Lecture 3

1 How to construct invariants for a given group action?

Let the action given by a single vector field $\mathbf{X}$ in $\mathbb{R}^m$. If $\mathbf{X} \neq 0$ everywhere then there exists exactly $m-1$ functionally independent invariants $\phi_1, \ldots \phi_{m-1}$ for this action.

If s vector fields are given which span $s-$dimensional orbits of a group action, then we expect to find $m-s$ functionally dependent invariants.

Examples:

- $SO(2)$ acting on $\mathbb{R}^2$ via $\mathbf{X} = x\partial_y - y\partial_x$. Then x^2+y^2 is an invariant, and any other invariant is a function thereof.

- $SO(2)$ acting on $\mathbb{R}^3$ via $\mathbf{X} = x\partial_y - y\partial_x + (1+z^2)\partial_z$. Then we have to solve

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{dz}{1+z^2}$$

Since $r^2 = x^2+y^2$ is an invariant, then we can eliminate $x = \sqrt{r^2-y^2}$ so

$$\frac{dy}{\sqrt{r^2-y^2}} = \frac{dz}{1+z^2}$$

which implies

$$\arcsin(y/r) = \arctan(z) + k$$

but $y/r = \sin\theta$ implies $x/y = \tan\theta$ so $\arcsin(y/r) = \arctan(y/x)$, hence

$$\arctan(z) - \arctan(y/x)$$

is a second invariant. So

$$\tan\left(\arctan(z) - \arctan(y/x)\right) = \frac{z - y/x}{1 + zy/x} = \frac{zx-y}{yz+x} := q \tag{1.1}$$

is a second invariant (functionally dependent on the former). It is defined on the domain $z \neq -x/y$. Alternatively:

$$\tilde{q} = \frac{r}{\sqrt{1+q^2}} = \frac{x+yz}{\sqrt{1+z^2}}$$

is an invariant defined everywhere.

- $\mathbf{X}_1 = -y\partial_x + x\partial_y$, $\mathbf{X}_2 = 2xz\partial_x + 2yz\partial_y + (z^2+1-x^2-y^2)\partial_z$. We have $[\mathbf{X}_1, \mathbf{X}_2] = 0$ so these generate an abelian group. We already know an invariant r of $\mathbf{X}_1$. Is there another invariant? Yes, it is ∂_z. Since any invariant of $(\mathbf{X}_1, \mathbf{X}_2)$ is, in particular, an invariant of $\mathbf{X}_1$. it must be a function $\phi(r,z)$. So, we may push $\mathbf{X}_2$ to the coordinates (r,z), that is, consider $F(x,y,z) = (r,z)$ and let $\mathbf{Y} := F_*(\mathbf{X}_2)$. We calculate $\mathbf{Y}$ easily via

$$\mathbf{Y} = \mathbf{X}_2(r)\partial_r + \mathbf{X}_2(z)\partial_z = 2rz\partial_r + (z^2+1-r^2)\partial_z$$

The function we are looking satisfies

$$2rz\frac{\partial\phi}{\partial r}+(z^2+1-r^2)\frac{\partial\phi}{\partial z}=0$$

which verifies the characteristic equation

$$\frac{dr}{2rz}=\frac{dz}{z^2+1-r^2}$$

The solution is

$$\phi=\frac{z^2+r^2+1}{r}=\frac{x^2+y^2+z^2+1}{\sqrt{x^2+y^2}}$$

2 Invariance of Differential equations: Overview

Definition 2.1. *An ODE of order* n *is a relation*

$$H(x,y,y^{'},\ldots y_n)=0 \tag{2.1}$$

where $H(x_1,\ldots x_{n+1}):\mathbb{R}^{n+1}\to\mathbb{R}$.

A point transformation is an action $\Psi:=(\psi_1,\psi_2)$ *of a Lie group* G *acting on the space* $\mathbb{R}^2$ *identified with* (x,y)*:* $\Psi(g,x,y)=(\psi_1(g,x,y),\psi_2(g,x,y)):=(x_g,y_g)$.

The ODE (2.1) is invariant under the point transformation iff for any solution $y=y(x)$ *of the ODE, the graph* $(x,y(x))$ *is transformed under* Ψ *to a graph of another solution of the same ODE:* $(\psi_1(g,x,y(x)),\psi_2(g,x,y(x)))$ *is a graph of a solution* $y_{(g)}=y_{(g)}(x)$ *for any* $g\in G$.

Example: The action of $SO(2)$ on $\mathbb{R}^2$ via

$$\Psi(t,x,y)=(x\cos(t)-y\sin(t),x\sin(t)+y\cos(t))$$

preserves the linear functions $y=ax+b$. Indeed,

$$(x_{(t)},y_{(t)})=(x\cos(t)-(ax+b)\sin(t),x\sin(t)+(ax+b)\cos(t))$$

and we eliminate

$$x=\frac{x_{(t)}+b\sin(t)}{\cos(t)-a\sin(t)}$$

and

$$y_{(t)}=x\sin(t)+(ax+b)\cos(t)=\frac{\sin(t)+a\cos(t)}{\cos(t)-a\sin(t)}x_{(t)}+\frac{b}{\cos(t)-a\sin(t)}$$

Hence: *The equation* $y^{''}=0$ *is invariant under the* $SO(2)$ *action.*

If we replace the slop $a=y^{'}$ by y_1, we see that the action $SO(2)$ is extended to an action on $\mathbb{R}^3$ parameterized by $x,y,y_1)$ as:

$$(x,y,y_1)\to\left(x\cos(t)-y\sin(t),x\sin(t)+y\cos(t),\frac{\sin(t)+y_{(1)}\cos(t)}{\cos(t)-y_{(1)}\sin(t)}\right) \tag{2.2}$$

which is induced by the vextorfield

$$\mathbf{X} = -y\partial_x + x\partial_y + (1+(y_1)^2)\partial_{y_1}$$

Let now $y = y(x)$ be any function. Suppose $y = ax + b$ is the tangent to its graph at (x_0, y_0). Then

$$y_{1,(t)} = a_{(t)} := \frac{\sin(t) + a\cos(t)}{\cos(t) - a\sin(t)}$$

is the slop to the tangent of the transformed graph at $(x_{(t)}, y_{(t)})$. In particular, we obtained that the derivative $y^{'}$ is transformed as y_1 in the extended action.

We shall see later that the invariants of this action on $\mathbb{R}^3$, namely

$$r^2 = x^2 + y^2 \quad , \quad q = \frac{y^{'}x - y}{yy^{'} + x}$$

(see (1.1)) are constants for any solution of $SO(2)$ invariant equation:

$$r^2(x, y(x)) = C_1, \qquad q(x, y(x), y^{'}(x)) = C_2 \ ,$$

and that any such equation of first order is given by a functional relation between r^2 and q:

$$H(x, y, y^{'}) = F(r^2, q) = 0 \ .$$

3 Prolongation of vector fields

Motivated by the extension of functional to differential invariance we wish to extend the graph of a function $y = f(x)$ in $\mathbb{R}^2$ to a graph of a *prolonged* function $(y, y^{'}) = Pr^{(1)}f(x) := (f(x), f^{'}(x))$ in $\mathbb{R}^3$. More generally,

$$Y^{(n)} = (y, y^{'}, y^{''}, \dots y_n) = (f(x), f^{'}(x), \dots f^{(n)}(x))$$

What is the image of $Y^{(n)} = Pr^{(n)}f(x)$ under the flow of a vector field $\mathbf{X} = \xi\partial_x + \eta\partial_y$?

$$\tilde{y}^{'} = \frac{d\tilde{y}}{d\tilde{x}} \quad ; \quad \widetilde{y_{k+1}} = \frac{d\widetilde{y_k}}{d\tilde{x}}$$

and so on, where

$$\tilde{x} = x + \varepsilon\xi(x, y) + \dots \quad , \quad \tilde{y} = y + \varepsilon\eta(x, y) + \dots \quad ; \quad \tilde{y}_1 = y_1 + \varepsilon\eta^{(1)}(x, y, y_1) + \dots$$

Here we make, for any $h = h(x, y, y_1, \dots$

$$dh = (h_x + h_y y_1 + \dots h_{y_n} y_{n+1})\, dx$$

implies

$$d\tilde{y} = dy + \varepsilon d\eta \quad , \quad d\tilde{x} = dx + \varepsilon d\xi$$

$$\widetilde{y_1} = \frac{d\tilde{y}}{d\tilde{x}} = \frac{dy}{dx} + \varepsilon\left(\frac{d\eta}{dx} - y_1\frac{d\xi}{dx}\right) := y_1 + \varepsilon\eta^{(1)} +$$

as well as

$$\widetilde{y_n} = \frac{d\tilde{y}_{n-1}}{d\tilde{x}} = \frac{dy_{n-1}}{dx} + \varepsilon\left(\frac{d\eta^{(n-1)}}{dx} - y_n\frac{d\xi}{dx}\right) := y_n + \varepsilon\eta^{(n)} +$$

Definition 3.1. *The prolongation of a vector field* $\mathbf{X} = \xi\partial_x + \eta\partial_y$ *to order* n *is given by*

$$Pr^{(n)}\mathbf{X} := \xi\partial_x + \eta\partial_y + \eta^{(1)}\partial_{y_1} + \ldots + \eta^{(n)}\partial_{y_n}$$

where

$$\eta^{(0)} \equiv \eta \ , \quad \eta^{(k)} = \frac{d\eta^{(k-1)}}{dx} - y_k\frac{d\xi}{dx}$$

We shall sometimes write $\mathbf{X}$ *instead of* $Pr^{(n)}\mathbf{X}$, *when no confusion is expected.*

Examples:

(A) $-y\partial_x + x\partial_y + \left(1 + (y_1)^2\right)\partial_{y_1} + 3y_1y_2\partial_{y_2} + \left(3(y_2)^2 + 4y_1y_3\right)\partial_{y_3} + \ldots$

(B) $x\partial_x + y\partial_y - y_2\partial_{y_2} - 2y_3\partial_{y_3}\ldots$

(C) $\partial_x, \quad x\partial_x - y_1\partial_{y_1} - 2y_2\partial_{y_2} - 3y_3\partial_{y_3} + \ldots \quad , x^2\partial_x - 2xy_1\partial_{y_1} - (2y_1 + 4xy_2)\partial_{y_2} - 6(xy_3 - y_2)\partial_{y_3} + \ldots$

Theorem 1. *The prolongation of vector fields is compatible with the Lie product, i.e*

$$[Pr^{(n)}(\mathbf{X}_1), Pr^{(n)}(\mathbf{X}_2)] = Pr^{(n)}[\mathbf{X}_1, \mathbf{X}_2] \ .$$

In particular, it follows that the prolongation of a Lie algebra is an isomorphic Lie algebra.

Definition 3.2. *The* $n-$ *prolongation of action* $\Psi : G \times \mathbb{R}^2 \to \mathbb{R}^2$ *is denoted by* $Pr^{(n)}\Psi : G \times \mathbb{R}^{2+n} \to \mathbb{R}^{2+n}$. *The invariants vectorfields are the prolonged Lie algebra corresponding to* Ψ.

Remark 3.1. *Note that, in general, the prolonged action is only local (even if the original action is global- see (2.2)).*