Home Exercise 1

1. Consider the vectorfields

$$\mathbf{X}_1 = x\partial_y - y\partial_x + z\partial_w - w\partial_z \ , \qquad \mathbf{X}_2 = z\partial_x - x\partial_z + w\partial_y - y\partial_w$$

   in the unite sphere $S^3 := \{x^2 + y^2 + z^2 + w^2 = 1\}$.

   a) Prove that this is an integrable system, and calculate the integrable surfaces in S^3.

   b) Find the projection of $\mathbf{X}_1, \mathbf{X}_2$ on $\mathbb{R}^3$ given by $P_*\mathbf{X}_1, P_*\mathbf{X}_2$ where $P : S^3 \to \mathbb{R}^3$ is given by $P(x, y, z, w) = (x, y, z)$. Show that this system is integrable on $\mathbb{R}^3$, and find the corresponding integral surface in $\mathbb{R}^3$

2. Prove that the set of non-singular, upper triangular $n \times n$ matrices is a Lie group, and find the corresponding Lie algebra.

3. Show that $[\mathbf{X}, \mathbf{Y}]_{(x)} = -\frac{d}{dt}\big|_{t=0} \exp(t\mathbf{X})_*(Y_{(\exp(-t\mathbf{X})\circ x)})$.

4. (*) A map $\psi : \mathbb{R}^m \to \mathbb{R}^m$ is *conformal* if for any $x \in \mathbb{R}^m$ and any vectorfield $\mathbf{X}$ on $\mathbb{R}^m$, $|\psi_*(\mathbf{X})| = \lambda(x)|\mathbf{X}|$ where λ independent of $\mathbf{X}$. Prove that if $m \geq 3$:

   (a) Any conformal v.f $\mathbf{X} = \sum \xi_i \partial_{x_i}$ must satisfy

$$\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i} = 0 \ , \quad i \neq j \ , \quad \frac{\partial \xi_i}{\partial x_i} = \mu(x) \quad , \ i = 1, \ldots m \ .$$

   (b) Show that the set of all conformal mappings on $\mathbb{R}^m$ is a finite dimensional Lie group, isomorphic to $SO(m+1)$ if $m \geq 3$, and find its generators. What about $m = 2$?

5. Prove that, for $\phi = \phi(x, y, \ldots, y_n)$ and $\mathbf{X} = \xi\partial_x + \eta\partial_y$ on $\mathbb{R}^2$, where ξ, η are functions of x, y:

$$Pr^{(n+1)}\mathbf{X}(D_x\phi) = D_x\left(Pr^{(n)}\mathbf{X}(\phi)\right) - D_x\phi \cdot D_x\xi \ .$$

6. Use this to prove Theorem 1 in Lec. note # 4.