Home Exercise 2

1. i) Determine the (8th parameter) transformation group of the equation $y^{''} = 0$. See Lecture 5, p.2 for the corresponding vector fields $\mathbf{X}_1, \ldots \mathbf{X}_8$.

   ii) Prove that $y^{(n)} = 0$ have $n+4$ dimensional point symmetry for $n > 2$.

   iii) Prove that there are at most $n+4$ dimensional point symmetry for $n-th$ order ODE if $n > 2$. Why does it differ from $n = 2$?

   iv) Prove that $y^{''} = xy + e^{y^{'}} + e^{-y^{'}}$ have no point symmetry whatsoever!

2. Find the point symmetries of the equations below and use them to reduce the order. Whenever possible, give the general solutions.

   i) $3yy^{''} = 5y^{'2}$.
   ii) $4y^2y^{'''} - 18yy^{'}y^{''} + 15y^{'3} = 0$.
   iii) $y^{'''}(1 + y^{'2}) = (3y^{'} + a)y^{''2}$.
   iv) $y^{''} = 2(xy^{'} - y)/x^2$.

3. (lecture 10)

   i) Find the generators of the point symmetries of the PDE $u_t = u_{xx} + u_x^2$. Compare it to the symmetries of the heat equation (lecture 10).

   ii) The non-linear heat equation is $u_t = (h(u)u_x)_x$. Find relations between the point-symmetries of this equation and the function h.

   iii) Show that the equation $u_t = u_{xxx} + uu_x$ has a symmetry group of 4 generators. Find these generators and the transformations induced by them.

4. (lecture 5-6)

   Find an action on $\mathbb{R}^2$ which realizes the group generated by

   $$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_3 \ , \quad [\mathbf{X}_2, \mathbf{X}_3] = \mathbf{X}_1 \ , \quad [\mathbf{X}_3, \mathbf{X}_1] = \mathbf{X}_2$$

   Hint: use the generators of $SO(3)$ on $\mathbb{R}^3$: $x\partial_y - y\partial_x$, $x\partial_z - z\partial_x$, $z\partial_y - y\partial_z$.

5. (Lect. 9): Prove that any function $H = H(x, y, \tilde{x}, \tilde{y})$ induces a contact transformation, provided $\tilde{x}$, $\tilde{y}$, $\tilde{y}_1$ can be factored out from

   $$H = 0 \ , \quad H_x + y_1 H_y = 0, \quad H_{\tilde{x}} + \tilde{y}_i H_{\tilde{y}_1} = 0 \ .$$

   Prove that a v-f $\mathbf{X}$ generates a contact transformation if

   $$\mathbf{X} = \xi(x, y, y_1)\partial_x + \eta(x, y, y_1)\partial_y + \eta^{(1)}(x, y, y_1)\partial_{y_1}$$

   where

   $$\xi = \frac{\partial \Omega}{\partial y_1} \ , \quad \eta = y_1 \frac{\partial \Omega}{\partial y_1} - \Omega \ , \quad \eta^{(1)} = -\frac{\partial \Omega}{\partial x} - y_1 \frac{\partial \Omega}{\partial y}$$

   for some function $\Omega(x, y, y_1)$.