Lecture 4

1 Differential invariants

Definition 1.1. *Given a point transformation Ψ acting on $\mathbb{R}^2$, a differential invariant of order n is an invariant function of the prolonged action $Pr^{(n)}\Psi$.*

Examples (referred to group actions on the previous page) :

(A): $\omega_0 = \sqrt{x^2+y^2}$, $\omega_1 = \frac{xy_1-y}{x+yy_1}$.

(B) $\omega_0 = y/x$, $\omega_1 = y_1$.

(C) $\omega_0 = y$, $\omega_3 = 2(y_1)^{-3}y_3 - 3(y_1)^{-4}(y_2)^2$ is a complete set of third order invariants.

Are there higher order invariants?

Theorem 1. *For a group action $\Psi : G \times \mathbb{R}^2 \to \mathbb{R}^2$, if α, β are invariants of order n then $D_x\alpha/D_x\beta$ is an invariant of order $n+1$. Here D_x is a complete derivative:*

$$D_x = \partial_x + y_1\partial_y + \ldots + y_{n+1}\partial_{y_n} \ .$$

Exercises:

1. Prove that, for $\phi = \phi(x, y, \ldots, y_n)$ and $\mathbf{X} = \xi\partial_x + \eta\partial_y$ on $\mathbb{R}^2$:

$$Pr^{(n+1)}\mathbf{X}(D_x\phi) = D_x\left(Pr^{(n)}\mathbf{X}(\phi)\right) - D_x\phi \cdot D_x\xi \ .$$

2. Use this to prove Theorem 1.

Example:

(A) For the prolonged action of $SO(2)$ we know that $r = \sqrt{x^2+y^2}$ is an invariant and $q = \frac{y^{'}x-y}{yy^{'}+x}$ is a second order invariant. Then

$$D_x(q)/D_x(r) = \frac{\sqrt{x^2+y^2}}{(x+yy_1)^2}\left[(x^2+y^2)y_2 - (1+y_1)(xy_1-y)\right)$$

is a third order invariant. We can replace it by

$$\kappa = \frac{D_x(q)/D_x(r)}{(1+q^2)^{3/2}} + \frac{q}{r(1+q^2)^{1/2}} = \frac{y_2}{(1+(y_1)^2)^{3/2}}$$

which is an expression for the curvature of the graph of the function $(x, y(x))$.

(B) For the prolonged action of the scaling group we know the invariants $w_0 = y/x$, $w_1 = y_1$. Then a second order invariant is:

$$D_x(w_1)/D_x(w_0) = \frac{x^2y_2}{xy_1 - y}$$

Corollary 1.1. *If w_0 is a (zero order) invariant, w_1 first order invariant of an action derived by a single symmetry (vectorfield) , than* <u>all</u> *differential invariants of order n can be obtain, recursively, by*

$$w_n = \frac{D_x w_{n-1}}{D_x w_0} = \frac{D_x^{n-1} w_1}{D_x^{n-1} w_0} \ .$$

In particular, <u>any</u> *n order invariant is of the form*

$$G\left(w_0, w_1, \frac{D_x w_1}{D_x w_0}, \frac{D_x^2 w_1}{D_x^2 w_0} \cdots \frac{D_x^{n-1} w_1}{D_x^{n-1} w_0}\right)$$

1.1 Infinitesimal formulation of invariance for ODE

Theorem 2. *An ODE $H = 0$ of order n is invariant under the action of the flow $\psi(t, x, y)$ generated by $\mathbf{X}$ if and only if*

$$Pr^{(n)}\mathbf{X}(H) = 0 \qquad \mathrm{mod}\ H = 0$$

provided $H_x^2 + H_y^2 + \ldots H_{y_n}^2 \neq 0 \quad \mathrm{mod}\ H = 0$.

Counter-Example: $H = (y_2 + y)^2$ verifies $Pr\mathbf{X}H = 0 \mod H$ for any $\mathbf{X}$!
In particular, the explicit ODE of order n

$$y_n = w(x, y, \ldots y_{n-1})$$

is $\mathbf{X}$ invariant iff

$$Pr^{(n-1)}\mathbf{X}(w) = \eta^{(n)}\left(x, y, \ldots, y_{n-1}, w(x, y, \ldots, y_{n-1})\right) \tag{1.1}$$

(take $H = w(x, y, \ldots, y_{n-1}) - y_n$).
Examples:

- A linear ODE:
$$y_n = \sum_0^{n-1} w^{(i)}(x) y_i \ , \quad y_0 := y$$
and the transformation $(\tilde{x}, \tilde{y}) = (x, e^t y)$. Its prolongation
$$\mathbf{X} = y\partial_y + y_1\partial_{y_1} + y_2\partial_{y_2} + \ldots$$
so $\eta^{(n)} = y_n$ and $Pr^{(n-1)}\mathbf{X}(w) = w$.

- Under $(\tilde{x}, \tilde{y}) = (x, y + t)$. Then $\mathbf{X} = \partial_y$ so $w_y = 0$ or $w = w(x, y_1, \ldots y_{n-1})$. In that case the order of the equation can be reduced by using the variable $z = y_1$.

- $(\tilde{x}, \tilde{y}) = (x + t, y)$ then $\mathbf{X} = \partial_x$ and $w = w(y, y_1, \ldots y_{n-1})$. The order can be reduced again by inverting the dependent and independent variables $x \to y$, $y \to x$.

1.2 First order ordinary differential equations

Recall that, for $\mathbf{X} = \xi\partial_x + \eta\partial_y$,

$$\eta^{(1)} = \eta_x + (\eta_y - \xi_x)y_1 - \xi_y(y_1)^2$$

so, from (1.1) we get for a first order ODE $y' = F(x,y)$

$$\eta_x + (\eta_y - \xi_x)F - \xi_y F^2 = \xi F_x + \eta F_y \tag{1.2}$$

This is a PDE for ξ, η. Note that any vectorfield where

$$\eta/\xi = F \tag{1.3}$$

is a solution (not a big deal!)

Proposition 1.1. *If $Pdx + Qdy = 0$ is an ODE and $\xi\partial_x + \eta\partial_y$ is a generator of its symmetry, then*

$$R(x,y) := (\xi P + \eta Q)^{-1}$$

is an integration factor, that is, $RPdx + RQdy$ is an exact differential.

Remark 1.1. *In case (1.3) R is not defined!*

In some cases we may find other, more helpful solutions. For example

$$y' = F(y/x)$$

which is invariant under the scaling group action $(x,y) \to e^t(x,y)$. Here $\mathbf{X} = x\partial_x + y\partial_y$ is a solution of (1.2). As we know, $z_1 := y/x$ is an invariant, i.e $\mathbf{X}(z_1) = 0$. So, if we take any other function z_2 of x or y as a second coordinate, then $\mathbf{X}$ is transformed into

$$\mathbf{X}(z_1)\partial_{z_1} + \mathbf{X}(z_2)\partial_{z_2} = \mathbf{X}(z_2)\partial_{z_2}$$

and the ODE is transformed into

$$dz_2/dz_1 = \tilde{F}(z_1)\ .$$

For example, if $z_2 = \ln x$ then $y = z_1 e^{z_2}$, $x = e^{z_2}$ so

$$F(z_1) = \frac{dy}{dx} = \frac{d(z_1 e^{z_2})}{de^{z_2}} = \frac{1 + z_1 dz_2/dz_1}{dz_2/dz_1}$$

and

$$\frac{dz_2}{dz_1} = \frac{1}{F(z_1) - 1}\ .$$

Another example:

$$y' = \frac{y + xH(\sqrt{x^2+y^2})}{x - yH(\sqrt{x^2+y^2})} \tag{1.4}$$

is invariant under the action of $\mathbf{X} = -y\partial_x + x\partial_y$. Indeed, it is the most general equation of this form: Any such equation must be of the form $H(w_0) = w_1$ where w_0, w_1 are the two first invariant of order ≤ 1 by Theorem 1 (see (A)). Again, we take the zero-order invariant $r = \sqrt{x^2 + y^2}$ as the first coordinate, and (most conveniently) θ to be the second one. So $\mathbf{X} = \partial_\theta$ in the new coordinate and the equation must be reduced into $d\theta/dr = F(r)$. We calculate, using this and (1.4),

$$\frac{dy}{dx} = \frac{d(r\sin\theta)/dr}{d(r\cos\theta)/dr} = \frac{\sin\theta + r\theta^{'}\cos\theta}{\cos\theta - r\theta'\sin\theta} = \frac{\sin\theta + H(r)\cos\theta}{\cos\theta - H(r)\sin\theta}$$

which yields

$$\theta^{'} = \frac{H(r)}{r}$$

1.3 Differential invariants revisited

Consider the firsr order linear PDE on $\mathbb{R}^{n+1}$:

$$\sum a_i \partial_{x_i} f = 0$$

We can associate with this a vector field $\mathbf{A} = \sum a_i \partial_{x_i}$. We already know that it can be transformed to a new set of coordinates $s, \phi_i, i = 1, \ldots n$ where

$$\mathbf{A}\phi_i = 0, \qquad \mathbf{A}s = 1$$

so $\mathbf{A}$ is transformed into ∂_s in this system.

Now, we consider an n order ODE:

$$y_n = w(x, y, y^{'}, y^{''}, \ldots, y^{(n-1)})$$

and the corresponding vector field on $\mathbb{R}^{n+1}$

$$\mathbf{A} = \partial_x + y_1\partial_y + y_2\partial_{y_1} + \ldots + w\partial_{y_{n-1}}$$

in the coordinates $x, y_0, y_1, \ldots y_{n-1}$. Here $w = w(x, y, y_1, \ldots y_{n-1})$.

There is a deep relation between the ODE and $\mathbf{A}$ so defined. A solution of $\mathbf{A}\phi = 0$ is an *invariant of motion* to the ODE, that is

$$\frac{d}{dx}\phi(x, y, y^{'}, \ldots y_{n-1}) = \mathbf{A}\phi = 0$$

If we found such a non-constant invariant ϕ then $\phi_{y_{n-1}} \neq 0$ (why?) and we solve for the implicit function $y_{n-1} = \hat{w}(x, y, y_1, \ldots, y_{n-2})$ and reduce the order of the ODE:

$$y^{(n-1)} = \hat{w}\left(x, y, y^{'}, \ldots, y_{n-2}\right) \ .$$

Lemma 1.1. *If $\mathbf{A} \neq 0$ is a v-f in $\mathbb{R}^{n+1}$ then there are (locally) n functionally independent invariant functions $\phi_1, \ldots \phi_n$ verifying $\mathbf{A}\phi_i = 0$. Moreover, any invariant function f is given by $f = F(\phi_1, \ldots \phi_n)$ for some smooth $F : \mathbb{R}^n \to \mathbb{R}$.*

Moreover, if $\mathbf{B}$ is another vector field with the same invariants as $\mathbf{A}$, then $\mathbf{B} = \lambda(x_1, \ldots x_{n+1})\mathbf{A}$.

Proof. Since we can find new variables $s_0, \dots s_n$ in which $\mathbf{A} = \partial_{s_0}$, then all invariants are functions of $s_1, \dots s_n$. Also, any other vector field with the same invariants is of the form $\tilde{\lambda}\partial_{s_0}$ for some $\tilde{\lambda}(s_0, \dots s_n)$. □

Moreover, if we find a complete set of n functionally independent invariants $\phi_1, \dots \phi_n$ then we may eliminate $y = y(x, \phi_1^0, \dots \phi_n^0)$ form the system

$$\phi_1 = \phi_1^0 \quad \dots \phi_n = \phi_n^0$$

and get n parameter family of solutions!

Example: consider $y^{''} = -y$ then $\mathbf{A} = \partial_x + y_1\partial_y - y_0\partial_{y_1}$ and the invariants are

$$\phi_1 = y_0^2 + y_1^2 \quad , \quad \phi_2 = x - \arctan(y_0/y_1)$$

We can eliminate the solution $y_0 = y_1 \tan(x - \phi_2^0) \to (y_0)^2 = (y_1)^2 \tan^2(x - \phi_2^0) \to (y_0)^2 + (y_1)^2 = (y_1)^2 \cos^{-2}(x - \phi_2^0) \to (y_1)^2 = \phi_1^0 \cos^2(x - \phi_2^0) \to$

$$y_1 := y = (\phi_1^0)^{1/2} \sin(x - \phi_2^0)$$

2 Symmetry of ODE: second formulation

Let $\mathbf{A} = \partial_x + y_1\partial_{y_0} + \dots + w\partial_{y-1}$ and $\mathbf{X} = \xi\partial_x + \eta\partial_y + \eta_1\partial_{y_1} + \dots + \eta_{n-1}\partial_{y_{n-1}}$ be a prolonged symmetry. Let $\phi_1, \dots \phi_n$ be a set of functionally independent invariants. Since $\mathbf{X}(\phi)$ is also an invariant if ϕ is (prove!) and since the set of invariants are complete by Lemma 1.1, then

$$\mathbf{A}\mathbf{X}(\phi_i) = 0 \to \mathbf{X}(\phi_i) = \Omega_i(\phi_1, \dots \phi_n)$$

It follows

$$[\mathbf{X}, \mathbf{A}]\phi_i = \mathbf{X}(\mathbf{A}(\phi_i)) - \mathbf{A}(\mathbf{X}(\phi_i)) = 0 \ , i = 1, \dots n \ .$$

So, by Lemma 1.1 again:

$$[\mathbf{X}, \mathbf{A}] = \lambda(\mathbf{x}, \mathbf{y}, \mathbf{y_1}, \dots \mathbf{y_{n-1}})\mathbf{A} \tag{2.1}$$

Writing explicitly:

$$[\mathbf{X}, \mathbf{A}] = -(\mathbf{A}\xi)\partial_x + [\mathbf{X}(y_1) - \mathbf{A}(\eta)]\partial_y + \dots + [\mathbf{X}(w) - \mathbf{A}(\eta_{n-1})]\partial_{y_{n-1}}$$

so the first component (coefficient of ∂_x) yields

$$\lambda = -\mathbf{A}\xi = -\xi_x - y_1\xi_y := -\frac{d\xi}{dx}$$

Now, from (2.1)

$$\left(\eta^{'} \frac{d\eta}{dx}\right)\partial_y + \left(\eta^{''} - \frac{d\eta_1}{dx}\right)\partial_{y_1} + \dots + \left(\mathbf{X}(w) - \frac{d\eta_{n-1}}{dx}\right)\partial_{y_{n-1}} = -\frac{d\xi}{dx}\left(y_1\partial_y + y_2\partial_{y_1} + \dots + w\partial_{y_{n-1}}\right) \tag{2.2}$$

Recall from the definition of the prolongation

$$\eta_k = \frac{d\eta_{k-1}}{dx} - y_k\frac{d\xi}{dx}$$

we obtain that (2.2) is equivalent to

$$\mathbf{X}(w) = \eta_n \quad \mod \ y_n = w$$

which is precisely $\mathbf{X}(H) = 0$ for $H = y_n - w(x, y, y_1, \dots y_{n-1})$.