Lecture 8

1 Lagrangian dynamics

1.1 Overview

We now consider the important case of equations (or, rather, system of equations) derived from a Lagrange principle. Let $L = L(x, y_1, \dots y_p, y_1^{(1)}, \dots y_p^{(n)})$ be any function of x and p dependent variables $Y := \{y_1, \dots y_p\}$ and their x derivative to order n. We shall denote such a function by $L = L(x, Y^{(n)})$. This Lagrangian induces an *action* on the set of orbits $\mathcal{Y} = Y(x)$ defined on the interval $x \in (\alpha, \beta)$ which verify the end conditions $\mathcal{Y}(\alpha) = Y_1, \mathcal{Y}(\beta) = Y_2$:

$$\mathcal{L}(\mathcal{Y}) = \int_\alpha^\beta L\left(x, \mathcal{Y}^{(n)}(x)\right) dx \ .$$

A function $\mathcal{Y}$ in this set is called stationary of this Lagrangian if

$$\delta_{\mathcal{Y}}\mathcal{L}(\Gamma) := \frac{d}{dt}\mathcal{L}(\mathcal{Y}^{(n)} + t\Gamma^{(n)})\Big|_{t=0} = 0 \tag{1.1}$$

for any $n-$differentiable $\Gamma = \{\gamma_1(x), \dots \gamma_p(x)\}$ which verifies

$$\Gamma^{(n)}(\alpha) = \Gamma^{(n)}(\beta) = 0 \ . \tag{1.2}$$

Now, let D_i is the complete derivative with respect to y_i:

$$D_j = \partial_x + y_j^{(1)}\partial_{y_j} + \dots \ .$$

Let

$$E_j := \sum_{k=0}^{n}(-D_j)^k \partial_{y_j^{(k)}} \ .$$

Here $D_j^0 = 1$.

Proposition 1.1. *$\mathcal{Y}$ is a stationary solution of the Lagrangian L iff it satisfies the system of equations (Euler-Lagrange equations)*

$$E_i(L) = 0 \quad , i = 1, \dots p \ . \tag{1.3}$$

Proof. From (1.1) we get that

$$\delta_Y\mathcal{L}(\Gamma) = \sum_{j=1}^{p}\sum_{i=0}^{n}\int_\alpha^\beta \frac{\partial L}{\partial y_j^{(i)}}\frac{d^i\gamma_j}{dx^i}dx$$

By integration by parts and (1.2) we get

$$\delta_Y\mathcal{L}(\Gamma) = \sum_{j=1}^{p}\sum_{i=1}^{n}\int_\alpha^\beta (-1)^i D_j^i\left(\frac{\partial L}{\partial y_j^{(i)}}\right)\gamma_j dx = 0$$

which implies the result (since Γ is, apart from condition (1.2), arbitrary). □

Example A *mechanical Lagrangian* is of the form

$$L(x, Y^{(2)}) = \sum_1^p m_i (y_i^{'})^2 - U(x, Y)$$

where m_i are constants (inertia masses) and U the interaction potential. the equation $E_i(L) = o$ takes the form

$$m_i y_i^{''} + U_{y_i} = 0$$

In particular, the equation for the single pendulum of mass m and length l in a gravitational field g, given by $L = m(y^{'})^2/2 - (1 - gl\cos(y))$ takes the form

$$y^{''} = (gl/m)\sin(y) \ .$$

1.2 Transformations of the Lagrangian

Let now a symmetry group G acting on a domain in $\mathbb{R}^{p+1}$ parameterized by $(x, Y) = (x, y_1, \ldots y_p)$. The action is denoted by $\Psi : \mathbb{R}^{p+1} \to \mathbb{R}^{p+1}$ where $\Psi := (\psi_{(x)}, \psi_{(Y)})$.

$$x \to \tilde{x} = \psi_{(x)}(x, Y) \quad , \quad Y \to \tilde{Y} = \psi_{(Y)}(x, Y) \ .$$

This action induces an action on the graph of a dependent variable $Y = Y(x)$ as defined in lecture 3. It can also be prolonged into the graph of $Y^{(n)}$, The prolonged action is denoted by $\Psi^{(n)} : (x, Y^{(n)}) \to \tilde{x}, \tilde{Y}^{(n)}$.

Such an action induces a transformation on a Lagrangian L in the following, natural way:

$$\tilde{\mathcal{L}}(\tilde{Y}) = \mathcal{L}(Y) \tag{1.4}$$

It follows that the Lagrangian function $L : (\alpha, \beta) \times \mathbb{R}^{p(n+1)} \to \mathbb{R}$ is transformed into $\tilde{L} : (\tilde{\alpha}, \tilde{\beta}) \times \mathbb{R}^{p(n+1)} \to \mathbb{R}$ where $(\tilde{\alpha}, \tilde{\beta}) = \left(\psi_{(x)}(\alpha, Y_1), \psi_{(x)}(\beta, Y_2)\right)$. From (1.4) we get the form of the transformed Lagrangian by change of variables formula:

$$\tilde{L}\left(\tilde{x}, \tilde{Y}^{(n)}\right) D_x \psi_{(x)}(x, Y) = L(x, Y^{(n)}) \tag{1.5}$$

where $D_x := \partial_x + \sum_i^p y_i^{(1)} \partial_{y_i}$.

How does the Euler-Lagrange equations look like for the transformed Lagrangian? Let us consider a variation $Y \to Y + t\Gamma$ and $x_{(t)}, Y_{(t)}$ given by

$$\tilde{x} = \psi_{(x)}\left(x_{(t)}, Y(x_{(t)}) + t\Gamma(x_{(t)})\right), \ \ \tilde{Y}_{(t)} = \psi_{(Y)}\left(x_{(t)}, Y(x_{(t)}) + t\Gamma(x_{(t)})\right) \ .$$

By this convention, $\tilde{x}$ is independent of t. Hence

$$0 = D_x \psi_{(x)} \frac{dx_{(t)}}{dt} + \nabla_Y \psi_{(x)} \cdot \Gamma \ .$$

So

$$\frac{dx_{(t)}}{dt} = -\nabla_Y \psi_{(x)} \cdot \Gamma / D_x \psi_{(x)}$$

$$\frac{d\tilde{Y}_{(t)}}{dt} = \nabla_Y \psi_{(Y)} \cdot \Gamma + D_x \psi_{(Y)} \frac{dx_{(t)}}{dt} = \nabla_Y \psi_{(Y)} \cdot \Gamma - (D_x \psi_{(x)})^{-1} D_x \psi_{(Y)} \nabla_Y \psi_{(x)} \cdot \Gamma$$

Consider now

$$F_{i,j} := \begin{pmatrix} D_x \psi_{(x)} & \partial_{y_i} \psi_{(x)} \\ D_x \psi_{(y_j)} & \partial_{y_i} \psi_{(y_j)} \end{pmatrix}$$

Then

$$\frac{d\tilde{y}_j}{dt} = (D_x \psi_{(x)})^{-1} \sum_{i,j} \det(F_{i,j}) \gamma_i \ .$$

Since, by definition

$$\frac{d}{dt} \tilde{\mathcal{L}}(\tilde{Y}) \bigg|_{t=0} = \int_{\tilde{\alpha}}^{\tilde{\beta}} \left(\sum_{i=1}^{p} \tilde{E}_i(\tilde{L}) \frac{d\tilde{y}_i}{dt} \right) d\tilde{x} = \int_{\alpha}^{\beta} D_x \psi_{(x)} \left(\sum_{i=1}^{p} \tilde{E}_i(\tilde{L}) \frac{d\tilde{y}_i}{dt} \right) dx$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{p} \int_{\alpha}^{\beta} \det(F_{i,j}) \tilde{E}_i(\tilde{L}) \gamma_i dx = \frac{d}{dt} \mathcal{L}(Y) \bigg|_{t=0} = \sum_{i=1}^{p} \int_{\alpha}^{\beta} E_i(L) \gamma_i dx \ .$$

We finally get the relation between the Euler-Lagrange equations:

$$\sum_{j=1}^{p} \det(F_{i,j}) \tilde{E}_j(\tilde{L}) = E_i(L) \ . \tag{1.6}$$

Example: In the case $p = 1$ we obtain

$$E(L) = \det \begin{pmatrix} \partial_x \psi_{(x)} & \partial_y \psi_{(x)} \\ \partial_x \psi_{(y)} & \partial_y \psi_{(y)} \end{pmatrix} \tilde{E}(\tilde{L})$$

Why did we replaced $D_x \psi_{(x)}$ by $\partial_x \psi_{(x)}$?

Let now $(x, y) \to (y, x)$ so $\psi_{(x)}(x, y) = y$, $\psi_{(y)}(x, y) = x$. In this case we obtain

$$E(L) = -\tilde{E}(\tilde{L}) \ .$$

Can you verify it directly from the Lagrangian formulation?

1.3 Lagrangian preserving transformations

A transformation ψ is said to preserve a given Lagrangian L if $\tilde{L}$, as defined in (1.5), is equel to L:

$$L\left(\tilde{x}, \tilde{Y}^{(n)}\right) D_x \psi_{(x)}(x, Y) = L(x, Y^{(n)}) \ . \tag{1.7}$$

It is possible to generalize this definition and request the transformed Lagrangian to induce the same *Lagrangian action*. Suppose there exists a function $V = V(x, Y^{(n-1)})$ such that the transformed Lagrangian takes the form $L + D_x V$. Evidently, this Lagrangian generates the same action as L, up to an irrelevant constant, for

$$\int_{\alpha}^{\beta} [L + D_x V] dx = \int_{\alpha}^{\beta} L dx + V(\beta, Y^{(n)}(\beta)) - V(\alpha, Y^{(n)}(\alpha))$$

Exercise: Prove directly, using (1.1), that the Euler Lagrange equations for L and $L + D_x V$ are identical. For this, you need to show the identity $E_i D_x \equiv 0$ for any i.

We now generalize the notion of a transformation which preserves the Lagrangian, to a transformation which preserves the Lagrangian action. Thus

$$L\left(\tilde{x}, \tilde{Y}^{(n)}\right) D_x \psi_{(x)}(x, Y) = L(x, Y^{(n)}) + D_x V \tag{1.8}$$

for some function $V = V(x, Y^{(n-1)})$.

Next, consider a symmetry generated by a vector field

$$\mathbf{X} := \xi \partial_x + \sum_{j=1}^{p} \eta_j \partial_{y_j}$$

acting on the domain (x, Y) of a Lagrangian action induced by $L(x, Y^{(n)})$.
Example: The v-f ∂_x preserves any Lagrangian of the form $L = L(Y^{(n)})$.
The v-f ∂_{y_j} preserves any Lagrangian which is independent of y_j (but may depend on its derivatives).

If we substitute $x + t\xi$ for $\tilde{x} := \psi_{(x)}(x, Y)$, $y_j^{(k)} + t\eta_j^{(k)}$ for $\tilde{y}_j^{(k)}$ and tV for V in (1.8) and differentiate the equality at $t = 0$ we get

$$Pr^{(n)}\mathbf{X}(L) + LD_x(\xi) = D_x V \ .$$

Example: Let $L = \sqrt{1 + (y')^2}$ and consider $\mathbf{X} = -y\partial_x + x\partial_y$. We already met the prolongation of this field $Pr^{(1)}\mathbf{X} = -y\partial_x + x\partial_y + (1 + y_1^2)\partial_{y_1}$. Then

$$Pr^{(1)}\mathbf{X}(L) + LD_x\xi = \left(1 + (y')^2\right) L_{y'} - y' L \equiv 0$$

as we could expect. Since the action of this Lagrangian is the arc-length of the graph of y, it is not surprising that a rotation in the plane preserves this arc-length.

A natural conclusion is

Corollary 1.1. *If* $\mathbf{X}$ *preserves the Lagrangian action, it also induces a symmetry of the corresponding Euler-Lagrange equation* $E(L) = 0$*.*

The inverse claim is not true, in general.
Example: The Kepler problem (point mass under inverse square law): We consider the Lagrangian

$$L(Y, Y^{(1)}) = \frac{1}{2}\left((y_1')^2 + (y_2')^2 + (y_3')^2\right) - \frac{M}{\sqrt{y_1^2 + y_2^2 + y_3^2}}$$

The EL equations are

$$y_i'' = -\frac{My_i}{y_1^2 + y_2^2 + y_3^2} \quad , \quad i = 1, 2, 3 \ .$$

Direct computation implies that this equation is invariant under the first prolongation of

$$\mathbf{X}_{i,j} = y_i \partial_{y_j} - y_j \partial_{y_i} \quad , \quad i \neq j \ ,$$

$$\mathbf{X}_4 = \partial_x \ ,$$

$$\mathbf{X}_5 = x\partial_x + \frac{2}{3}\sum_1^3 y_i \partial_{y_i}$$

The v.f $\mathbf{X}_{i,j}$ represents the symmetry of the system with respect to rotation in space $SO(3)$. We readily see that its prolongation $Pr^{(1)}\mathbf{X}_{i,j} = y_i\partial_{y_j} - y_j\partial_{y_i} + y_i^{'}\partial_{y_j^{'}} - y_j^{'}\partial_{y_i^{'}}$ preserves the Lagrangian

$$Pr^{(1)}\mathbf{X}_{i,j}(L) = 0 \ .$$

Evidently, the prolongation of $\mathbf{X}_4$ (which is $\mathbf{X}_4$ itself) preserves to Lagrangian as well. However, $Pr^{(1)}\mathbf{X}_5 = x\partial_x + \frac{2}{3}\sum_1^3 y_i\partial_{y_i} - \frac{1}{3}\sum_1^3 y_i^{(1)}\partial_{y_i^{(1)}}$ verifies

$$Pr^{(1)}\mathbf{X}_5(L) - D_x(x)L = \frac{2}{3}\left(M - \sum_1^3 \left(y_j^{(1)}\right)^2\right) \neq D_x V \ .$$

We shall see later that $\mathbf{X}_{i,j}$ induces an invariant $L_{i,j} = y_i y_j^{'} - y_j y_i^{'}$ of the Kepler system. It is nothing but the angular momentum of this system in the direction perpendicular to the (y_i, y_j) plan. Similarly, $\mathbf{X}_4$ induces the invariant

$$\mathcal{E} = \frac{1}{2}\left((y_1^{'})^2 + (y_2^{'})^2 + (y_3^{'})^2\right) + \frac{M}{\sqrt{y_1^2 + y_2^2 + y_3^2}}$$

which is to energy of the system. The v-f $\mathbf{X}_5$ is a manifestation of the third law of Kepler: It induces the transformation $x \to e^t x$, $y_i \to e^{2t/3} y_i$ which implies, between other, the power law of 2/3 between the period and the radius of a planet's orbit. It does not correspond, however, to a conservation law.