Lecture 2

1 Surfaces and their tangents

There are several ways to define n dimensional surface in $\mathbb{R}^m$. The first is by a mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^m$. The second is as the level surface of a function $F : \mathbb{R}^m \to \mathbb{R}^{m-n}$.

Definition 1.1. *A surface N in $\mathbb{R}^m$, given as the image of $\Phi(U)$ for $U \subset \mathbb{R}^n$, is of dimension n if the degree of the Jacobian matrix $D\Phi$ is n (its maximal possible degree). A vector field tangent to N is given by $\Phi_*(\mathbf{X})$ where $\mathbf{X}$ is a vectorfield on U in $\mathbb{R}^n$.*

In the other case,

Definition 1.2. *A surface N in $\mathbb{R}^m$, given as the level set of $F = z_0 \in \mathbb{R}m - n$ is of dimension n if the degree of the Jacobian matrix DF is $m - n$. A vectorfield $\mathbf{X}$ in $\mathbb{R}^m$ is tangent to N if its image $F_*(\mathbf{X})$, which is a vectorfield in $\mathbb{R}^{m-n}$, attains a zero at z_0.*

Example: $N = \mathbb{S}^2 \subset \mathbb{R}^3$. $F(x, y, z) = 1 - x^2 - y^2 - z^2 = 0$. The tangent are all vector fields of the form

$$\{a\partial_x + b\partial_y + x\partial_z \quad ; \quad ax + by + cz = 0\}$$

Lemma 1.1. *If $\mathbf{X}, \mathbf{Y}$ are tangent to N at any point, so is $[\mathbf{X}, \mathbf{Y}]$.*

For $\mathbf{X} = z\partial_z - z\partial_z$, $\mathbf{Y} = z\partial_y - y\partial_z$ then $[\mathbf{X}, \mathbf{Y}] = y\partial_x - x\partial_y$ is also tangent.

Definition 1.3. *Let $(\mathbf{X}_1, \ldots \mathbf{X}_k)$ a family of vector fields in $\mathbb{R}^m$. Then N is an integral surface if its tangent at any poind it spanned by $(\mathbf{X}_1, \ldots \mathbf{X}_k)$ at his point. $(\mathbf{X}_1, \ldots \mathbf{X}_k)$ is integrable iff there exists an integral manifold trough any point.*

Definition 1.4. *$(\mathbf{X}_1, \ldots \mathbf{X}_k)$ is said to be in involution if there exists functions $c^k_{i,j}(x)$ so that $[\mathbf{X}_i, \mathbf{X}_j] = \sum_k c^k_{i,j}\mathbf{X}_k$.*

Theorem 1. *(Frobinous) $(\mathbf{X}_1, \ldots \mathbf{X}_k)$ are linearly independent. Then they are in involution iff they are integrable.*

Idea of proof: let

$$\psi(t_1, \ldots t_k) = \exp(t_1\mathbf{X}_1)\exp(t_2\mathbf{X}_2)\ldots\exp(t_k\mathbf{X}_k)x_0$$

where $x_0 \in \mathbb{R}^m$. We then show that this defines, locally, an integrable surface tangent to the vector fields. This is evident for x_0, and also for $\psi(0, 0, ..., t_j, 0...)$ by Lemma 0.1....

2 Lie algebras of vector fields

Definition 2.1. *If* $(\mathbf{X}_1, \dots \mathbf{X}_k)$ *are linearly independent and* $[\mathbf{X}_i, \mathbf{X}_j] = \sum_k c^k_{i,j} \mathbf{X}_k$ *where* $c^k_{i,j}$ *are constants, then they form a Lie algebra under the Lie derivative.*

Theorem 2. *Lie algebra generate an action of a Lie group via*

$$\psi(t_1, \dots t_k, x) = \exp(t_1 \mathbf{X}_1) \exp(t_2 \mathbf{X}_2) \dots \exp(t_k \mathbf{X}_k) \circ x_0 \tag{2.1}$$

or

$$\psi(t_1, \dots t_k, x) = \exp\left(\sum_1^k t_k \mathbf{X}_k \right) \circ x_0 \tag{2.2}$$

or any combination thereof.

Problem

1. Show that $[\mathbf{X}, \mathbf{Y}]_{(x)} = - \frac{d}{dt}\Big|_{t=0} \exp(t\mathbf{X})_* (Y_{(\exp(-t\mathbf{X}) \circ x)})$.
2. Show that, at any point $(t_1, \dots t_k)$, $\psi_*(\partial_{t_j})$ is tangent to the orbit (2.1) and (2.2) at that point (that is, spanned by $\mathbf{X}_1, \dots \mathbf{X}_k$ at $x = \psi(t_1, \dots t_k)$.

Examples

1. $\mathbf{X}_1 = \partial_x$, $\mathbf{X}_2 = x\partial_x + y\partial_y$. Then $[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_1$. Let $(x_0, y_0) \in \mathbb{R}^2$, $y_0 > 0$.

   $$\psi(t_1, t_2) := \exp(t_2 \mathbf{X}_2) \exp(t_1 \mathbf{X}_1) \circ (x_0, y_0) = \left(e^{t_2} x_0 + t_1 e^{t_2}, y_0 e^{t_2} \right) \tag{2.3}$$

   this is an action of the group $G := (a, b)$, $b > 0$ defined by $(a_2, b_2) \circ (a_1, b_1) := (a_2 e^{-b_1} + a_1, b_1 + b_2)$.

   Let us compute the vectorfields induced on G. Let $\theta(a, b) := \phi\left(e^b (x_0 + a), y_0 e^b \right)$. Then $\partial_a \theta = e^b \mathbf{X}_1(\phi)$, $\partial_b \theta = \mathbf{X}_2(\phi)$ so the vectorfields

   $$e^{-b} \partial_a = [\psi_\#]^{-1} (\mathbf{X}_1) \quad , \partial_b = [\psi_\#]^{-1} (\mathbf{X}_2) \tag{2.4}$$

   are the Lie algebra of the group G on the upper half plane.

   Problem: Calculate the Lie group and the corresponding Lie algebra for $\psi(t_1, t_2) := \exp(t_1 \mathbf{X}_1) \exp(t_2 \mathbf{X}_2) \circ (x_0, y_0)$ and $\psi(t_1, t_2) := \exp\left(t_1 \mathbf{X}_1 + t_2 \mathbf{X}_2 \right) \circ (x_0, y_0)$

2. $\mathbf{X}_1 = \partial_x$, $\mathbf{X}_2 = \partial_y$, $\mathbf{X}_3 = y\partial_x - x\partial_y$. Then

   $$\exp(t_1 \mathbf{X}_1) \exp(t_2 \mathbf{X}_2) \exp(t_3 \mathbf{X}_3) z = e^{it_3} (z + t_1 + it_2)$$

   which is a transformation group under

   $$(t, t_3) \cdot (\tau, \tau_3) := (t_3 + \tau_3, e^{it_3} \tau + t) \ .$$

2.1 Invariant functions

$I : M \to \mathbb{R}$ is $G-$ invariant if $I(\psi(,x)) = I(x)$ for all $g \in G$ and $x \in M$. Equivalent conditions:

- I is constant on orbits of G.
- The level sets $\{I(x) = c\}$ are G invariant subsets of M.

Examples:

- Action of $O(n)$ on $\mathbb{R}^n$ are all radial functions $I = I(r)$, $r = \sqrt{x_1^2 + \dots x_n^2}$.
- Only constants are invariant functions of a transivite action (e.g $GL(\mathbb{R}, n)$ on $\mathbb{R}^n$).
- Invariant functions of the co-adjoint action of $O(n)$ on $Sym(n)$: All symmetric polynomials of the eigenvalues $tr(A^k)$, $k = 1, \dots n$.

Functionally dependence of a set $f_1, \dots f_n$ if for any x there exists a neighborhood U and a function H of n variables so $H(f_1, \dots f_n) \equiv 0$ on U.
A *fundamental problem* is to determine a complete set of independent invariant functions for a certain group action.

Local invariants: For open $U \subset M$, $I : U \to \mathbb{R}$ is a local invariant iff there exists $V_x \subset G$, neighborhood of the identity, and $I(g \cdot x) = I(x)$ for any $x \in U$ and $g \in V_x$ for which $g \cdot x \in U$. Example: the action of $\mathbb{R}$ on the 2-torus $g(\theta, \phi) = (\theta + g, \phi + \kappa g)$ has a local invariant $I = \phi - \kappa\theta$. It is, indeed, local if $\kappa \notin \mathbb{Q}$.

2.2 Vectorfields induced by action

Let $\boldsymbol{v}$ be a vector tangent to G at the identity $e \in G$. Then $\boldsymbol{v}$ induces a vector-field $\mathbf{X}^{(v)}$ on M as follows:

$$\mathbf{X}^{(v)}_{(x)} = \Psi^{(x)}_*(\boldsymbol{v}) \ \ \forall \, x \in M \tag{2.5}$$

where $\Psi^{(x)} := \Psi(x, \cdot) : G \to M$. That is, for any $\phi \in C^\infty(M)$

$$\mathbf{X}^{(v)}(\phi)_{(x)} = \boldsymbol{v} \left(\phi(\Psi^{(x)}(h) \right)_{h=0} .$$

Examples:

1. Using (2.3) we take

$$\Psi^{(x,y)}(t_1, t_2) = \left(e^{t_2} x + t_1 e^{t_2}, y e^{t_2} \right)$$

The identity of the group is $t_1 = t_2 = 0$ so

$$\mathbf{X}^{(\partial_{t_1})}(\phi)_{(x,y)} = \partial_{t_1} \left(\phi(\Psi^{(x,y)}(t_1, t_2) \right)_{t_1 = t_2 = 0} = \partial_x \phi = \mathbf{X}_1(\phi)$$

$$\mathbf{X}^{(\partial_{t_2})}(\phi)_{(x,y)} = \partial_{t_2} \left(\phi(\Psi^{(x,y)}(t_1, t_2) \right)_{t_1 = t_2 = 0} = (x\partial_x + y\partial_y)\phi = \mathbf{X}_2(\phi)$$

2. The action of $\mathbb{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$: Let $\boldsymbol{v}_{i,j} := \partial_{t_{i,j}}$ at the identity of $\mathbb{GL}(n,\mathbb{R})$ (the identity matrix $t_{i,j} = \delta_{i,j}$). Then

$$\partial_{t_{i,j}} \phi \left(\sum_l t_{1,l} x_l, \sum_l t_{2,l} x_l \ \ldots \sum_l t_{n,l} x_l \right)_{t_{i,j}=\delta_{i,j}} = x_j \phi_{x_i}$$

so $\mathbf{X}^{(\partial_{t_{i,j}})} = x_j \partial_{x_i}$.

2.3 Right action on Lie groups

In case $M = G$ then we define $\Psi(g,h) = R_g(h) = h \cdot g$ where $g, h \in G$. Any vector $\boldsymbol{v}$ tangent to the identity of G is extended to a vectorfield $\boldsymbol{V}$ on G as follows:

$$(R_g)_*(\boldsymbol{v})_{(h=0)} = \boldsymbol{V}_g,$$

that is,

$$\boldsymbol{V}_g(\phi) = \boldsymbol{v}_{(h=e)}(\phi(h \cdot g))$$

This way we obtain *Right-invariant vector fields* on G, each is uniquely determined by its value at the identity. In fact

$$R_{g,*}\boldsymbol{V} = \boldsymbol{V}$$

since

$$R_{g,*}(\boldsymbol{V}_h) = R_{g,*}\left(R_{h,*}\boldsymbol{v}\right) = \left(R_g R_h\right)_* \boldsymbol{v} = R_{h \cdot g,*}\boldsymbol{v} = \boldsymbol{V}_{R_g(h)}$$

Corollary 2.1. *The right-invariant v.f is a finite dimensional Lie algebra, which induces a Lie algebra structure on the tangent of G at the identity as*

$$[\boldsymbol{v}_1, \boldsymbol{v}_2] := [\boldsymbol{V}_1, \boldsymbol{V}_2]_{(e)} \ .$$

Examples:

1. The vector fields (2.4) are right invariant for the group $(h_1, h_2) \circ (a,b) := (h_1 e^{-b} + a, b + h_2)$. Indeed, $(R_{(a,b)})_* \partial_{h_1} = e^{-b}\partial_a$ and $(R_{(a,b)})_* \partial_{h_2} = \partial_b$

2. The right action of $\mathbb{GL}(m,\mathbb{R})$: Let $\boldsymbol{v}^{(i,j)} := \partial_{x_{i,j}}$ be a vector at the identity $x_{i,j} = \delta_{i,j}$. Then, for $T = \{t_{i,j}\} \in \mathbb{GL}(m,\mathbb{R})$

$$\boldsymbol{V}_T^{(i,j)} = \partial_{x_{i,j}} \left(\sum_{k,m} x_{l,k} t_{k,m} \right) \partial_{x_{l,m}} = \sum_m t_{j,m} \partial_{x_{i,m}}$$

If we replace $t_{i,j}$ by $x_{i,j}$ we obtain all the right-invariant vector fields

$$\boldsymbol{V}^{(a)} = \sum_{i,j,m} a_{i,j} x_{j,m} \partial_{x_{i,m}}$$

for any $m \times m$ real matrix $a = \{a_{i,j}\}$. We can now calculate

$$[\boldsymbol{V}^{(a)}, \boldsymbol{V}^{(b)}] = \boldsymbol{V}^{([a,b])}$$

where $[a,b] = ab - ba$, the matrix cumutator.

2.4 Relation between action induced and right-invariant vectorfields

Lemma 2.1. *If $\Psi : G \times M \to M$ is an action on M, $\boldsymbol{v}$ a vector in the tangent of G at the identity, $\mathbf{X}^{(v)}$ the induced vector via (2.5) and $\boldsymbol{V}^{(v)}$ a right invariant on G, then*

$$\Psi_*^{(x)}\left(\boldsymbol{V}^{(v)}\right) = \mathbf{X}^{(v)} \ .$$

Proof. Since $\Psi^{(x)} \circ R_g(h) = \Psi^{(\Psi^{(x)}(g))}(h)$ (prove) and $\left[\Psi^{(x)} \circ R_g\right]_* = \Psi_*^{(x)} \circ R_{g,*}$ we get

$$\Psi_*^{(x)}\left(\boldsymbol{V}_g^{(v)}\right) = \Psi_*^{(x)} \circ R_{g,*}\,(v) = \Psi_*^{(\Psi^{(x)}(g))}(v) = \mathbf{X}^{(v)}_{(\psi^{(x)}(g))}$$

□

Corollary 2.2. *The vectorfields $\mathbf{X}$ induced by an action $\Psi : G \times M \to M$ is a finite Lie algebra, where*

$$\left[\mathbf{X}^{(\boldsymbol{v}_1)}, \mathbf{X}^{(\boldsymbol{v}_2)}\right] = \mathbf{X}^{([\boldsymbol{v}_1, \boldsymbol{v}_2])}$$

and $[\boldsymbol{v}_1, \boldsymbol{v}_2]$ is the Lie multiplication induced on the tangent of G at the identity by the right-invariant extension.

Remark 2.1. *In general,*

$$\Psi(g, \cdot)_* \mathbf{X}^{(v)} \neq \mathbf{X}^{(v)} \ .$$

Indeed, Let $g \in G$ and $\Psi_{(g)} = \Psi(\cdot, g) : M \to M$.

Then

$$\Psi_{(g),*}\mathbf{X}^{(v)}(\phi)_{\Psi_{(g)}(x)} \equiv \mathbf{X}^{(v)}\left(\phi \circ \Psi_{(g)}\right)_{(x)} \equiv \boldsymbol{v}\left(\phi \circ \Psi_{(g)} \circ \Psi^{(x)}(h)\right)_{h=0}$$

On the other hand

$$\mathbf{X}^{(v)}(\phi)_{\Psi_{(g)}(x)} \equiv \boldsymbol{v}\left(\phi \circ \Psi^{(\Psi_{(g)}(x))}(h)\right)_{h=0}$$

but $\Psi^{(\Psi_{(g)}(x))}(h) \equiv \Psi\left(h, \Psi_{(g)}(x)\right) \equiv \Psi\left(h, \Psi(g, x)\right) \equiv \Psi(hg, x)$ while $\Psi_{(g)} \circ \Psi^{(x)}(h) = \Psi(g, \Psi(h, x)) = \Psi(gh, x)$ and, in general, $\Psi(gh, x) \neq \Psi(hg, x)$.

2.5 Infinitesimal invariants

Lemma 2.2. *$f : M \to \mathbb{R}$ is invariant under the action Ψ of G iff $\mathbf{X}(f) = 0$ for any vectorfiels $\mathbf{X}$ induced by the action.*

Proof. Assume $f(x) = f(\Psi, g, x))$ for any $g \in G$. In particular, $\boldsymbol{c}_{(h=0)}\left(f \circ \Psi(h, x)\right) = 0$ for any $\boldsymbol{v}$ at the identity of G. But, according to definition, it is just $\mathbf{X}^{(v)}(f) = 0$ where $\mathbf{X}^{(v)}$ is induced on M by the action.

Conversely, if $\mathbf{X}(f) = 0$ everywhere then also

$$\frac{d}{dt} f\left(\exp(t\mathbf{X})x\right) = 0$$

for any t so $f(x) = f\left(\exp(\mathbf{X})\right)$ for any $\mathbf{X}$ in the Lie algebra. But any element $\psi(g, \cdot)$ is represented in this way. □

Corollary 2.3. *f is an invariant iff it is a solution of the linear first order PDE*

$$\sum_1^m \xi_i \frac{\partial f}{\partial_{x_i}} = 0$$

for any $\mathbf{X} = \sum \xi_i \partial_{x_i}$ in the Lie algebra.

Example: the Lie albebra of the translation $(x, y) \to (x + ct, y + t)$ is $\mathbf{X} = c\partial_x + \partial_y$, and $f(x, y) = x - cy$ is an invariant. Is it the only invariant?

Theorem 3. *Let N be a surface determined by $f_1 = \ldots f_{m-n} = 0$ and $(\nabla f_1, \ldots \nabla f_{m-n})$ is of rank $m - n$. Then N is G invariant under the action ψ of G on $\mathbb{R}^m$ iff $\mathbf{X}(f_j) = 0$ for any $\mathbf{X}$ in the Lie algebra of ψ, $1 \leq j \leq m - n$ and $x \in N$.*

Proof. Without limit of generality we can transform the coordinated in $\mathbb{R}^m$ into $y_1, \ldots y_m$ where $y_i = f_i$ for $i = 1, \ldots n$. Then N is (locally) given by $y_1 = \ldots = y_m = 0$. If $\mathbf{X}$ in the Lie algebra then $\mathbf{X} = \sum \xi_i \partial_{y_i}$, so $\mathbf{X}(f_j) = \xi_j$. The condition $\mathbf{X}(f_j) = 0$ implies that

$$\xi_j = 0 \quad 1 \leq j \leq n \quad \text{whenever} \quad y_1 = \ldots = y_n = 0 \ .$$

The flow $\phi(t, x)$ is a solution of

$$\frac{d\phi_i}{dt} = \xi_i(\phi(x, t)) \quad , \phi_i(0) = 0 \quad for \ i = 1 \ldots n \ .$$

Hence $\phi_i = 0$ for $i = 1 \ldots n$. □

Example: $f(x, y) = x^4 + x^2y^2 + y^2 - 1$. The Lie algebra of the rotation group is $\mathbf{X} = -x\partial_y + y\partial_x$, and

$$\mathbf{X}(f) = -4x^3y - 2xy^3 + 2x^3y + 2xy = -2xy(x^2 + 1)^{-1} f(x, y)$$

so the zero level of f is rotation invariant. Indeed $f(x, y) = (x^2 + 1)(x^2 + y^2 - 1)$.

Note that $H(x, y) = y^2 - 2y + 1$ also verifies $\mathbf{X}(H) = 2x(y - 1) = 0$ for $H(x, y) = 0$ but is *not* rotationally invariant. In this case, $\nabla H = 0$ on $H = 0$!

Definition 2.2. *If the dimension of the space spanned by all vectorfields induced by an action Ψ at a point $x \in M$ is is independent of x (in a neighborhood of x), then the action is called regular (locally regular).*

Proposition 2.1. *If the action Ψ of a group is locally regular, then there exists a complete set of invariant functions $f_1, \ldots f_n$ under the action, in the sense that any invariant function g is of the form $g = G(f_1, \ldots f_n)$ for some function G of n variables.*