Lecture 7

1 ODE of higher order

1.1 Successive order reductions

Given an ODE

$$y^{(n)} = w(x, y, y^{'}, \dots y^{(n-1)})$$

and an action of a symmetry group generated by Gr generated by $\mathbf{X}_1, \dots \mathbf{X}_r$, we can always reduce the order by one in the usual way: transform $\mathbf{X}_1$, say, to ∂_s and set s as the dependent variable. Then, we know that in the new coordinates t, s (say), the ODE takes the form

$$s^{(n)} = \tilde{w}(t, s^{'}, \dots s^{(n-1)})$$

so the corresponding operator

$$\mathbf{A} = \partial_t + s_1 \partial_s + s_2 \partial_{s_1} + \dots + \tilde{w} \partial_{s_{n-1}}$$

while the prolongations of the transformed symmetry generators

$$\mathbf{X}_1 = \partial_s, \quad \mathbf{X}_j = \tilde{\xi}_j \partial_t + \tilde{\eta}_j \partial_s + \tilde{\eta}_j^{(1)} \partial_{s_1} + \dots + \tilde{\eta}_j^{(n-1)} \partial_{s-1} \;\; j = 2, \dots r \ .$$

Since $\tilde{w}$ is independent of s we may now reduce the system by removing the coefficients of ∂_s from $\mathbf{A}$ and $\mathbf{X}_j$, $2 \leq j \leq r$. Thus, we end up with

$$\hat{\mathbf{A}} = \partial_t + s_2 \partial_{s_1} + \dots \tilde{w}(x, s_1, \dots s_{n-1}) \partial_{s_{n-1}}$$

$$\hat{\mathbf{X}}_j = \tilde{\xi}_j \partial_t + \tilde{\eta}_j^{(1)} \partial_{s_1} + \dots + \tilde{\eta}_j^{(n-1)} \partial_{s_{n-1}} \quad , \quad j = 2 \dots r \ .$$

In order to go on with the reduction process, we need to verify that $\hat{\mathbf{X}}_j$ $2 \leq j \leq r$, generate a symmetry group for $\hat{\mathbf{A}}$, that is

$$[\hat{\mathbf{X}}_j, \hat{\mathbf{A}}] = \hat{\lambda}_j \hat{\mathbf{A}} \quad , 2 \leq j \leq r \ ? \tag{1.1}$$

Remark 1.1. *Note that, at this stage, we do not know if $\hat{\mathbf{X}}_j$ is an algebra of v-f on the coordinate space $t, s_1, \dots s_{n-1}$.*

Since

$$[\mathbf{X}_j, \mathbf{A}] = [\hat{\mathbf{X}}_j + \tilde{\eta}_j \mathbf{X}_1, \hat{\mathbf{A}} + s_1 \mathbf{X}_1] = \lambda_j \mathbf{A}$$

by assumption, using $[\tilde{\mathbf{X}}_j, s_1 \mathbf{X}_1] + [\tilde{\eta}_j \mathbf{X}_1, s_1 \mathbf{X}_1] = [\mathbf{X}_j, s_1 \mathbf{X}_1] = -s_1 [\mathbf{X}_j, \mathbf{X}_1] + \tilde{\eta}_j^{(1)} \mathbf{X}_1$ we arrive at

$$\tilde{\eta}_j^{(1)} \mathbf{X}_1 - \hat{\mathbf{A}}(\tilde{\eta}_j) \mathbf{X}_1 - s_1 [\mathbf{X}_1, \mathbf{X}_j] + [\hat{\mathbf{X}}_j, \hat{\mathbf{A}}] = \lambda_j \left(\hat{\mathbf{A}} + s_1 \mathbf{X}_1 \right) \tag{1.2}$$

so a necessary condition for (1.1) is

$$[\mathbf{X}_1, \mathbf{X}_j] = \alpha \mathbf{X}_1 + \beta \hat{\mathbf{A}}$$

for some functions α, β. From this we get

$$\partial_s(\tilde{\xi}_j)\partial_t + \sum_{k=0}^{n-1} \partial_s(\tilde{\eta}_j^{(k)})\partial_{s_k} = \alpha\partial_s + \beta\partial_t + \beta \sum_{k=1}^{n-1} s_{k+1}\partial_{s_k}$$

so $\beta s_{k+1} = \partial_s(\tilde{\eta}_j^{(k)})$ for $k \geq 1$, but this is impossible since $\tilde{\eta}_j^{(k)}$ is independent of s_{k+1}, hence $\beta = 0$. Also, $\alpha = \tilde{\eta}_{j,s}^{(0)}$, as well as $\partial_s(\tilde{\eta}_j^{(k)}) = 0 = \partial_s(\tilde{\xi}_j)$. It follows that α cannot depend on neither s nor x, so it is a constant $\alpha = \alpha_0$:

$$[\mathbf{X}_1, \mathbf{X}_j] = \alpha_0 \mathbf{X}_1 \tag{1.3}$$

Remark 1.2. *Note that, with condition (1.3), $\hat{\mathbf{X}}_2, \ldots \hat{\mathbf{X}}_r$ is an algebra for each fixed value of s. Indeed*

$$[\hat{\mathbf{X}}_i, \hat{\mathbf{X}}_j] = [\mathbf{X}_i - \eta_i \mathbf{X}_1, \mathbf{X}_j - \eta_j \mathbf{X}_1] = [\mathbf{X}_i, \mathbf{X}_j] - \eta_i [\mathbf{X}_1, \mathbf{X}_j] + \eta_j [\mathbf{X}_1, \mathbf{X}_i] \mod \mathbf{X}_1$$

so (1.3) we get

$$[\hat{\mathbf{X}}_i, \hat{\mathbf{X}}_j] = [\mathbf{X}_i, \mathbf{X}_j] \mod \mathbf{X}_1$$

Conclusion: If there exists j_0 so that the structure constants $C^i_{j_0,k} = 0$ for any $i, k \neq j_0$ then we may continue the reduction process.

1.2 Solutions by first integrals

If we can find a first integral $\phi = \phi(x, y, y_1, \ldots y_{n-1})$ of the equation, then we can use it in the solution process. In some cases we can use the symmetry group, if exists, to compute such an integral without solving any ODE. For this, assume that there is a Lie algebra of n symmetric v-f $\mathbf{X}_j$, $j = 1, \ldots n$ for the given ODE. We also suppose that the corresponding action on the extended space $\mathbb{R}^{n+1} := (x, y, y_1, \ldots y_{n-1})$ is transitive, that is, the $n+1$ v-f $\mathbf{X}_1, \ldots \mathbf{X}_n$ and $\mathbf{A}$ are linearly independent at each point. Our object is to solve

$$\mathbf{X}_1(\phi) = 1, \quad \mathbf{A}(\phi) = \mathbf{X}_2(\phi) = \ldots = \mathbf{X}_n(\phi) = 0 \ . \tag{1.4}$$

If such ϕ can be found, we could change to a new set of variables by which $\mathbf{X}_1 = \partial_\phi$ which is invariant of the ODE. In particular, the homogeneous system

$$\mathbf{A}(\phi) = \mathbf{X}_2(\phi) = \ldots = \mathbf{X}_n(\phi) = 0 \tag{1.5}$$

should have a non-trivial solution. We recall Theorem 1, Lecture 2, to obtain the condition for such a solution: These vector fields should be in involution. In fact we know that

$$[\mathbf{X}_i, \mathbf{X}_j] = \sum_{k=2}^{n} C^k_{i,j} \mathbf{X}_k + C^1_{i,j} \mathbf{X}_1$$

and

$$[\mathbf{X}_j, \mathbf{A}] = \lambda_j \mathbf{A}$$

so we get the solvability of (1.5) provided

$$C^1_{i,j} = 0 \tag{1.6}$$

for any $2 \le i, j \le n$.

Next, the solvability of $\mathbf{X}_1(\phi) = 1$ within the set of homogeneous solutions of (1.5) needs also a compatibility condition: $[\mathbf{X}_1, \mathbf{X}_j](\phi) = 0$ and $[\mathbf{X}_1, \mathbf{A}](\phi) = 0$. The last condition is satisfied since $[\mathbf{X}_1, \mathbf{A}] = \lambda_1 \mathbf{A}$. The first condition requires

$$\sum_{k=1}^{n} C^k_{1,j} \mathbf{X}_k(\phi) = C^1_{1,j} \mathbf{X}_1(\phi) = C^1_{1,j} = 0 \quad 2 \le j \le n \ . \tag{1.7}$$

If (1.6, 1.7) are satisfied then the system (1.4) can be solved and ϕ can be found from integration of

$$\phi_x dx + \phi_y dy + \sum_1^n \phi_{y_j} dy_j = \frac{\begin{vmatrix} dx & dy & dy_1 & \dots & dy_{n-1} \\ \xi_2 & \eta_2 & \eta_2^{(1)} & \dots & \eta_2^{(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_n & \eta_n & \eta_n^{(1)} & \dots & \eta_n^{(n-1)} \\ 1 & y_1 & y_2 & \dots & w \end{vmatrix}}{\begin{vmatrix} \xi_1 & \eta_1 & \eta_1^{(1)} & \dots & \eta_1^{(n-1)} \\ \xi_2 & \eta_2 & \eta_2^{(1)} & \dots & \eta_2^{(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_n & \eta_n & \eta_n^{(1)} & \dots & \eta_n^{(n-1)} \\ 1 & y_1 & y_2 & \dots & w \end{vmatrix}}$$

Once it was done, we factor out $y^{(n-1)} = y^{(n-1)}(\phi, x, y, y_1, \dots y_{n-2})$ and obtain the system in new coordinates $\phi, x, y, \dots, y_{n-1}$ as

$$\mathbf{X}_j = \xi_j \partial_x + \eta_j \partial_y + \dots + \eta_j^{(n-2)} \partial_{n-2} \quad j = 2 \dots n \ , \mathbf{A} = \partial_x + y_1 \partial_y + \dots + y^{(n-1)}(\phi, x, y, \dots, y_{n-2}) \partial_{y_{n-2}} \ .$$

where ϕ is now a *parameter*. We may proceed with this process if we could find a v-f $\mathbf{X} \in \mathbf{X}_2, \dots \mathbf{X}_n$, say $\mathbf{X}_2$, for which

$$[\mathbf{X}_2, \mathbf{X}_j] = \sum_{k=3}^{n} C^k_{2,j} \mathbf{X}_k \quad , \quad [\mathbf{X}_i, \mathbf{X}_j] = \sum_{k=3}^{n} C^k_{i,j} \mathbf{X}_k$$

for any $i, j \ge 3$.

Definition 1.1. *Given a Lie algebra $\mathcal{G} := Span\{\mathbf{X}_1, \dots \mathbf{X}_r\}$, the commutator algebra $C(\mathcal{G})$ is spanned by the set of all commutators $[\mathbf{X}_i, \mathbf{X}_j] \in \mathcal{G}$. (why it is a subalgebra of $\mathcal{G}$?)*

Then there is a hierarchy and there exists $m \ge 1$ for which $\mathcal{G} \supset C(\mathcal{G}) \supset C^2(\mathcal{G}) \supset \dots \supset C^m(\mathcal{G}) = C^{m+1}(\mathcal{G})$.

A Lie algebra is called solvable if $C^m(\mathcal{G}) = \{e\}$.

Conclusion (**Lie Theorem**): If the algebra of the symmetry group with n generators is solvable and transitive on the space of first integrals of the ODE, then we may find these invariants (and solve the equation) by carrying out n line integrals.