}, \Psi(g, x)) = x$.

Examples of action of $(\mathbb{R},+)$ $(m=1)$

- $\Psi(t,x) = x + ta$ for $x, a \in \mathbb{R}^d, t \in \mathbb{R}$.
- Scaling transformations $\Psi(\lambda, x) = (e^{\alpha_1 t} x_1, \ldots e^{\alpha_n} x_d)$ where $\alpha_i \in \mathbb{R}$ and $x \in \mathbb{R}^d$.
- $\Psi(t,(x,y)) = \left(\frac{x}{1-tx}, \frac{y}{1-tx}\right)$
- $\Psi(t,x,y,z) = (x\cos t + y \sin t, -y\cos t + x \sin t, z + t)$.

Examples of other group actions

1. $\mathbb{GL}(\mathbb{R}, m)$ acts on $\mathbb{R}^m$.
2. Same group acting on $\mathbb{PR}^{n-1}$. For $A := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbb{GL}(\mathbb{R}, 2)$

$$p \to A(p) := \frac{\alpha p + \beta}{\gamma p + \delta}$$

   where $\mathbb{PR}^1$ is identified with $\mathbb{R}^+ \cup \{\infty\}$.

3. The affine group acting on $\mathbb{R}^m$ via

$$(A, a)(x) := Ax + a$$

4. Left $L_g(h) = g^{-1} \cdot h$, right $R_g(h) = h \cdot g$ and adjoint $Ad_g(h) = g^{-1} \cdot h \cdot g$ actions of a group $\mathbb{GL}(n, \mathbb{R})$ on itself.

5 Vector fields

On open domain $M \subset \mathbb{R}^m$: A directional derivative for functions on:

$$\mathbf{X} = \sum_1^m \xi_i \partial_{x_i} \ , \quad \xi_i \in C^\infty(M)$$

$$\mathbf{X} : C^\infty(M) \to C^\infty(M) \quad : \quad \phi \to \mathbf{X}(\phi) := \sum_1^m \xi_i \frac{\partial \phi}{\partial x_i} \ .$$

Theorem 1. *If $L : C^\infty(M) \to C^\infty(M)$ is linear and satisfies Libnitz rule*

$$L(\phi_1 \phi_2) = \phi_1 L(\phi_2) + \phi_2 L(\phi_1)$$

then $L(\phi) = \mathbf{X}(\phi)$ for some v-f $\mathbf{X}$.

The v-f $\mathbf{X}$ generate the action:

$$\frac{d}{dt}\Psi(t,x) = \mathbf{X}\left(\Psi(x,t)\right) \ , \quad \Psi(x,0) = x \ .$$

We also denote $\Psi(t,x) := \exp(t\mathbf{X}) \circ x$.

$$\mathbf{X}(\phi) = \frac{d}{dt}\phi\left(\exp(t\mathbf{X})\right)_{t=0}$$

We also have

$$\phi\left(\exp(t\mathbf{X})_{(x)}\right) = \phi(x) + \mathbf{X}(\phi)(x) + \frac{1}{2}\mathbf{X}^2\phi(x) + \dots$$

Examples

- $\sum a_i \partial_{x_i}$.
- $\sum \alpha_i x_i \partial_{x_i}$.
- $v = x^2\partial_x + xy\partial_y$.
- $y\partial_x - x\partial_y + \partial_z$.

Properties of v-f: Linearity $\mathbf{X}(f+g) = \mathbf{X}(f) + \mathbf{X}(g)$ and Leibniz' rule $\mathbf{X}(fg) = g\mathbf{X}(f) + f\mathbf{X}(g)$.
Transformation law: A mapping $F : M \to M'$ implies a mapping from the vectorfields on M to vectorfields on M' as follows: For $\phi \in C^\infty(M')$ we have $\phi \circ F \in C^\infty(M)$ and for a vectorfield $\mathbf{X}$ on M we define

$$F_*[\mathbf{X}](\phi)_{(F(x))} \equiv \mathbf{X}(\phi \circ F)_{(x)}$$

In coordinates form (we shall use Einstein convention) $x^{i'} = x^{i'}(x^i) \to \partial_{x^i} = \frac{\partial x^{i'}}{\partial x^i}\partial_{x^{i'}}$:

$$\mathbf{X} \to \mathbf{X'} = b^{i'}\partial_{i'}, \quad b^{i'} = \frac{\partial x^{i'}}{\partial x^i} b^i$$

A practical way to do it:

$$\mathbf{X}x^n = b^i \partial_{x^i} x^n = b^n, \quad \mathbf{X}x^{n'} = b^{n'}$$

so

$$\mathbf{X} = (\mathbf{X}x^i)\partial_{x^i} = (\mathbf{X}x^{i'})\partial_{x^{i'}}$$

Example $\mathbf{X} = x\partial_x + y\partial_y$ under $u = y/x$, $v = xy$:

$$\mathbf{X}u = 0 \ , \ \mathbf{X}v = 2xy$$

so $\mathbf{X}' = 2v\partial_v$.
Another example: Rotation $\mathbf{X} = -y\partial_x + x\partial_y$, $r = (x^2+y^2)^{1/2}$, $\phi = \arctan(y/x)$.

Lemma 5.1. *Given a vectorfield* $\mathbf{X}$ *on* M *consider* $\exp(t\mathbf{X}) \circ x$ *as a mapping from* $M \times \mathbb{R}$ *to* M*. Then*

$$\exp(t\mathbf{X})_*(\mathbf{X}) = \exp(t\mathbf{X})_*(\partial_t) = \mathbf{X}$$

Theorem 2. *There are local coordinates* $s_1, \ldots, s_m$ *transforming* $\mathbf{X}$ *into* ∂_{s_1}*.*

To prove we have to solve

$$\mathbf{X}s_1 = 1, \quad \mathbf{X}s_j = 0 \ \ , \ j = 2, \ldots n \ .$$

which is a set of n independent PDE. Geometrical intuition...

Lie brackets $[\mathbf{X}, \mathbf{Y}](\phi) = \mathbf{X}\mathbf{Y}\phi - \mathbf{Y}\mathbf{X}\phi$.
Explicit writing:

$$\mathbf{X} = \sum \xi_i \partial_i, \mathbf{Y} = \sum \eta_i \partial_i, \quad [\mathbf{X}, \mathbf{Y}] = \sum_i \sum_j (\xi_j \partial_j \eta_i - \eta_j \partial_j \xi_i) \, \partial_i$$

Example: $\mathbf{X} = y\partial_x, \mathbf{Y} = x^2\partial_x + xy\partial_y$,

$$[\mathbf{X}, \mathbf{Y}] = xy\partial_x + y^2\partial_y$$

Properties: Bilinearity, Skew-symmetry and Jaconi-identity

$$[\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] + [\mathbf{Y}, [\mathbf{Z}, \mathbf{X}]] + [\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]] = 0$$

Proof: We may assume $\mathbf{X} = \partial_{x_i}$. Then $[\mathbf{X}, \mathbf{Y}] = \sum_j (\partial_i \eta_j) \, \partial_j$ and the rest is Libnitz.

Invariance under transformation of coordinates: $[\mathbf{X}, \mathbf{Y}]' = [\mathbf{X}', \mathbf{Y}']$. If $y = F(x)$ then

$$[\mathbf{X}, \mathbf{Y}]'(\phi)_{(y)} = [\mathbf{X}, \mathbf{Y}](\phi \circ F)_{(x)} = \mathbf{X}\left(\mathbf{Y}(\phi \circ F)\right) - ... = \mathbf{X}\left(\mathbf{Y}'(\phi) \circ F\right)_{(x)} = \mathbf{X}'\mathbf{Y}'(\phi)_{(y)} - ...$$

Theorem 3. *Let*

$$\Psi(t, x) = \exp(-\sqrt{t}\mathbf{W}) \exp(-\sqrt{t}\mathbf{V}) \exp(\sqrt{t}\mathbf{V}) \exp(\sqrt{t}\mathbf{W})$$

Then

$$[\mathbf{V}, \mathbf{W}] = \frac{d}{dt}\Psi(x, t)_{t=0} \ .$$

Proof. Let $y = \exp(\sqrt{t}\mathbf{V})x$, $z = \exp(\sqrt{t}\mathbf{W})y$, $u = \exp(-\sqrt{t}\mathbf{V})z$ hence

$$\Psi(t, x) = \exp(-\sqrt{t}\mathbf{W})u$$

so

$$\phi(\Psi(t, u)) = \phi(u) - \sqrt{t}\mathbf{W}\phi(u) + \frac{t}{2}\mathbf{W}^2\phi(u) + O(t^{3/2})$$

$$\phi(u) = \phi(z) - \sqrt{t}\mathbf{V}\phi(z) + \frac{t}{2}\mathbf{Z}^2\phi(z) + O(t^{3/2})$$

so

$$\phi(\Psi(t, u)) = \phi(z) - \sqrt{t}\left(\mathbf{W}\phi(z) + \mathbf{V}\phi(z)\right) + \frac{t}{2}\left(\mathbf{W}^2\phi(z) + \mathbf{V}^2\phi(z) + 2\mathbf{V}\mathbf{W}\phi(z)\right) + O(t^{3/2})$$

Next

$$\phi(z) = \phi(y) + \sqrt{t}\mathbf{W}\phi(y) + \frac{t}{2}\mathbf{W}^2\phi(y) + O(t^{3/2})$$

so

$$\phi(\Psi(t,u)) = \phi(y) - \sqrt{t}\mathbf{V}\phi(y) + \frac{t}{2}\left(\mathbf{V}^2\phi(y) + 2\mathbf{V}\mathbf{W}\phi(y) - 2\mathbf{W}\mathbf{V}\phi(y)\right) + O(t^{3/2})$$

Next

$$\phi(y) = \phi(z) + \sqrt{t}\mathbf{V}\phi(x) + \frac{t}{2}\mathbf{V}^2\phi(x) + O(t^{3/2})$$

so

$$\phi(\Psi(t,u)) = \phi(x) + t\left(\mathbf{V}\mathbf{W}\phi(x) - \mathbf{W}\mathbf{V}\phi(x)\right) + O(t^{3/2}) = \phi(x) + t[\mathbf{V},\mathbf{W}]\phi(x) + O(t^{3/2})$$

□

Theorem 4. $[\mathbf{X},\mathbf{Y}] = 0$ *iff*

$$\exp(t\mathbf{X})\exp(t\mathbf{Y}) = \exp(t\mathbf{Y})\exp(t\mathbf{X})$$

for any x, t.

Proof. Suppose $\mathbf{X} = \partial_{x_1}$. Then $\exp(t\mathbf{X})(x_1,\dots x_n) = (x_1 + t, x_2, \dots, x_m)$. For $\mathbf{Y} = \sum \eta_i \partial_{x_i}$ it follows that $[\mathbf{X},\mathbf{Y}] = \sum \left(\partial_{x_1}\eta_i\right)\partial_{x_i} = 0$, so $\eta_i = \eta_i(x_2,\dots x_m)$ for all i. Then the system of equations

$$dx_i/dt = \eta_i(x_2,\dots,x_m) \quad 1 \le i \le n$$

defines $\exp(t\mathbf{Y})$. In particular $x_1(t) = x_1 + \int_0^t \eta_1(x_2(t),\dots x_n(t))$ under $\exp(t\mathbf{Y})$. It follows that

$$\exp(t\mathbf{Y})\exp(t\mathbf{X})(x_1,\dots x_n) = \left(x_1 + t + \int_0^t \eta_1(x_2(\tau),\dots x_n(\tau))d\tau, x_2(t),\dots x_n(t)\right)$$

$$= \exp(t\mathbf{X})\exp(t\mathbf{Y})(x_1,\dots x_n)$$

□