Lecture 10

1 Applications to PDE

There is a direct generalization of the symmetry methods yo PDE, and even system of PDE's. Here the independent variable $x = (x_1, \ldots x_q)$ is a vector in $\mathbb{R}^q$ and the dependent variable $u = (u_1, \ldots u_p)$ is in $\mathbb{R}^p$. For a multi-index vector $J = (i_1, \ldots i_q)$, $i_j \in \mathbb{N} \cup \{0\}$ we denote the derivative

$$\partial_J u := \frac{\partial^{|J|} u}{\partial_{x_1}^{i_1} \ldots \partial_{x_q}^{i_q}}$$

where $|J| = i_1 + \ldots + i_q$.

A system of PDE of order n is an equation of the form

$$H_k(x, \partial_{J_1} u_1, \ldots \partial_{J_s} u_k) = 0 \ \ , \ |J| \leq n \ , \ k = 1 \ldots p \ .$$

We now generalizes the notion of a symmetry point transformation on such a PDE. It is a transformation $\psi = (\psi_{(x)}, \psi_{(u)}) : \mathbb{R}^{p+q} \to \mathbb{R}^{p+q}$ given by

$$\psi_{(x)} = \psi_{(x)}(x_1 \ldots x_q, u_1, \ldots u_p) \in \mathbb{R}^q, \quad \psi_{(u)} = \psi_{(u)}(x_1 \ldots x_q, u_1, \ldots u_p) \in \mathbb{R}^p$$

which maps the graph of any solution $u = u(x)$ of the PDE to another solution $\tilde{u}(\tilde{x})$ of the same PDE.

Let now

$$\mathbf{X} = \sum_1^q \xi_i \partial_{x_i} + \sum_1^p \eta_j \partial_{u_j}$$

be a v-f on a domain in $\mathbb{R}^{p+q}$. Its $n-$ prolongation is defined on a domain of dimension $q \begin{pmatrix} p+n \\ n \end{pmatrix}$ coordinates labeled $(x_1, \ldots x_q, \ldots u_{1,(J)}, \ldots u_{p,J})$ where $0 \leq |J| \leq n$:

$$Pr^{(n)}\mathbf{X} = \sum_1^q \xi_i \partial_{x_i} + \sum_{j=1}^p \eta_j \partial_{u_j} + \sum_{j=1}^p \sum_{|J| \leq n} \eta_{j,(J)} \partial_{u_{j,J}}$$

We now generalize the calculation of the prolongation $\eta^{(n)}$ from lecture 3 to the coefficient $\eta_{(J)}$. Recall that the recursion formula

$$\eta^{(n)} = \frac{d}{dx}\eta^{(n-1)} - y_n \frac{d}{dx}\xi \ .$$

This leads to a direct generalization for the case of multiple variables. Suppose, for simplicity, that $p = 1$, so we have only one dependent variable. Then

$$\eta_{(j)} = \frac{d\eta}{dx_j} - \sum_i u_i \frac{d\xi_i}{dx_j} \quad , \quad \eta_{(j,k)} = \frac{d\eta_{(j)}}{dx_k} - \sum_i u_{j,i} \frac{d\xi_i}{dx_k}$$

and so on... When we write these, apparently innocent expressions in details we get

$$\eta_{(n)} = \eta_{,n} + \eta_{,u} u_{,n} - \sum_i \xi_{i,n} u_{,i} - \sum_i \xi_{i,u} u_{,n} u_{,i} \tag{1.1}$$

$$\eta_{(nm)} = \eta_{,nm} + \eta_{,nu} u_m + \eta_{,mu} u_n - \xi_{i,nm} u_{,i} + \eta_{,uu} u_{,n} u_{,m} - \sum_k (\xi_{k,nu} u_{,m} + \xi_{k,mu} u_{,n})$$
$$- \sum_k \xi_{k,uu} u_{,n} u_{,m} u_{,k} + \eta_{,u} u_{,nm} - \sum_i (\xi_{i,n} u_{,mi} - \xi_{i,m} u_{,ni}) - \sum_k \xi_{k,u} (u_{,k} u_{,mn} + u_{,n} u_{,mk} + u_{,nk} u_{,m}) \tag{1.2}$$

The condition that a given v-f $\mathbf{X}$ generates a symmetry of the PDE $H = 0$ of order n is

$$Pr^{(n)} \mathbf{X}(H) = 0 \qquad \mathrm{mod}\ H = 0$$

with obvious generalization for a system.
Examples:

1. The heat equation $u_{,11} - u_{,2} = 0$. From (1.1,1.2)

$$\eta_{(2)} - \eta_{(11)} = \eta_{,2} + \eta_{,u} u_{,2} - \xi_{1,2} u_{,1} - \xi_{2,2} u_{,2} - \xi_{1,u} u_{,2} u_{,1} - \xi_{2,u} u_{,2} u_{,2} -$$
$$\eta_{,11} - 2\eta_{,1u} u_{,1} + \xi_{1,11} u_{,1} + \xi_{2,11} u_{,2} - \eta_{,uu} u_{,1}^2 + 2 \sum_k u_{,k} \xi_{k,u} u_{,1}$$
$$+ \sum_k \xi_{k,uu} u_{,1}^2 u_{,k} - \eta_{,u} u_{,11} + 2 \sum_i \xi_{i,1} u_{,1i} - \xi_{1,u} u_{,1} u_{,11} - 2\xi_{2,u} u_{,1} u_{,12} + \xi_{2,u} u_{,2} u_{,11} \tag{1.3}$$

The coefficient of $u_{1,2}$ is $2\xi_{2,1} - \xi_{2,u} u_{,1}$. It does not contain a multiple of the equation, so it must be zero identically, so $\xi_{2,1} = \xi_{2,u} = 0$ hence

$$\xi_2 = \alpha(x_2) \ . \tag{1.4}$$

The coefficient of $u_{,1}$ is $-\xi_{1,2} - 2\eta_{1,u} + \xi_{1,11}$ and it must be zero since u_1 does not appear in the equation, so

$$-\xi_{1,2} - 2\eta_{,1u} + \xi_{1,11} = 0 \ . \tag{1.5}$$

The coefficient of $u_{,1}^2$ is $2\xi_{1,u} - \eta_{,uu}$ and it must be zero as well:

$$2\xi_{1,u} - \eta_{,uu} = 0 \ . \tag{1.6}$$

We now substitute $u_2 = u_{11}$ in (1.3). The coefficient of u_2 is $2\xi_{1,1} - \xi_{2,2} - 2\xi_{1,u} u_{,1}$. It must be identically zero, so, with (1.6),

$$2\xi_{1,1} = \xi_{2,2} \ \ , \quad \xi_{1,u} = \eta_{uu} = 0 \ . \tag{1.7}$$

Finally, the terms which are free of derivatives of u give, as expected, the heat equation for η:

$$\eta_{,11} = \eta_{,2} \ . \tag{1.8}$$

From (1.4, 1.7) we get $\xi_1 = x_1\alpha^{'}(x_2)/2 + \beta(x_2)$, and

$$\xi_{1,2} = \alpha^{''}(x_2)x_1/2 + \beta^{'}(x_2) = 2\eta_{,1u} \ ,$$

while $\eta = \gamma(x_1, x_2) + u\delta(x_1, x_2)$.... The symmetry fields are:

$$\mathbf{X}_1 = \partial_{x_1} \ , \quad \mathbf{X}_2 = \partial_{x_2} \ , \quad \mathbf{X}_3 = x_1\partial_{x_1} + 2x_2\partial_{x_2} \quad , \quad \mathbf{X}_4 = x_2\partial_{x_1} - (1/2)x_1 u\partial_u$$

$$\mathbf{X}_5 = x_1x_2\partial_{x_1} + x_2^2\partial_{x_2} - u(x_2/2 + x_1^2/4)\partial_u \quad , \quad \mathbf{X}_6 = u\partial_u, \quad \mathbf{X}_7 = g\partial_u$$

where $g = g(x_1, x_2)$ is a solution of the heat equation. These are, in fact, *all* the symmetry generators, so they form an (infinite dimensional) Lie algebra (check!).

Let us now use the symmetries of the heat equation to obtain special solutions. From $\mathbf{X}_5$ we get the transformation

$$\tilde{x_1} = \frac{x_1}{1 - tx_2} \quad , \quad \tilde{x_2} = \frac{x_2}{1 - tx_2} \quad , \tilde{u} = u\sqrt{1 - tx_2}e^{-tx_1^2/4(1-tx_2)} \ .$$

From the trivial solution $\tilde{u} = 1$ we get the family of solutions (setting e.g. $t = -1$),

$$u = \frac{e^{-x_1^2/4(1+x_2)}}{\sqrt{1 + x_2}} \ .$$

From here we can apply the shift in x_2 via $\mathbf{X}_2$ to obtain the fundamental solution

$$u(x_1, x_2) = \frac{e^{-x_1^2/4x_2}}{\sqrt{x_2}}$$

The "trivial" field $\mathbf{X}_7$ can also generate non-trivial solutions. For, let g_1, g_2 any two solutions of the heat equation. Any $\mathbf{X}, \mathbf{Y}$ vectorfields spanned by $\mathbf{X}_1, \ldots \mathbf{X}_6$ we get

$$[\mathbf{X} + g_1\partial_u, \mathbf{Y} + g_2\partial_u] = [\mathbf{X}, \mathbf{Y}] + (\mathbf{X}(g_2) - \mathbf{Y}(g_1))\,\partial_u + g_1\partial_u\mathbf{Y} - g_2\partial_u\mathbf{X}$$

which must be a symmetry as well. Let us take $\mathbf{X} = \mathbf{X}_4$ and $\mathbf{Y} = \mathbf{X}_5$. Since $\partial_u\mathbf{X}_4 = -1/2x_2\partial_u$ and $\partial_u\mathbf{X}_5 = -(x_2/2 + x_1^2/4)$ we get that

$$h := -x_2 g_{2,x_1} - x_1x_2 g_{1,x_1} - x_2^2 g_{1,x_2} - (x_2/2 + x_1^2/4)g_1 + 1/2x_2 g_2$$

must be a solution of the heat equation as well!

2. Laplace equation in $\mathbb{R}^3$:

$$\Delta u := u_{,11} + u_{,22} + u_{,33} = 0$$

We need to prolong the v-f

$$\mathbf{X} = \xi_1\partial_1 + \xi_2\partial_2 + \xi_3\partial_3 + \eta\partial_u$$

into second order, and find the coefficients $\eta_{(11)}, \eta_{(22)}, \eta_{(33)}$. Then $Pr^{(2)}\mathbf{X}(H) = 0$ implies

$$\eta_{(11)} + \eta_{(22)} + \eta_{(33)} = (u_{,11} + u_{,22} + u_{,33})Q$$

where Q is any function of u and its derivatives up to second order (why?).

We now compare the coefficients of the mixed derivatives of u. We obtain as a result of somewhat long but straightforward calculation, that

$$\xi_{1,2} + \xi_{2,1} = 0 \ , \quad \xi_{3,1} + \xi_{1,3} = 0 \ , \quad \xi_{3,2} + \xi_{2,3} = 0 \tag{1.9}$$

as well as that neither ξ_1, ξ_2, ξ_3 are independent of u. In addition

$$\xi_{1,1} = \xi_{2,2} = \xi_{3,3} \tag{1.10}$$

Now, (1.9, 1.10) implies that the vectors $(\xi_1, \xi_2, \xi_3$ generate a conformal mapping in the Euclidian metric $dx_1^2 + dx_2^2 + dx_3^2$ (why?). It can be shown that these are polynomials of order 2 (show it!). Then, a direct computation yields

$$\mathbf{X}_1 = \partial_1, \mathbf{X}_2 = \partial_2, \mathbf{X}_3 = \partial_3 \tag{1.11}$$

as well as the rotations

$$\mathbf{R}_{1,2} = x_2\partial_1 - x_1\partial_2 \ , \quad \mathbf{R}_{1,3} = x_1\partial_3 - x_3\partial_1, \quad \mathbf{R}_{2,3} = x_2\partial_3 - x_3\partial_2 \tag{1.12}$$

and the dilation

$$\mathbf{D} = x_1\partial_1 + x_2\partial_2 + x_3\partial_3 \tag{1.13}$$

An conformal mapping in $\mathbb{R}^3$ is the inversion. For $\boldsymbol{r} := (x_1, x_2, x_3)$, it is given by $I(\boldsymbol{r}) := \boldsymbol{r}/|r|^2$. Then the action

$$q_i := I \circ \exp(t\partial_i) I \tag{1.14}$$

induces a v-f

$$\left.\frac{d}{dt} q_i\right|_{t=0} = (x_1^2 + x_2^2 + x_3^2)\partial_i - 2x_i x_1 \partial_1 - 2x_i x_2 \partial_2 - 2x_i x_3 \partial_3 := I_i \tag{1.15}$$

are additional generators of the conformal group in $\mathbb{R}^3$.

If we now compare the coefficients of u_x^2, say, then we obtain that $\eta_{uu} = 0$, hence

$$\eta = \beta(x_1, x_2, x_3)u + \alpha(x_1, x_2, x_3)$$

Then, the coefficients in the linear derivatives yield the following equations

$$2\partial_i \beta = \Delta \xi^{(i)}$$

and $\Delta\alpha = 0$, that is, α is, by itself, a solution of the Laplace equation. Thus

$$\mathbf{X}_\alpha := \alpha\partial_u \quad , \quad \Delta u = 0 \tag{1.16}$$

is a generator of a symmetry group. From (1.11, 1.12, 1.15) we get that the coefficients $\xi^{(i)}$ are all linear so $\Delta\xi^{(i)} = 0$ and β is a constant. Thus

$$\mathbf{X}_u := \partial_u \tag{1.17}$$

is another symmetry, as well as (1.11, 1.12, 1.15). From (1.15) we have that $\Delta\xi^{(i)} = 2$ so $\beta = x_i$ for any of the v.f. (1.15). Thus, we get the additional 3 generators

$$\mathbf{X}_{Ii} := (x_1^2 + x_2^2 + x_3^2)\partial_i - 2x_i x_1 \partial_1 - 2x_i x_2 \partial_2 - 2x_i x_3 \partial_3 + x_i \partial_u \ , \quad i = 1, 2, 3 \ . \qquad (1.18)$$

Let us try to understand the meaning of the transformation induced by $\mathbf{X}_{Ii}$. From (1.14) we get

$$\boldsymbol{r} \rightarrow \frac{\boldsymbol{r} + tr x_i}{|\boldsymbol{r}/r + t x_i|}$$

and

$$u \rightarrow |\boldsymbol{r}/r + t x_i| u$$

so, for $i = 1, 2, 3$,

$$\tilde{u} = \frac{u\left(\frac{\boldsymbol{r}+tr x_i}{|\boldsymbol{r}/r + t x_i|}\right)}{|\boldsymbol{r}/r + t x_i|}$$

is a solution of the Laplace equation, if $u(\boldsymbol{r})$ is.

3. If we consider only 2 coordinates x_1, x_2, then (1.11-2.6) and (1.18) is reduced to the 5 generators of the *Mobiuos* group on $\mathbb{R}^2$. In complex notation $z = x_1 + i x_2$, $\overline{z} = x_1 - i x_2$, $\partial_z = \partial_{x_1} - i\partial_{x_2}$, $\partial_{\overline{z}} = \partial_{x_1} + i\partial_{x_2}$ we get the generators

$$\partial_z, \quad 2z\partial_z, \quad -z^2\partial_z + z\partial_u$$

which induce, respectively (here t is a complex "time" parameter)

$$z \rightarrow z + t, \quad z \rightarrow e^{2tz}, \quad z \rightarrow \frac{z}{tz + 1} \ .$$

However, (1.9,1.10), when reduced to the two dimensional case, are the C-R equation which are satisfied for *any* transformation induced by an analytic function. In particular, any v-f of the form

$$\mathbf{X} = f(z)\partial_z + \overline{f}(\overline{z})\partial_{\overline{z}} + \left[W(z) + \overline{W}(\overline{z})\right]\partial_u$$

induces such a symmetry group.

4. The case of the *wave equation*

$$\Box u := -u_{,11} + u_{,22} + u_{,33} = 0 \ .$$

By an equivalent way we get the conditions (1.9, 1.10) replaced by

$$\xi_{1,2} - \xi_{2,1} = 0 \ , \quad \xi_{3,1} - \xi_{1,3} = 0 \ , \quad \xi_{3,2} + \xi_{2,3} = 0 \qquad (1.19)$$

$$\xi_{1,1} = \xi_{2,2} = \xi_{3,3} \qquad (1.20)$$

This is the condition for the generators of conformal transformation *with respect to the Lorentz metric* $-dx_1^2 + dx_2^2 + dx_3^2$. In a completely analogous way to the Laplace equation we get the generators (1.11). The rotations (1.12) are replaced by the hyperbolic rotations

$$\mathbf{H}_{1,2} = x_2\partial_1 + x_1\partial_2 \ , \quad \mathbf{H}_{1,3} = x_1\partial_3 + x_3\partial_1, \quad \mathbf{R}_{2,3} = x_2\partial_3 - x_3\partial_2 \tag{1.21}$$

while the dilation (2.6) preserves its form. The action of $\mathbf{H}_{12}$, for example, is

$$(x_1, x_2, x_3) \to (x_2 \cosh(t) + x_1 \sinh(t), x_3, x_2 \sinh(t) + x_1 \cosh(t)) \ ,$$

so $u\left(x_2 \cosh(t) + x_1 \sinh(t), x_3, x_2 \sinh(t) + x_1 \cosh(t)\right)$ is a solution of the wave equation for any t, if $u(x_1, x_2, x_3)$ is.

The generators of inversion with respect to the Lorenz metric are, analogously to (1.18),

$$\mathbf{X}_{Hi} := (-x_1^2 + x_2^2 + x_3^2)\partial_i + 2x_i x_1\partial_1 - 2x_i x_2\partial_2 - 2x_i x_3\partial_3 + x_i\partial_u \ , \quad i = 1, 2, 3 \ . \tag{1.22}$$

Let us demonstrate now a way to obtain similarity solutions of the wave equation. A trivial one is using the symmetry $\mathbf{X} = \partial_{x_1}$. This implies that we can find solutions $u = u(x_2, x_3)$ of the wave equation. The equation these solutions must satisfy is, evidently, the Laplace equation $u_{,22} + u_{,33} = 0$, that is, the potential equation in x_2, x_3.

Another, less trivial example is to use the dilation symmetry $\mathbf{D} = \partial_{x_1} + \partial_{x_2} + \partial_{x_3}$. We use the transformation of the variables x_1, x_2, x_3 to the variables $y_1, y_2\tau$ by which $\mathbf{D} = \partial_\tau$. To obtain this we take $\tau = \ln(x_1)$ and $y_1 = x_2/x_2$, $y_2 = x_3/x_1$ (check). Then we look for solutions of the form $u = u(y_1, y_2)$. For this, we calculate

$$\partial_1 = e^{-\tau}\left(-y_1\partial_{y_1} - y_2\partial_{y_2} + \partial_\tau\right) \ ; \quad \partial_2 = e^{-\tau}\partial_{y_1} \quad , \quad \partial_3 = e^{-\tau}\partial_{y_2}$$

The wave equation is transformed into

$$-\partial_\tau u + \partial_{y_1}^2 u + \partial_{y_2}^2 u - y_1^2\partial_{y_1}^2 u - y_2^2\partial_{y_2}^2 u - 2y_1y_2\partial_{y_1}\partial_{y_2} u - 2y_2\partial_{y_2} u - 2y_1\partial_{y_1} u = 0 \tag{1.23}$$

and we can easily obtain the equation for $u = u(y_2, y_3)$ by omitting the τ derivative from the equation.

5. If, as in the Laplace equation, we remove x_3 from the game, we get

$$\xi_{1,2} - \xi_{2,1} = 0 \quad , \xi_{1,1} = \xi_{22} \tag{1.24}$$

hence $\xi_{1,11} - \xi_{1,22} = \xi_{2,11} - \xi_{2,22} = 0$. So, both ξ_1, ξ_2 verify the wave equation in two independent variables. The general solution is a function of $x_1 + x_2$ or a function of $x_1 - x_2$. So, we get

$$\mathbf{X}_1 = \alpha(x_1 - x_2)(\partial_{x_1} - \partial_{x_2}) \quad , \quad \mathbf{X}_2 = \beta(x_1 + x_2)(\partial_{x_1} + \partial_{x_2})$$

where α and β are arbitrary functions. Indeed, under these transformations,

$$\tilde{x}_1 - \tilde{x}_2 = x_1 - x_2 + t\alpha(x_1 - x_2) + \ldots, \quad \tilde{x}_1 + \tilde{x}_2 = x_1 + x_2 + t\beta(x_1 - x_2)$$

while $\tilde{u} = u$. Thus, if we omit the "tilde" from x_1, x_2,

$$\tilde{u}(\tilde{x}_1, \tilde{x}_2) = u\left(x_1 + t\alpha(x_1 - x_2) + t\beta(x_1 + x_2), x_2 - t\alpha(x_1 - x_2) + t\beta(x_1 + x_2)\right) \ . \tag{1.25}$$

Indeed, if u is a solution of the wave equation then $u = \psi(x_1 + x_2)$ or $u = \phi(x_1 - x_2)$. In both cases the structure of the solition is preserved under (1.25).

2 Multiple reduction

When is it possible to use further symmetry generators? Suppose we utilized the symmetry $\mathbf{X}$. It means that we dropped the variable τ which verifies $\mathbf{X} = \partial_\tau$. In the new variables $\tau, y_1, \dots y_k$ the equation takes the form

$$H(y_1, \dots y_k, u_{(J)}) = 0$$

where $u_{(J)}$ stands, as usual, for the derivatives of u. Since we consider only functions $u = u(y_1, \dots y_k)$, the condition for another symmetry $\mathbf{Y}$ to be a symmetry of the reduced equation is that

$$\mathbf{Y} = \alpha(\tau, y_1, \dots y_k)\partial_\tau + \sum_1^k \beta_j(y_1, \dots y_k)\partial_{y_i}$$

that is, the components in the direction different from τ must be independent of τ. In coordinate-invariant form this implies that $[\mathbf{X}, \mathbf{Y}]$ should be in the direction of $\mathbf{X}$.

Let us consider again the wave equation. We used the dilation $\mathbf{D}$ to reduce it to the form (1.23). Now, we my use the rotation symmetry $\mathbf{R}_{23}$. Indeed, $[\mathbf{D}, \mathbf{R}_{2,3}] = 0$. In the new coordinates it takes the form

$$\mathbf{R}_{23} = \mathbf{R}_{23}(y_1)\partial_{y_1} + \mathbf{R}_{23}(y_2)\partial_{y_2} = y_2\partial_{y_1} - y_1\partial_{y_2} \ .$$

The invariants of $\mathbf{R}_{23}$ are $\rho := y_1^2 + y_2^2$ and u, so we look for solutions of the form $u = u(\rho)$. We can find

$$\partial_{y_1}^2 u + \partial_{y_2}^2 u - y_1^2\partial_{y_1}^2 u - y_2^2\partial_{y_2}^2 u - 2y_1y_2\partial_{y_1}\partial_{y_2} u - 2y_2\partial_{y_2} u - 2y_1\partial_{y_1} u = 4\rho(1-\rho)\partial_\rho^2 u + (4-6\rho)\partial_\rho u = 0$$

which is an ODE.

2.1 Separation of variables

We consider the Helmhotz equation

$$\Psi_{xx} + \Psi_{yy} + \omega^2\Psi = 0 \tag{2.1}$$

and look for symmetries of the form

$$\mathbf{X} = X(x, y, z)\partial_x + Y(x, y, z)\partial_y + Z(x, y, z)\partial_z$$

and $\mathbf{Q} = \partial_x^2 + \partial_y^2 + \omega^2 z\partial_z$. Then the condition for a symmetry is

$$[\mathbf{X}, \mathbf{Q}] = R\mathbf{Q} \tag{2.2}$$

where R is some function of x, y, z and its prolonged variables. The set of symmetries verifying (2.2) is a Lie algebra. It follows that u is a solution of (2.1) then $\mathbf{X}(u)$ is a solution as well. We find out that

$$\mathbf{X}_1 = \partial_x \ , \quad \mathbf{X}_2 = \partial_y \ , \quad \mathbf{X}_3 = y\partial_x - x\partial_y$$

are the generators of this group, and the structure of this algebra is

$$[\mathbf{X}_1, \mathbf{X}_2] = 0 \ , \quad [\mathbf{X}_3, \mathbf{X}_1] = \mathbf{X}_2 \ , \quad [\mathbf{X}_3, \mathbf{X}_2] = -\mathbf{X}_1 \ .$$

This is the *Euclidian group*, parameterized by g_1, g_2, g_3 where

$$T_1(g_1)(\phi) := \exp(g_1\mathbf{X}_1)(\phi) = \phi(x + g_1, y) \ , \ \ T_2(g_2)(\phi) := \exp(g_2\mathbf{X}_2)(\phi) = \phi(x, y + g_2)$$

$$T_3(g_3)(\phi) := \exp(g_1\mathbf{X}_3)(\phi) = \phi\left(x\cos(g_3) + y\sin(g_3), -x\sin(g_3) + y\cos(g_3)\right)$$

Then, any action on the set of functions $\phi = \phi(x, y)$ is given by

$$T(g_1, g_2, g_3) = T_2(g_2)T_1(g_1)T_3(g_3)$$

We may look for second order operators which generate a symmetry. Such an operator

$$\mathbf{S} = A_{11}\partial_x^2 + A_{12}\partial_{xy}^2 + A_{22}\partial_y^2 + B_1\partial_x + B_2\partial_y + C$$

is a symmetry operator if

$$[\mathbf{S}, \mathbf{Q}] = U\mathbf{Q} \tag{2.3}$$

where U is a first order operator (so that the r.h.s of (2.3) is of third order, as it should ne). Again, it follows that if u is a solution of (2.1) then so is $\mathbf{S}(u)$.

Note that $\mathbf{X}_1^2 + \mathbf{X}_2^2 = -\omega^2$ is identified by a zero order operator. Let $\mathcal{S}$ be the space of all symmetry operators of second order (including the first order ones), *modulo* $\mathbf{X}_1^2 + \mathbf{X}_2^2$.

Let $\mathcal{S}^{(2)} \subset \mathcal{S}$ be the set of "pure" second order operators, and $\mathcal{S}^{(1)}$ the first order operators in $\mathcal{S}$ (namely, those generating the Lie algebra). It follows that $\mathcal{S}^{(2)}$ is generated by the five operators

$$\mathbf{X}_2^2, \ \ \mathbf{X}_1\mathbf{X}_2, \ \ \mathbf{X}_3^2, \ \ \{\mathbf{X}_3, \mathbf{X}_1\}, \ \ \{\mathbf{X}_3, \mathbf{X}_2\} \tag{2.4}$$

where $\{\mathbf{X}, \mathbf{Y}\} := \mathbf{XY} + \mathbf{YX}$. Recall that $\mathbf{X}_1^2$ is not included, since we take the space modulo $\mathbf{X}_1^2 + \mathbf{X}_2^2$.

We know two ways to separate variables for the Helmhotz equation. The first is by eigenvalues of $\mathbf{X}_1, \mathbf{X}_2$, namely $\psi_k := e^{i[kx + \sqrt{\omega^2 - k^2}y]}$. Here

$$\mathbf{X}_1\psi_k = ik\psi_k \ \ , \ \mathbf{X}_2\psi_k = i\sqrt{\omega^2 - k^2}\psi_k \ . \tag{2.5}$$

The second one is $\psi_k := J_k(\omega r)e^{ik\theta}$ where J_k is the Bessel function

$$r^2 J_k^{''} + rJ_k^{'} + (r^2\omega^2 - k^2)J_k = 0$$

Here, also, ψ_k is given by

$$\mathbf{X}_3\psi_k = ik\psi_k \ . \tag{2.6}$$

More generally, if $\mathbf{X}$ is a symmetry v-f of order one, and $u(x, y), v(x, y)$ is a new set of variables for which $\mathbf{X} = \partial_u$, then $\psi_k(u, v) = e^{iku}V(v)$ is a solution of (2.1) provided V solves a second order ODE whose coefficients depend only on v. So $\mathbf{X}(\psi_k) = ik\psi_k$.

Note also that if $\mathbf{X}$ is a symmetry (either first or second order), then $\mathbf{X}^g := T(g)\mathbf{X}T(g^{-1})$ is also a symmetry, and $\psi_k^g := T(g)\psi_k$ verifies $\mathbf{X}^g\psi_k^g = ik\psi_k^g$. In particular, it follows that

$\mathbf{X}^{g_1g_2} = (\mathbf{X}^{g_2})^{g_1}$ is a representation of the group G in the Lie algebra. A direct computation yields

$$\mathbf{X}_1^{g_1} = \mathbf{X}_1, \quad \mathbf{X}_2^{g_1} = \mathbf{X}_2, \quad \mathbf{X}_3^{g_1} = \mathbf{X}_3 - g_1\mathbf{X}_2$$
$$\mathbf{X}_1^{g_2} = \mathbf{X}_1, \quad \mathbf{X}_2^{g_2} = \mathbf{X}_2, \quad \mathbf{X}_3^{g_2} = \mathbf{X}_3 + g_1\mathbf{X}_1$$
$$\mathbf{X}_1^{g_3} = \cos(g_3)\mathbf{X}_1 + \sin(g_3)\mathbf{X}_2 \ , \ \mathbf{X}_2^{g_3} = -\sin(g_3)\mathbf{X}_1 + \cos(g_3)\mathbf{X}_2 \ , \quad \mathbf{X}_3^{g_3} = \mathbf{X}_3 \quad (2.7)$$

This implies a representation (the so called *adjoint representation*) on the Lie algebra $\mathcal{S}^{(1)} := Span\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$. Moreover, if $\mathbf{S} \in \mathcal{S}$ and ψ is an eigenfunction of $\mathbf{S}$, then ψ^g is an eigenfunction of $\mathbf{S}^g$ as well. So, we identify the eigenfunctions modulo the group action $\psi \sim \psi^g$. From (2.7) we get that there are only two orbits of the adjoint action on $\mathcal{S}^{(1)}$. The first is spanned by $\mathbf{X}_1, \mathbf{X}_2$ and can be identified with either of them. the second is composed of all the fields which have a nonzero component of $\mathbf{X}_3$. It can be identified with $\mathbf{X}_3$. Thus, (2.5, 2.6) are the *only* representatives of the eigenspaces associated with the operators in $\mathcal{S}^{(1)}$.

We can also verify that, for any $\mathbf{X}, \mathbf{Y}$, $\{\mathbf{X}, \mathbf{Y}\}^g = \{\mathbf{X}^g, \mathbf{Y}^g\}$ so we extend the adjoint representation from $\mathcal{S}^{(1)}$ to the whole of $\mathcal{S}$. To verify this, we only have to notice that the space spanned by $\mathbf{X}_1^2 + \mathbf{X}_2^2$ commutes with all the elements of $\mathcal{S}$.

Using this, we may exted the adjoint action from $\mathcal{S}^{(1)}$ to $\mathcal{S}^{(2)}$. We find out, in this way, that there are only 4 orbits of this group acting on $\mathcal{S}^{(2)}$. These are given by

$$(a)\ \mathbf{X}_2^2 \ , \quad (b)\ \mathbf{X}_3^2, \quad (c)\ \{\mathbf{X}_3, \mathbf{X}_2\} \ , \quad (d)\ \mathbf{X}_3^2 + \alpha\mathbf{X}_1^2 \qquad (2.8)$$

where $\alpha \in \mathbb{R}$ is a free parameter. We see that (2.8-a) yields the separation variables x, y with the separation function given as in (2.5). Next, (2.8-b) yields the polar variables r, θ with the separation function given as in (2.6).

The next two cases give us new set of variables. For (2.8-c) we get the *parabolic coordinates*

$$x = \frac{1}{2}(u^2 - v^2) \ , \quad y = uv$$

in terms of which the equation (2.1) takes the form

$$\partial_u^2\psi + \partial_v^2\psi + (u^2 + v^2)\omega^2\Psi = 0$$

and

$$\{\mathbf{X}_3, \mathbf{X}_2\} = (u^2 + v^2)^{-1}\left(v^2\partial_u^2 - u^2\partial_v\right)$$

The eigenfunctions $\{\mathbf{X}_3, \mathbf{X}_2\}\Psi_k = k^2\Psi_k$ are $\Psi_k = U(u)V(v)$ where

$$U'' + (\omega^2u^2 - k^2)U = 0 \ , \quad V'' + (\omega^2v^2 + k^2)V = 0$$

For (2.8-d) we get the *elliptic coordinates*

$$x = \alpha\cosh u\cos v \ , \quad y = \alpha\sinh u\sin v$$

in terms of which the equation (2.1) takes the form

$$\partial_u^2\psi + \partial_v^2\psi + \alpha^2\omega^2(\cosh^2 u - \cos^2 v)\Psi = 0$$

and

$$\mathbf{X}_3^2 + \alpha \mathbf{X}_1^2 = (\cosh^2 u - \cos^2 v)^{-1} \left(\cos^2 v \partial_u^2 + \cosh^2 u \partial_v^2 \right)$$

The eigenfunctions $(\mathbf{X}_3^2 + \alpha \mathbf{X}_1^2)\Psi_k = k^2 \Psi_k$ are $\Psi_k = U(u)V(v)$ where

$$U^{''} + (\alpha^2 \omega^2 \cosh^2 u + k^2)U = 0 \quad , \quad V^{''} - (\alpha^2 \omega^2 \cosh^2 v + k^2)V = 0$$

For further information see the book of W. Miller, *Symmetry and separation of variables*, ch. 1.3.