Lecture 9

1 Differential forms (survival kit)

Let $D \subset \mathbb{R}^m$. We already know that a vector field $\mathbf{X}$ on D is an operator on $C^\infty(D)$. So, we can view a function $f \in C^\infty(D)$ as a co-vector, namely a linear operator on the space of vector fields: $df(\mathbf{X}) := \mathbf{X}(f)$. In local coordinates $\mathbf{X} = \sum \xi_i \partial_{x_i}$ we get

$$df(\mathbf{X}) = \sum_1^m \xi_i \partial_{x_i} f \quad .$$

The linear space $\Omega^{(1)}$ is composed of all linear functionals on the space of vector-fields. In particular, every $f \in C^\infty(D)$ induces an element of $\Omega^{(1)}$ via df.

More generally, for any $\alpha, f \in C^\infty(D)$, $\alpha df \in \Omega^{(1)}$ is defined by the multiplication $\alpha df(\mathbf{X}) = \alpha \mathbf{X}(f)$. If we choose f to be a coordinate x_j for $j \in \{1, \dots m\}$ we get the representation $\omega := \sum_1^m \alpha_j(x) dx_j$, acting via

$$\omega(f) = \sum_1^m \alpha_j(x) \partial_{x_i} f \ .$$

The space $\Omega^{(2)}$ is defined as follows: Given $\omega \in \Omega^{(2)}$ and a vector field $\mathbf{X}$, $\omega(\mathbf{X}) \in \Omega^{(1)}$ is given by

$$\omega(\mathbf{X})(\mathbf{Y}) := \omega(\mathbf{X}, \mathbf{Y})$$

is a bi-linear operator on the pairs of fields $\mathbf{X}, \mathbf{Y}$ which is, in addition- anti symmetric

$$\omega(\mathbf{X}, \mathbf{Y}) = -\omega(\mathbf{Y}, \mathbf{X}) \ .$$

The way to obtain elements of $\Omega^{(2)}$ is via the *wedge product*. Given $\alpha, \beta \in \Omega^{(1)}$ then $\alpha \wedge \beta \in \Omega^{(2)}$ is defined via

$$\alpha \wedge \beta(\mathbf{X}, \mathbf{Y}) := \alpha(\mathbf{X})\beta(\mathbf{Y}) - \alpha(\mathbf{Y})\beta(\mathbf{X}) \ .$$

In local coordinates,

$$\alpha \wedge \beta = \sum_{i,j} \alpha_i \beta_j dx_i \wedge dx_j = \sum_{i>j} (\alpha_i \beta_j - \beta_i \alpha_j) dx_i \wedge dx_j$$

where $dx_i \wedge dx_j(\mathbf{X}, \mathbf{Y}) := \mathbf{X}(x_i)\mathbf{Y}(x_j) - \mathbf{X}(x_j)\mathbf{Y}(x_i) = -dx_j \wedge dx_i(\mathbf{X}, \mathbf{Y})$.

In a progressive way we define $\sigma \in \Omega^{(k)}$ as multi-linear anti-symmetric k form. In local coordinates

$$\alpha = \sum_{i_1, \dots i_k} \alpha_{i_1, \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

and the wedge product $\alpha \wedge \beta \in \Omega^{(k+l)}$ for $\alpha \in \Omega^{(k)}$, $\beta \in \Omega^{(m)}$

$$\alpha \wedge \beta(\mathbf{X}_1, \dots \mathbf{X}_{k+m}) = \sum \pi(i_1, \dots i_{k+m}) \alpha(\mathbf{X}_{i_1}, \dots \mathbf{X}_{i_k}) \beta(\mathbf{X}_{k+1}, \dots \mathbf{X}_{k+m})$$

where $\pi = +1$ for an even permutation, $\pi = -1$ for odd permutation.

In addition, we extend the operator $d : \Omega^{(k)} \to \Omega^{(k+1)}$ from its definition on C^∞ (identified with $\Omega^{(0)}$) to $\Omega^{(1)}$ as follows:

a) $d^2 \equiv 0$

b) If $\alpha \in \Omega^{(k)}$ then $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$.

c) For $\phi \in C^\infty$,$\alpha \in \Omega^{(k)}$, $d(\phi\alpha) = d\phi \wedge \alpha + \phi d\alpha$.

So, the exterior differential d always *raise* the order of a differential form form $\Omega^{(k)}$ to $\Omega^{(k+1)}$. There is a natural operation which *lower* the order of the differential form: If $\alpha \in \Omega^{(k)}$ and $\mathbf{X}$ is some vector field, then $\mathbf{X} \lrcorner \alpha \in \Omega^{(k-1)}$ where

$$\mathbf{X} \lrcorner \alpha(\mathbf{X}_1, \dots, \mathbf{X}_{k-1}) := \alpha(\mathbf{X}, \mathbf{X}_1, \dots \mathbf{X}_{k-1}) \ .$$

Next, we define the derivative of a differential form in the direction of a given vector field $\mathbf{X}$. This is the *Lie derivative*, denoted as $\mathcal{L}_\mathbf{X} : \Omega^{(k)} \to \Omega^{(k)}$. First, we define it on the space $\Omega^{(0)} := C^\infty(D)$ as follows:

$$\mathcal{L}_\mathbf{X}(f) := \mathbf{X}(f) \ .$$

Then, for a given form $\alpha \in \Omega^{(k)}$ we define

$$\mathcal{L}_\mathbf{X}\alpha := \mathbf{X} \lrcorner d\alpha + d\mathbf{X} \lrcorner \alpha \ . \tag{1.1}$$

From (1.1) we get that the Lie derivative commutes with d:

$$\mathcal{L}_\mathbf{X} d\alpha = d\mathcal{L}_\mathbf{X}\alpha$$

We may also define the lie derivative on another vectorfield

$$\mathcal{L}_\mathbf{X}\mathbf{Y} = [\mathbf{X}, \mathbf{Y}]$$

that is our old friend the commutator. With this definition, there is an alternative definition to (1.1) for the Lie derivative:

$$[\mathcal{L}_\mathbf{X}\alpha](\mathbf{X}_1, \dots \mathbf{X}_k) := \mathbf{X}\left(\alpha(\mathbf{X}_1, \dots \mathbf{X}_k)\right) - \sum_{j=1}^k \alpha(\mathbf{X}_1, \dots, \mathcal{L}_\mathbf{X}\mathbf{X}_j, \dots, \mathbf{X}_k) \tag{1.2}$$

The Lie derivative verifies the Leibnitz rule applies for both exterior (wedge) and contraction products:

$$\mathcal{L}_\mathbf{X}(\alpha \wedge \beta) = (\mathcal{L}_\mathbf{X}\alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_\mathbf{X}\beta) \ , \quad \mathcal{L}_\mathbf{X}(\alpha \lrcorner \beta) = (\mathcal{L}_\mathbf{X}\alpha) \lrcorner \beta + \alpha \lrcorner (\mathcal{L}_\mathbf{X}\beta) \ .$$

Consider a mapping $\Psi : D \to \tilde{D}$ where,as before, D is a domain in $\mathbb{R}^n$. We do *not* assume that this mapping is a diffeomorphism, neither that $\tilde{D}$ had the same dimension as D. We assume, however, that Ψ is surjective on $\tilde{D}$, and that $\tilde{D}$ is a domain in $\mathbb{R}^m$ for $m \leq n$. Now, if $\alpha \in \Omega^{(k)}(\tilde{D})$, we can *pull it back* to a form in $\Omega^{(k)}(D)$ by Ψ as follows:

$$\Psi^*(\alpha)(\mathbf{X}_1, \dots \mathbf{X}_k) := \alpha\left(\Psi_*(\mathbf{X}_1), \dots, \Psi_*(\mathbf{X}_k)\right) \quad , \tag{1.3}$$

where Ψ_* is the *push forward*, defined on vector field as in Lecture 1. It follows by definition that Ψ^* commutes with the exterior differential d:

$$d\Psi^*(\alpha) = \Psi^*(d\alpha) \ . \tag{1.4}$$

There is another useful interpretation of a differential form. Consider a smooth orbit $\boldsymbol{x} : [0,1] \to \Gamma \subset \mathbb{R}^m$. We view this orbit as a mapping from $[0,1]$ to $\mathbb{R}^m$, and use it to pull back a one form $\omega \in \Omega^{(1)}(\mathbb{R}^m)$ to a form $\boldsymbol{x}^*(\omega) = f(x)dx \in \Omega^{(1)}([0,1])$. Then, the integral of ω on Γ is defined by

$$\int_\Gamma \omega := \int_0^1 f(x)dx \ . \tag{1.5}$$

It can be shown that this definition is independent of the particular parameterizations $\boldsymbol{x}$ of Γ. Similarly, $k - forms$ can be integrated on $k-$dimensional surfaces.

Remark 1.1. *Differential forms act on vector fields only* locally*. In coordinate representation it means that a differential form acts linearly only on the coefficients of the vector fields (and not, e.g., on the spacial derivatives of these coefficients). Note, in particular, that the definitions of $d\alpha$, $\mathbf{X}\lrcorner\alpha$ and $\mathcal{L}_\mathbf{X}\alpha$ preserve this property, as both are differential forms by definition. However, $\mathcal{L}_\mathbf{X}\alpha$* does *depend on the derivatives if the coefficients of $\mathbf{X}$ (as this form is not* acting *on $\mathbf{X}$).*

2 Contact forms

We now consider differential forms on the space $\mathbb{R}^{n+2}$ obtained from the prolongation of $\mathbb{R}^2$ into the $n - th$ order derivatives. In particular, $\alpha \in \Omega^{(1)}$ is represented in these coordinates as

$$\alpha = \alpha_{(x)}dx + \alpha_{(y)}dy + \sum_{j=1}^n \alpha_j dy_j$$

where we identify, as usual, y_j with the $j - th$ derivative of a function $y = y(x)$. The coefficients are functions of the variables $x, y, y_1, \ldots, y_n$.

Any such function induces a natural local embedding of the real line $\mathbf{X}$ into $\mathbb{R}^{n+2}$ vis

$$Y^{(n)}(x) := \left(x, y(x), y^{'}(x), \ldots, y^{(n)}(x)\right) \ .$$

By (1.3) we may pull this α back into a one-form on $\mathbb{R}$:

$$Y_*^{(n)}(\partial_x) = \partial_x + y^{'}\partial_y + y^{''}\partial_{y_1} + \ldots + y^{(n+1)}\partial_{y_n}$$

so

$$\left[T^{(n)}\right]^*(\alpha) = \left[\alpha_{(x)} + \alpha_{(y)}y^{'} + \sum_{j=1}^n \alpha_j y^{(j+1)}\right] dx \ . \tag{2.1}$$

Definition 2.1. *A contact form is a one-form $\alpha \in \Omega^{(1)}(\mathbb{R}^{n+2})$ which is pulled back by* any *embedding $x \to Y^{(n)}(x)$ of $\mathbb{R}$ into $\mathbb{R}^{n+1}$, to the zero form in $\Omega^{(1)}(\mathbb{R})$.*

From (2.1) we obtain that a contact form of first order must satisfy $\alpha_{(x)} = -y_1\alpha_{(y)}$, that is, a multiple of $dy - y_1dx$. Similarly, any $n - th$ order contact form must be a combination (with function coefficients) of the forms $\theta_k := dy_{k-1} - y_kdx$.

Definition 2.2. *A contact transformation* $\Phi : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$ *is such that preserves contact forms.*

We claim that a point transformation, that is, a transformation prolonged into $\mathbb{R}^{n+2}$ from a transformation in the point coordinates x, y of $\mathbb{R}^2$, is a contact transformation. Let us check this for the case $n = 1$. In that case we know that

$$\tilde{x} = \tilde{x}(x, y) \ , \quad \tilde{y} = \tilde{y}(x, y), \quad \tilde{y}_1 = \frac{d\tilde{y}}{d\tilde{x}} = \frac{\tilde{y}_y y_1 + \tilde{y}_x}{\tilde{x}_y y_1 + \tilde{x}_x}$$

Hence $d\tilde{y} = \tilde{y}_x dx + \tilde{y}_y dy$, $d\tilde{x} = \tilde{x}_x dx + \tilde{x}_y dy$ so

$$\Phi^* \left(d\tilde{y} - \tilde{y}_1 d\tilde{x}\right) = \frac{\tilde{x}_x \tilde{y}_y - \tilde{x}_y \tilde{y}_x}{\tilde{x}_y y_1 + \tilde{x}_x} \left(dy - y_1 dx\right) \ .$$

However, there are contact transformations which are not point transformation.
Example: Legendre transform: $\tilde{x} = y_1, \tilde{y} = y - x y_1, \tilde{y}_1 = -x$. It satisfies

$$d\tilde{y} - \tilde{y}_1 d\tilde{x} = dy - y_1 dx \ .$$

Exercise: Prove that any function $H = H(x, y, \tilde{x}, \tilde{y})$ generates a contact transformation, provided $\tilde{x}$, $\tilde{y}$, $\tilde{y}_1$ can be factored out from

$$H = 0 \quad , \quad H_x + y_1 H_y = 0, \quad H_{\tilde{x}} + \tilde{y}_i H_{\tilde{y}_1} = 0 \ .$$

As an example, $H = x\tilde{x} + y\tilde{y}$ generates the Legendre transformation.
Exercise: Prove that a v-f $\mathbf{X}$ generates a contact transformation if

$$\mathbf{X} = \xi(x, y, y_1)\partial_x + \eta(x, y, y_1)\partial_y + \eta^{(1}(x, y, y_1)\partial_{y_1}$$

where

$$\xi = \frac{\partial \Omega}{\partial y_1} \ , \quad \eta = y_1 \frac{\partial \Omega}{\partial y_1} - \Omega \ , \quad \eta^{(1)} = -\frac{\partial \Omega}{\partial x} - y_1 \frac{\partial \Omega}{\partial y}$$

for some function $\Omega(x, y, y_1)$.

2.1 Symmetry transformations

Let us consider now $1-$form in $\mathbb{R}^{n+2}$ which has only non-zero component in dx. Such a form is called *horizontal* (or *Lagrangian density*) and has the general form

$$\omega = L(x, y, y_1, \ldots, y_n)dx \ .$$

A contact transformation Ψ is said to be a symmetry of $\omega = Ldx$ if it preserves ω up to a contact transformation, that is,

$$\Psi^* \omega = \omega + \Theta$$

for some contact transformation Θ. In particular, for a point transformation $\tilde{x} = \tilde{x}(x, y)$, $\tilde{y} = \tilde{y}(x, y)$ we get $d\tilde{x} = \tilde{x}_x dx + \tilde{x}_y dy$. In particular we get

Theorem 1. *A point transformation $\tilde{x} = \tilde{x}(x, y)$, $\tilde{y} = \tilde{y}(x, y)$ is a symmetry for $\omega = Ldx$ iff*

$$L(x, y, y_1, \ldots, y_n) = L(\tilde{x}, \tilde{y}, \tilde{y}_1, \ldots, \tilde{y}_n)(\tilde{x}_x + \tilde{x}_y y_1)$$

where $\tilde{y}_1, \ldots \tilde{y}_n$ is determined from $\tilde{x}, \tilde{y}$ by prolongation.

Remark 2.1. *Compare this Theorem with (1.7) in Lecture 8.*

We now invert the question upside down. *Given* a contact transformation Ψ, what is the most general Lagrangian density which is preserved by this transformation?

Assume $I = I(x, y, y_1, \ldots, y_n)$ is an invariant function of Ψ, that is

$$\Psi^*(I) := I \circ \Psi = I \ . \tag{2.2}$$

Then, $L := D_x I$ is an example of a "trivial" Lagrangian which for which Ψ is a symmetry. To see this we first observe that, for *any* function $I \in C^\infty(\mathbb{R}^{n+2})$, the Lagrangian density $D_x I dx$ is the horizontal part of the one form dI. Indeed

$$\begin{aligned} dI := I_x dx + I_y dy + I_{y_1} dy_1 + \ldots + I_{y_n} dy_n \equiv \left(I_x + y_1 I_y + \ldots + y_n I_{y_{n-1}}\right) dx \\ + I_y(dy - y_1 dx) + I_{y_1}(dy_1 - y_2 dx) + \ldots + I_{y_n}(dy_n - y_{n+1} dx) = D_x I dx + \Theta \end{aligned} \tag{2.3}$$

where Θ is a contact form. From (2.2, 1.4) we get

$$\Psi^*(D_x I dx) = \Psi^*(dI - \Theta) = \Psi^*(dI) - \Psi^*(\Theta) = d\Psi^*(I) - \Psi^*(\Theta) = dI - \Psi^*(\Theta) = I_x dx + \Theta - \Psi^*(\Theta)$$

and $\Psi^*(\Theta)$ is a contact form since Ψ is a contact transformation.

Remark 2.2. *There is another point of view for the last claim. If Γ is an orbit in the space $\mathbb{R}^{n+2}$, then the action of a Lagrangian density $\omega = Ldx$ on Γ is just the integral $\int_\Gamma \omega$ as introduced in (1.5). If $L = D_x I$ then $\int_\Gamma \omega = I(\Gamma_1) - I(\Gamma_2)$ where Γ_1, Γ_2 are the end-points of the orbit. If Ψ is a contact transformation, then the Lagrangian action is transformed into*

$$\int_{\Psi(\Gamma)} \Psi^*(\omega) = I(\Psi(\Gamma_2)) - I(\Psi(\Gamma_1)) \ .$$

However, if I is an invariant function of the transformation Ψ then $I(\Psi(\Gamma_i)) = I(\Gamma_i)$ for $i = 1, 2$ so we understand the notion of the invariance of $D_x I$ as the invariance of the corresponding Lagrangian action on orbits (see Lect. 8).

We now turn to the general complete result which characterize *all* Lagrangian densities which are invariant with respect to a given contact transformation Ψ:

Theorem 2. *If Ψ is a contact transformation and ω_0 is a Lagrangian density invariant under Ψ (e.g. $\omega_0 = D_x I_0 dx$ for some invariant function I_0 of Ψ), then ω is a Ψ invariant Lagrangian density if and only if $\omega = I\omega_0$ for some invariant function I of Ψ.*

Example: Consider the action $SO(2)$ on $\mathbb{R}^2$. We know that $I_0 = x^2 + y^2$ and $I_1 = \frac{xy_1 - y}{x + yy_1}$. We define $\omega_0 = (1/2) D_x I_0 dx = (x + yy_1)dx$. So, the most general Lagrangian which is invariant under the action of $SO(2)$ is

$$L(x, y, y_1) = (x + yy_1) F\left(x^2 + y^2, \frac{xy_1 - y}{x + yy_1}\right)$$

for some function $F(,)$.

Remark 2.3. *Since we already know that the Euclidian arc-length* $L = \sqrt{1+y_1^2}$ *is a Lagrangian invariant under* $SO(2)$ *(see Lect. 8) then the example above shows that there must be a function* F *for which*

$$(x+yy_1)F\left(x^2+y^2, \frac{xy_1-y}{x+yy_1}\right) = \sqrt{1+y_1^2}\ .$$

Can you find this F*?*