Lecture 5 & 6

1 Second order and beyond

1.1 One parameter symmetry (G1)

A second order ODE is

$$y^{''} = w(x, y, y^{'})$$

and the corresponding equation for the symmetry generating vectorfield

$$\mathbf{X}(w) = \eta^{(2)}(x, y, y_1, w(x, y, y_1)) \tag{1.1}$$

where $\mathbf{X} = \xi\partial_x + \eta\partial_y + \eta^{(1)}\partial_{y_1}$. Recall

$$\eta^{(1)} = \eta_x + (\eta_y - \xi_x)y_1 - \xi_y(y_1)^2 \quad , \quad \eta^{(2)} = \eta_{xx} + (2\eta_{xy} - \xi_{xx})y_1 +$$

$$(\eta_{yy} - 2\xi_{xy})(y_1)^2 - \xi_{yy}(y_1)^3 + (\eta_y - 2\xi_x - 3\xi_y y_1)y_2.$$

Examples:

1.

$$y^{''} = x^n y^2 \ .$$

In this case we get from (1.1):

$$x^n y^2(\eta_y - 2\xi_x) - nx^{n-1}y^2\xi - 2yx^n\eta + \eta_{xx} + y_1(2\eta_{xy} - \xi_{xx} - 3x^n y^2\xi_y)$$

$$+y_1^2(\eta_y y - 2\xi_{xy}) - y_1^3\xi_{yy} = 0 \tag{1.2}$$

This is a polynomial in y_1. The coefficients of y_1^2 and y_1^3 yield

$$\xi_{yy} = 0 \quad , \quad \eta_{yy} = 2\xi_{xy}$$

so

$$\xi = y\alpha(x) + \beta(x), \quad \eta = y^2\alpha^{'}(x) + y\gamma(x) + \delta(x) \ .$$

From this and the coefficient of y_1 we get

$$-3x^n y^2\alpha + 3y\alpha^{''} + 2\gamma^{'} - \beta^{''} = 0$$

hence $\alpha = 0$ and $2\gamma^{'} = \beta^{''}$. Thus

$$\xi = \beta(x) \quad , \quad \eta = y(\beta^{'}/2 + c) + \delta(x)$$

for some constant c. We substitute it in the zero oder (in y_1) of (1.2) to obtain

$$-y^2[x^n(\frac{5}{3}\beta^{'} + c) + nx^{n-1}\beta] + y(\frac{1}{2}\beta^{'''} - 2x^n\delta) + \delta^{''} = 0 \ ,$$

so

$$\frac{5}{2}x\beta^{'} + n\beta + cx = 0 \quad , \quad \delta = \frac{1}{4}x^{-n}\beta^{'''} \quad , \quad \delta^{''} = 0 \ .$$

Take, for example, $\beta = -(c/n)x$. Then $\delta \equiv 0$ and we obtain the vectorfield

$$\mathbf{X} = x\partial_x - (n+2)y\partial_y$$

which generates the transformation group $(x, y) \to (e^t x, e^{-(n+2)t} y)$. We may use it to reduce the order of the equation.

Surprisingly, there are other symmetries *only* if $n = -5, -15/7, -20/7$ (!)

Next, we can use this to reduce the order of the equation. The function $z_1(x, y) = yx^{n+1}$ is clearly an invariant. If we choose $z_2 = \ln(x)$, so $x = e^{z_2}, y = z_1 e^{-(n+2)z_2}$. Then

$$y' = \frac{dy}{dx} = \frac{dy/dz_2}{dx/dz_2} = e^{-(n+3)z_2}\left(\frac{dz_1}{dz_2} - (n+2)z_1\right)$$

$$\begin{aligned} y'' &= \frac{dy'/dz_2}{dx/dz_2} = e^{-(n+4)z_2}\left(\frac{d^2z_1^2}{dz_2^2} - (n+2)\frac{dz_1}{dz_2} - (n+3)\left(\frac{dz_1}{dz_2} - (n+2)z_1\right)\right) \\ &= e^{-(n+4)z_2}\left(\frac{d^2z_1^2}{dz_2^2} - (2n+5)\frac{dz_1}{dz_2} + (n+3)(n+2)z_1\right) = x^n y^2 = z_1^2 e^{-(n+4)z_2} \end{aligned}$$

so

$$\frac{d^2z_1^2}{dz_2^2} = (2n+5)\frac{dz_1}{dz_2} - (n+3)(n+2)z_1 + z_1^2$$

which is, in fact, a first order equation under disguise (why?)

2.

$$y'' = 0$$

Then the symmetry equation is reduced into

$$\eta_{xx} + y_1(2\eta_{xy} - \xi_{xx}) + y_1^2(\eta_{yy} - 2\xi_{xy}) - y_1^3\xi_{yy} = 0 \ .$$

This reduces to $\eta_{xx} = 2\eta_{xy} - \xi_{xx} = \eta_{yy} - 2\xi_{xy} = \xi_{yy} = 0$. From this we obtain

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = x\partial_x, \quad \mathbf{X}_3 = y\partial_x \ , \quad \mathbf{X}_4 = xy\partial_x + y^2\partial_y \ , \quad \mathbf{X}_5 = x^2\partial_x + xy\partial_y$$

$$\mathbf{X}_6 = \partial_y, \quad \mathbf{X}_7 = x\partial_y, \quad \mathbf{X}_8 = y\partial_y \ .$$

So, we see that the dimension of the transformation groups for second order ODE varies from zero to 8. Can there be more than 8 dimensional transformation group for second order ODE? The answer is no. This follows from the fact that $\eta^{(1)}$ is at most quadratic in y_1 (why)?

Problems:

1. Determine the (8th parameter) transformation group generated on $\mathbb{R}^2$ by $\mathbf{X}_1, \ldots \mathbf{X}_8$. (Hint: It is the most general transformation which preserves linear functions).
2. Prove that $y^{(n)} = 0$ have $n+4$ dimensional symmetry for $n > 2$.
3. Prove that there are at most $n+4$ invariants for $n-th$ order ODE if $n > 2$. Why does it differ from $n = 2$?
4. Prove that $y'' = xy + e^{y'} + e^{-y'}$ have no symmetry whatsoever!

1.2 2-symmetry (G2)

Given a symmetry group generated by $\mathbf{Y}_1, \mathbf{Y}_2$, we get

$$[\mathbf{Y}_1, \mathbf{Y}_2] = a_1\mathbf{Y}_1 + a_2\mathbf{Y}_2 \ .$$

If $a_1 \neq 0$ we may choose another base $\mathbf{X}_1 = a_1\mathbf{Y}_1 + a_2\mathbf{Y}_2$, $\mathbf{X}_2 = \mathbf{Y}_2/a_1$ so

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_1 \qquad (a)$$

If $a_1 = 0$ then $\mathbf{X}_1 = \mathbf{Y}_1/a_2, \mathbf{X}_2 = \mathbf{Y}_2$ and get the same group structure. If both $a_1 = a_2 = 0$ then we, evidently, have an abelian group

$$[\mathbf{X}_1, \mathbf{X}_2] = 0 \ . \qquad (b)$$

What are the "canonical" forms of vectorfields realizing these abstract algebras? Consider

$$\mathbf{X}_1 = \xi_1\partial_x + \eta_1\partial_y \quad , \quad \mathbf{X}_2 = \xi_2\partial_x + \eta_2\partial_y \ .$$

There are, in fact, 2 cases for either (a) and (b):

$$(i) \ \ \delta := \xi_1\eta_2 - \xi_2\eta_1 \neq 0, \quad (ii) \ \delta = 0 \ .$$

a) We may always assume $\mathbf{X}_1 = \partial_x$. Then[1] $\mathbf{X}_2 = a(x,y)\partial_x + b(x,y)\partial_y$ and

$$[\mathbf{X}_1, \mathbf{X}_2] = a_x(x,y)\partial_x + b_x(x,y)\partial_y = \partial_x$$

so $a = x + a(y), b = b(y)$. We my further transform the variables $\tilde{x} = x + h(y), \tilde{y} = v(y)$ to obtain

$$\tilde{\mathbf{X}}_1 = \partial_{\tilde{x}} \quad , \quad \tilde{\mathbf{X}}_2 = \left[x + a(y) + h^{'}(y)b(y)\right]\partial_{\tilde{x}} + bv^{'}\partial_{\tilde{y}}$$

There are now two two cases

(i) $\delta = -b \neq 0$. Here we take h as a solution of $bh^{'} + a = h$, $bv^{'} = v$ and get (removing the tilda's)

$$\mathbf{X}_1 = \partial_x \ , \quad \mathbf{X}_2 = x\partial_x + y\partial_y \qquad (1.3)$$

(ii) $\delta = b = 0$, hence, with $h = a$,

$$\mathbf{X}_1 = \partial_x \ , \quad \mathbf{X}_2 = x\partial_x \qquad (1.4)$$

(b) By similar arguments we obtain the two possibilities:

(i) $\mathbf{X}_1 = \partial_x \ , \quad \mathbf{X}_2 = \partial_y$

(ii) $\mathbf{X}_1 = \partial_x \ , \quad \mathbf{X}_2 = y\partial_x$

We now ask what are the most general second order equations which realizes these symmetries. For this we must find the invariants of second degree in each case.

[1]Sometimes it is better to choose $\mathbf{X}_1 = \partial_y$, as in $a - ii, b - ii$ below.

(a-i) Recall the prolongation to second order,

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = x\partial_x + y\partial_y - y_2\partial_{y_2}$$

from which we obtain the 2 invariants y_1, xy_2. Then $H(x, y, y_1, y_2) = xy_2 - w(y_1) = 0$ stands for the most general invariant function, which gives us the canonical equation

$$y^{''} = x^{-1}w(y^{'})$$

(a-ii) From the prolongations

$$\partial_x, \quad x\partial_x - y_1\partial_{y_1} - 2y_2\partial_{y_2}$$

we readily get the invariants $y_2/(y_1)^2$, y, so $y^{''} = (y^{'})^2 w(y)$ is the most general canonical equation. However, if we change the role of x and y we obtain

$$\partial_y, \quad y\partial_y + y_1\partial_{y_1} + y_2\partial_{y_2} \ .$$

Here the invariants are $x, y_2/y_1$, and the most general equation

$$y^{''} = y^{'}w(x) \ .$$

(b-i) Here, clearly, y_1, y_2 are the invariants so the general equation is

$$y^{''} = w(y^{'}) \ .$$

(b-ii) Again, replacing the role of x and y

$$\partial_y, \quad x\partial_y + \partial_{y_1} \ .$$

The invariants x, y_2 yield

$$y^{''} = w(x) \ .$$

1.3 Full solutions for second order equations admitting G2

a) $\mathbf{X}_j := \xi_j\partial_x + \eta_j\partial_y$, $j = 1, 2$ where

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_1 \tag{1.5}$$

i) $\xi_1\eta_2 - \xi_2\eta_1 \neq 0$: We need to transform to new variables by which $\mathbf{X}_1 = \partial_s$ and $\mathbf{X}_2 = t\partial_t + s\partial_s$. For this, let $u = \ln(t)$ and

$$\mathbf{X}_1(u) = \xi_1 u_x + \eta_1 u_y = 0 \ , \qquad \mathbf{X}_2(u) = \xi_2 u_x + \eta_2 u_y = 1 \ .$$

which implies

$$u(x, y) = \int^{x,y} \frac{\eta_1 dx - \xi_1 dy}{\xi_1\eta_2 - \xi_2\eta_1}$$

which is well defined because of (1.8). Next we substitute $s = tv(s, t)$ and get (since $\mathbf{X}_1(t) = \mathbf{X}_1(u) = 0, \mathbf{X}_2(t) = t$)

$$\mathbf{X}_1(s) = t\mathbf{X}_1(v) = t\eta_1(y, t)v_y = 1 \tag{1.6}$$

and

$$\mathbf{X}_2(s) = v\mathbf{X}_2(t) + t\mathbf{X}_2(v) = s + t\mathbf{X}_2(v) \Longrightarrow \mathbf{X}_2(v) = 0$$

hence

$$tv_t + \eta_2 v_y = 0 \ . \tag{1.7}$$

implies

$$\mathbf{X}_1 = \partial_t \ , \quad \mathbf{X}_2 = t\partial_t + s\partial_s \ .$$

We get from (1.6)

$$s(y,t) = tv(y,t) = t\int^{y,t}\left(\frac{dy}{t\eta_1} - \frac{\eta_2 dt}{t^2\eta_1}\right) \ .$$

Finally, we get the equation $s^{''} = \tilde{w}(s^{'})/t$ whose solution

$$\int^{s^{'}} \frac{d\tau}{\tilde{w}(\tau)} = \ln(t) + c \Longrightarrow s^{'} = F^{(-1)}(\ln(t) + c)$$

where $F^{'} = 1/\tilde{w}$. This yields another integration $s(t) = \int^t F^{(-1)}(\tau, c)$. We get the solution after 4 integrations altogether.

ii) $\xi_1\eta_2 - \xi_2\eta_1 = 0$: Since $\mathbf{X}_2 = s(x,y)\mathbf{X}_1$ we get s immediately. We obtain t from

$$\mathbf{X}_1(t) = \xi_1 t_x + \eta_1 t_y = \xi_1\left(\partial_x + \frac{\eta_1}{\xi_1}\partial_y\right)t = 0 \ .$$

So, we solve the characteristic equation (ODE):

$$\frac{dy}{dt} = \eta_1/\xi_1$$

for the function $y = y(x,t)$. We factor out $t = t(x,y)$ to obtain the second variable.

Remark 1.1. *Note that it is the* only *case where we need to solve an ODE!*

Finally, the equation is reduced into $s^{''} = s^{'}w(t)$ so a pair of integration

$$s(t) = \int^t d\tau e^{\int^\tau w}$$

yields the result.

b) $\mathbf{X}_j := \xi_j\partial_x + \eta_j\partial_y$, $j = 1,2$ where

$$[\mathbf{X}_1, \mathbf{X}_2] = 0 \tag{1.8}$$

i) $\xi_1\eta_2 - \xi_2\eta_1 \neq 0$: The two integrations

$$t(x,y) = \int^{x,y} \frac{-\eta_1 dx + \xi_1 dy}{\xi_1\eta_2 - \xi_2\eta_1} \quad , \quad s(x,y) = \int^{x,y} \frac{-\eta_2 dx + \xi_2 dy}{\xi_1\eta_2 - \xi_2\eta_1}$$

reduces the system to $s^{''} = \tilde{w}(s^{'})$. An integration gives

$$t + c_0 = \int^{s^{'}} w^{-1}(\tau) d\tau$$

so

$$s(t) = \int^{t} F^{(-1)}(\tau + c_0) d\tau + d_0$$

where $F^{(-1)}$ is the inverse of F while F is the primitive of w^{-1}.

ii) $\xi_1\eta_2 - \xi_2\eta_1 = 0$: Then $\mathbf{X}_2 = t(x,y)\mathbf{X}_1$ so t is obtained immediately. Next, we take t, x as independent variables, hence $s = s(t,y)$ satisfies

$$1 = \mathbf{X}_1(s) = \mathbf{X}_1(y)s_y + \mathbf{X}_1(t)s_t = \eta_1(y,t)s_y$$

(recall $[\mathbf{X}_1, \mathbf{X}_1] = \mathbf{X}_1(t)\mathbf{X}_1 = 0$ so $\mathbf{X}_1(t) = 0$). So

$$s(y,t) = \int^{y} \eta_1^{-1}(\tau, t) d\tau \ .$$

The resulting equation $s^{''} = w(t)$ can be solved by two integrations:

$$s(t) = s_0 + t s_1 + \int^{t} w \ .$$

1.4 Solutions of G2-symmetric second order ODE in the space of invariants

We may look for two invariants ϕ, ψ of the ODE (functions of $x, y, y_1 \equiv y^{'}$). That is,

$$\mathbf{A}(\phi) = \mathbf{A}(\psi) = 0 \ .$$

The solutions are, then, obtained implicitly by

$$\phi(x,y,y_1) = \phi_0 \ , \quad \psi(x,y,y_1) = \psi_0 \ .$$

If the Jacobian derivative $\phi_y\psi_{y_1} - \psi_y\phi_{y_1} \neq 0$ then we can apply the Implicit Function Theorem to factor out

$$y = y(x, \phi_0, \psi_0)$$

and obtain the complete family of solutions.

A third function ρ satisfies

$$\mathbf{A}(\rho) = \rho_x + y_1\rho_y + w\rho_{y_1} = 1$$

and the trio ϕ, ψ, ρ forms a new set of independent variables, by which

$$\mathbf{A} = \partial_\rho \ , \quad \mathbf{X}_1 = \mathbf{X}_1(\phi)\partial_\phi + \mathbf{X}_1(\psi)\partial_\psi + \mathbf{X}_1(\rho)\partial_\rho \ , \quad \mathbf{X}_2 = \mathbf{X}_2(\phi)\partial_\phi + \mathbf{X}_2(\psi)\partial_\psi + \mathbf{X}_2(\rho)\partial_\rho \ .$$

The determinant of the coefficients

$$\Delta := \begin{vmatrix} 1 & y_1 & w \\ \xi_1 & \eta_1 & \eta_1^{(1)} \\ \xi_2 & \eta_2 & \eta_2^{(1)} \end{vmatrix}$$

verifies $\Delta \neq 0$ iff $\mathbf{X}_1(\phi)\mathbf{X}_2(\psi) - \mathbf{X}_1(\psi)\mathbf{X}_2(\phi) \neq 0$. Note that Δ and δ defined above are related if ϕ, ψ are functions of x, y only. In general, Δ may be zero while $\delta \neq 0$ and v.v.

The symmetry condition $[\mathbf{A}, \mathbf{X}] = \lambda \mathbf{A}$ implies that the coefficients of $\partial_\phi, \partial_\psi$ in $\mathbf{X}_1, \mathbf{X}_2$ are independent of ρ. Then, the truncation of the ∂_ρ component of $\mathbf{X}_1, \mathbf{X}_2$ does not change the Lie algebra structure. In particular we have the two cases (a,b) for the algebra representations, as well as the two cases i, ii corresponding to $\Delta \neq 0$ and $\Delta = 0$, respectively. Recall that we are now considering the *modified* fields

$$\tilde{\mathbf{X}}_i = \tilde{\xi}_i \partial_\phi + \tilde{\eta}_i \partial_\psi \ , \quad i = 1, 2$$

where $\tilde{\xi}_i := \mathbf{X}_i(\phi) = \tilde{\xi}_i(\phi, \psi)$, $\tilde{\eta}_i := \mathbf{X}_i(\psi) = \tilde{\eta}_i(\phi, \psi)$.

Let us consider the transitive, commutative case $[\tilde{\mathbf{X}}_1, \tilde{\mathbf{X}}_2] = 0$ and $\Delta \neq 0$. Recall that this case corresponds (but to identical to) (b-i). Then, by repeating the argument leading to (b-i) we get the existence of two invariants ψ, ψ for which

$$\tilde{\mathbf{X}}_1 = \partial_\psi, \quad , \quad \tilde{\mathbf{X}}_2 = \partial_\phi \ .$$

In particular, both

$$\mathbf{A}(\phi) = \mathbf{X}_1(\phi) = 0 \quad , \quad \mathbf{X}_2(\phi) = 1 \tag{1.9}$$

$$\mathbf{A}(\psi) = \mathbf{X}_1(\psi) = 0 \ , \ \mathbf{X}_2(\psi) = 0 \tag{1.10}$$

are solvable. Hence we solve for $\phi_x, \phi_y, \phi_{y_1}$ from

$$\phi_x + y_1 \phi_y + w \phi_{y_1} = 0 \ , \quad \xi_1 \phi_x + \eta_1 \phi_y + \eta_1^{(1)} \phi_{y_1} = 0 \ , \quad \xi_2 \phi_x + \eta_2 \phi_y + \eta_2^{(1)} \phi_{y_1} = 1$$

to obtain

$$\phi_x dx + \phi_y dy + \phi_{y_1} dy_1 = \Delta^{-1} \begin{vmatrix} dx & dy & dy_1 \\ 1 & y_1 & w \\ \xi_2 & \eta_2 & \eta_2^{(1)} \end{vmatrix} \tag{1.11}$$

which is a *exact differential* (!) So, we obtain ϕ by a line integral (without solving the characteristic equation or any first order ODE, for that matter).

In the same way

$$\psi_x dx + \psi_y dy + \psi_{y_1} dy_1 = \Delta^{-1} \begin{vmatrix} dx & dy & dy_1 \\ 1 & y_1 & w \\ \xi_1 & \eta_1 & \eta_1^{(1)} \end{vmatrix}$$

follows and we get $\mathbf{X}_1 = \partial_\psi, \quad \mathbf{X}_2 = \partial_\phi$.

The case corresponding to (a-i), that is $[\tilde{\mathbf{X}}_1, \tilde{\mathbf{X}}_2] = \tilde{\mathbf{X}}_1$ and $\Delta \neq 0$, we still have one invariant, say ϕ, corresponding to (1.9). Indeed, the normal form (1.8) can be converted into

$$\tilde{\mathbf{X}}_1 = \partial_\psi \ , \quad \tilde{\mathbf{X}}_2 = \psi \partial_\psi + \partial_\phi \tag{1.12}$$

and find ϕ by integrating (1.11).

To find the second invariant ψ we proceed as follows: Since $\phi_{y_1} \neq 0$ (why?) we can introduce $y_1 = y_1(x, y, \phi)$ and

$$\mathbf{A} = \partial_x + y_1(x,y,\phi)\partial_y + \mathbf{A}(\phi)\partial_\phi = \partial_x + y_1(x,y,\phi)\partial_y \ .$$

A second invariant ψ (related to x) should be found which satisfy

$$\mathbf{A}(\psi) = \psi_x + y_1(x,y,\phi)\psi_y = 0 \ , \quad \mathbf{X}_1(\psi) = 1 \ . \tag{1.13}$$

From (1.13) we get

$$\psi(x,y,\phi) = \int \frac{dy - y_1(x,y,\phi)dx}{\eta_1 - \xi_1 y_1} \ ,$$

up to a function of ϕ. In particular we circumvented the need to solve a first order ODE(!) (see remark 1.1).

In some cases we can also use the invariants method to solve the case $\Delta = 0$.

1.5 More is better?

If we have a symmetry group of more than 2 generators then we may find one of the types $G2$ as a subgroup and proceed as above. Between all Lie groups acting on $\mathbb{R}^2$, there is only one group which does not contain $G2$. Its Lie algebra is the same as this of $SO(3)$:

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_3 \ , \quad [\mathbf{X}_2, \mathbf{X}_3] = \mathbf{X}_1 \ , \quad [\mathbf{X}_3, \mathbf{X}_1] = \mathbf{X}_2 \ . \tag{1.14}$$

Exercise: Find an action on $\mathbb{R}^2$ which realizes this group. (Hint: use the generators of $SO(3)$ on $\mathbb{R}^3$: $x\partial_y - y\partial_x$, $x\partial_z - z\partial_x$, $z\partial_y - y\partial_z$).

Once we realized such a symmetry group for a given second order ODE $\mathbf{A}$, we must conclude that its prolongations to the 3-dimensional space x, y, y_1, together with $\mathbf{A}$, forms a (locally) linearly dependent system. That is, there exists functions $\alpha_1, \alpha_2, \alpha_3, \theta$ of x, y, y_1 such that

$$\alpha_1\mathbf{X}_1 + \alpha_2\mathbf{X}_2 + \alpha_3\mathbf{X}_3 + \theta\mathbf{A} = 0 \ . \tag{1.15}$$

Hence, we may write

$$\mathbf{X}_1 = \phi\mathbf{X}_2 + \psi\mathbf{X}_3 + \gamma\mathbf{A} \tag{1.16}$$

for some functions ϕ, ψ, θ. We claim that ϕ, ψ are nontrivial, independent invariants of the ODE:

$$\mathbf{A}(\phi) = \mathbf{A}(\psi) = 0 \ .$$

To show this, we first argue that there cannot be a linear dependence between $\mathbf{X}_2, \mathbf{X}_3, \mathbf{A}$, for assume

$$\beta_2\mathbf{X}_2 + \beta_3\mathbf{X}_3 + \gamma\mathbf{A} = 0 \ . \tag{1.17}$$

We make a change of variables to q, p, s where q and p are *first integrals* of the equation, namely

$$\mathbf{A}(q) = \mathbf{A}(p) = 0 \ , \mathbf{A}(s) = 1 \ . \tag{1.18}$$

so $\mathbf{A} = \partial_s$. Then, from $[\mathbf{X}_i, \mathbf{A}] = \lambda_i \mathbf{A}$ we get that the coefficients of ∂_p, ∂_q of $\mathbf{X}_i$ are independent of s. That is, $\mathbf{X}_i(p), \mathbf{X}_i(q)$, $i = 1, 2, 3$ are independent of s.

Next, we claim that p, q can be chosen in such a way that $\mathbf{X}_2(p) = 1$, $\mathbf{X}_2(q) = 0$. Otherwise we have $\mathbf{X}_2(p) = \mathbf{X}_2(q) = 0$ and $\mathbf{X}_2 = \mathbf{X}_2(s)\partial_s$. Hence $[\mathbf{X}_2, \mathbf{X}_i]$ is a vector field in the direction of ∂_s for $i = 1, 3$. It follows that all 3 fields $\mathbf{X}_i$, $i = 1, 2, 3$ are multiple of ∂_s, so the algebra (1.14) has a representation on $\mathbb{R}^1$. But this is impossible (show it!).

So, we have a representation of (1.14) as

$$\mathbf{X}_2 = \partial_p + \mathbf{X}_2(s)\partial_s, \quad \mathbf{X}_3 = \mathbf{X}_3(p)\partial_p + \mathbf{X}_3(q)\partial_q + \mathbf{X}_3(s)\partial_s \ , \quad \mathbf{A} = \partial_s$$

and its determinant is $\mathbf{X}_3(q)$. Since this determinant must be zero by (1.17), it follows that $\mathbf{X}_3(q) = 0$. Hence $[\mathbf{X}_2, \mathbf{X}_3](q) = \mathbf{X}_1(q) = 0$ as well, and can restrict the algebra of vectorfields (1.14) to $\mathbb{R}^1$ parametrized by the q coordinate (since the other confinements are independent of s). Again, we get a representation of (1.14) on $\mathbb{R}^1$ which is impossible. Hence

$$\beta_2 = \beta_3 = 0 \ . \tag{1.19}$$

Exercise: Prove that there is no one dimensional realization of (1.14) on $\mathbb{R}^1$.

From (1.16):

$$[\mathbf{X}_1, \mathbf{A}] = \phi[\mathbf{X}_2, \mathbf{A}] + \psi[\mathbf{X}_3, \mathbf{A}] + \mathbf{A}(\gamma)\mathbf{A} + \mathbf{A}(\phi)\mathbf{X}_2 + \mathbf{A}(\psi)\mathbf{X}_3$$

so, by the symmetry condition $[\mathbf{X}_i, \mathbf{A}] = \lambda_i \mathbf{A}$:

$$(-\lambda_1 + \lambda_2 + \lambda_2 - \mathbf{A}(\gamma))\mathbf{A} = \mathbf{A}(\phi)\mathbf{X}_2 + \mathbf{A}(\psi)\mathbf{X}_3 \ ,$$

and $\mathbf{X}_2, \mathbf{X}_2, \mathbf{A}$ verifies (1.17) where $\beta_2 = \mathbf{A}(\phi)$ and $\beta_3 = \mathbf{A}(\psi)$. Thus, $\mathbf{A}(\psi) = \mathbf{A}(\phi) = 0$ by (1.19) so ϕ, ψ are invariants of the ODE as claimed.

We may now show, by commuting (1.16) with $\mathbf{X}_i$, $i = 1, 2, 3$, that ϕ, ψ are independent invariants (show it!)