Kloosterman sums and Fourier coefficients

This note contains the slides of my lecture at Haifa, with some additional remarks.

Roberto Miatello and I have worked for years on the sum formula in various generalizations. At the miniconference, we both lectured on the sum formula. I have stressed the spectral side.

Spectral theory on the upper half plane
In $L^2\left(\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}\right)$ there is an orthonormal system u_0, u_1, u_2, ... with $u_0(z) = \sqrt{\frac{3}{\pi}}$, and for $j \geq 1$ *Maass cusp forms*

$$u_j(x+iy) = \sum_{n\neq 0} c_n(j) e^{2\pi i n x} W_{0,\nu_j}(4\pi|n|y).$$

$\nu_j \in i\mathbb{R}$ for $j \geq 1$. Ordered such that $j \mapsto \frac{1}{4} - \nu_j^2$ is increasing.
This system is not an orthonormal basis. The *Eisenstein series* span the orthogonal complement. These Maass forms depend on a complex parameter ν, and have a Fourier expansion with a "constant term":

$$\begin{aligned} E_\nu(x+iy) &= y^{\frac{1}{2}+\nu} + c_{0,0}(\nu) y^{\frac{1}{2}-\nu} \\ &\quad + \sum_{n\neq 0} c_{0,|N|}(\nu) e^{2\pi i n x} W_{0,\nu}(4\pi|n|y). \end{aligned}$$

The $W_{\cdot,\cdot}$ denotes the quickly decreasing Whittaker function, expressible in the K-Bessel function: $W_{0,\nu}(y) = \sqrt{\frac{y}{\pi}} K_\nu(y/2)$.

Sum formula

Kuznetsov, 1977, 1981; B. 1978

For suitably decreasing even holomorphic functions f, and $n, m \in \mathbb{Z} \setminus \{0\}$:

$$\sum_{j\geq 1} f(\nu_j) c_n(j) \overline{c_m(j)} + \frac{1}{4\pi i} \int_{\mathrm{Re}\nu=0} f(\nu) c_{0,n}(\nu) \overline{c_{0,m}(\nu)}\, d\nu$$
$$= \frac{1}{2\sqrt{|nm|}} \sum_{c=1}^{\infty} \frac{S(m,n;c)}{c} \tilde{f}\left(\frac{4\pi\sqrt{|nm|}}{c}\right) - \frac{\delta_{m,n}}{2\pi m} \frac{1}{2\pi i} \int_{\mathrm{Re}\nu=0} f(\nu) 2\nu \sin \pi\nu \, d\nu.$$

If $nm > 0$:

$$\tilde{f}(t) = \frac{1}{2\pi i} \int_{\mathrm{Re}\nu=0} f(\nu) J_{2\nu}(t) 2\nu \, d\nu.$$

The idea of the sum formula is that although the individual Maass cusp forms are not explicitly known, we can say much concerning their average.

In the left hand side of the sum formula, we see spectral data. The first term contains products of Fourier coefficients of Maass cusp forms. The second term contains Fourier coefficients of Eisenstein series. Since it corresponds to the continuous spectrum, this term is given by an integral.

In the right hand side, we see a sum of Kloosterman sums.

$$S(m,n;c) = \sum_{a \bmod c}^{*} e^{2\pi i(na+md)/c}, \qquad ad = 1 \bmod c.$$

The last term is simpler. It occurs only if n and m are equal.

In the case that n and m have the same sign, the formula for the transformation is as on the transparency. Otherwise, an I-Bessel function is used.

Kloosterman sums and Fourier coefficients of automorphic forms had been connected before. Petersson gave a formula for Fourier coefficients of one Poincaré series in terms of Kloosterman sums. As his Poincaré series converges there is no need to average over many cusp forms. Selberg also had used the relation between automorphic forms and Kloosterman sums to get a bound for the lowest cuspidal eigenvalue.

One should also note the similarity with the Selberg trace formula.

Weighted sum of eigenvalues
B. 1978

$$\sum_{j=1}^{N} \frac{|c_n(j)|^2}{\cos \pi \nu_j} \sim \frac{3}{\pi^2 |n|} N \quad (N \to \infty).$$

Non-trivial bound for sums of Kloosterman sums
Kuznetsov, 1980

$$\sum_{c=1}^{X} \frac{S(m,n;c)}{c} \ll_{m,n} X^{\frac{1}{6}} (\log X)^{\frac{1}{3}} \quad (X \to \infty).$$

(More on sums of Kloosterman sums in the lecture by Roberto Miatello.)

To obtain the weighted average of Fourier coefficients, take a Gaussian kernel as the test function f.

The latter result requires a bit more work. One has to understand the Bessel transformation. Actually, one can get quite far using only the behavior of $J_{2\nu}(t)$ as t approaches zero. Roberto Miatello will discuss that in his lecture.

Bessel transformation
Under suitable conditions on φ or on f:

$$\begin{aligned} f(\nu) &= \int_0^\infty \varphi(t) \frac{J_{-2\nu}(t) - J_{2\nu}(t)}{\sin \pi\nu} \frac{dt}{t}, \\ \varphi(t) &= \frac{i}{2} \int_{\mathrm{Re}\nu=0} f(\nu) \frac{J_{-2\nu}(t) - J_{2\nu}(t)}{\cos \pi\nu} \nu \, d\nu \\ &\quad + \sum_{b \geq 2,\, b \equiv 0 \bmod 2} (-1)^{\frac{b}{2}} (b-1) \\ &\qquad \cdot f\left(\frac{b-1}{2}\right) J_{b-1}(t). \end{aligned}$$

Note that $\frac{\nu}{\cos \pi\nu} = \frac{1}{\sin \pi\nu} \cdot \nu \tan \pi\nu$, where $\nu \tan \pi\nu$ is the density of the Plancherel measure on $i\mathbb{R}$.

In the expression for φ, we cannot restrict ourselves to values of f on the imaginary axis. In representational terms: we need one half of the discrete series besides the unitary principal series. So in a sense, one is forced by the Bessel transformation to look further than the Maass forms. In a natural version of the sum formula also the holomorphic cusp forms occur. Actually, it is natural to consider not individual modular forms, but the automorphic representation that they generate.

Automorphic representation
Let $G = \mathrm{PSL}_2(\mathbb{R}) \supset \Gamma = \mathrm{PSL}_2(\mathbb{Z})$. Each modular cusp form u generates an irreducible subspace of $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$. Instead of summing over an orthonormal system of cusp forms, it is more natural to sum over an orthogonal system $\{V_\varpi\}$ of irreducible subspaces of $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$.
Let $n \neq 0$. Each unitary irreducible representation V is isomorphic to a unique Whittaker model $W^n(V)$, consisting of functions on G that satisfy $F\left(\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right) g\right) = e^{2\pi i n x} F(g)$. Taking the Fourier coefficient determines an intertwining operator $F_n : V_\varpi \to W^n(\varpi)$. After normalization, this gives coefficients $c_n(\varpi)$, which determine the Fourier coefficients mentioned earlier.

In this way, we arrive at another formulation of the sum formula:

Sum formula
For f and φ related by the Bessel transformation, with suitable growth behavior, and for all $n, m \neq 0$:

$$\begin{aligned}
&\sum_{\varpi} \overline{c_m(\varpi)} c_n(\varpi) f(\nu_\varpi) \\
&\quad + \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{d_m(ir)} d_n(ir) f(ir)\, dr \\
&= 2 \sum_{c=1}^{\infty} \frac{S(m,n;c)}{c} \varphi\left(\frac{4\pi\sqrt{|mn|}}{c}\right) \\
&\quad + \frac{\delta_{m,n}}{\pi} \left(i \int_{\mathrm{Re}\nu = 0} f(\nu) \nu \tan \pi\nu \, d\nu \right. \\
&\qquad \left. + \sum_{b \geq 2,\, b \equiv 0(2)} (b-1) f\left(\frac{b-1}{2}\right) \right)
\end{aligned}$$

The first sum is over an orthogonal system of cuspidal automorphic representations. The integral in the second term corresponds to the continuous part of the spectrum. The terms on the right we have seen before. The delta term is given by the Plancherel measure.

Proofs

1. Scalar product of two Poincaré series

For function f on G satisfying $f\left(\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right) g\right) = e^{2\pi i n x} f(g)$ and suitable growth conditions,

$$P_{f,n}(g) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f(\gamma g)$$

determines element of $L^2(\Gamma \backslash G)$.

Each $P_{f,n}$ can be written according to the decomposition of $L^2(\Gamma \backslash G)$ as sum and integral of irreducible subspaces. This gives a description of the scalar product $(P_{f,n}, P_{f1,m})$.

This scalar product can also be computed by inserting the sum defining $P_{f1,m}$ and interchanging the order of integration and summation.

The resulting equality is the basis of the sum formula.

A different flavor has the proof by Cogdell and Piatetski-Shapiro:

2. Fourier coefficient of one Poincaré series

Cogdell and *Piatetski-Shapiro* have based a proof on the computation of the Fourier term of order m of the Poincaré series $P_{f,n}$. This can be computed in two different ways, yielding an equality that is formulated as an equality for distributions with variable f.

The spectral side of this equality is formulated in terms of the Kirillov model of the irreducible representations occurring in it. The matrix $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ acts as convolution with a certain distribution, called the *Bessel function* of the representation. It is used to describe the spectral terms in the equality.

In this way, one arrives first at a version of the sum formula with the independent test function in the sum of Kloosterman sums.

Some generalizations
Cofinite discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$ and their coverings inside the universal covering group of $\mathrm{PSL}_2(\mathbb{R})$ (Proskurin 1979, B. 1981)
Sum formula for $\Gamma\backslash\mathfrak{H}$ with Fourier coefficients along closed geodesics and power series coefficients at points of $\mathfrak{H}$ (Good, 1984)
Cofinite discrete subgroups of Lie groups of real rank one, and products of such groups (Miatello-Wallach, 1990)
This includes the case of discrete subgroups of $\mathrm{PSL}_2(\mathbb{C})$ (B.-Motohashi, 2003; Lokvenec, 2004)
Sum formula for PSL_2 over a totally real number field (B.-Miatello-Pacharoni, 2001)
Adelic version (Venkatesh, 2004)

As far as I know, the Bessel transformation has been inverted only for groups infinitesimally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$.

Bessel inversion for $\mathrm{SL}_2(\mathbb{C})$
The groups infinitesimally isomorphic to $\mathrm{PSL}_2(\mathbb{R})$ or $\mathrm{PSL}_2(\mathbb{C})$ are the sole ones for which the inversion of the Bessel transformation is known. For $\mathrm{PSL}_2(\mathbb{C})$, and $(\nu,p)\mapsto f(\nu,p)$ and $\varphi:\mathbb{C}^*\to\mathbb{C}$ suitable:

$$\begin{aligned}
\varphi(u) &= \frac{1}{2\pi i}\sum_{p\in\mathbb{Z}}\int_{\in\mu=0}\mathcal{K}^*_{\nu,p}(u)h(\nu,p)\\
&\qquad\cdot(p^2-\nu^2)\,d\nu,\\
f(\nu,p) &= \frac{\pi}{2}\int_{\mathbb{C}^*}\varphi(u)\mathcal{K}^*_{\nu,p}(u)\,d^*_{\mathbb{C}}u,\\
\mathcal{K}^*_{\nu,p}(u) &= \\
&\frac{\left|\frac{u}{2}\right|^{-2\nu}\left(\frac{iu}{|u|}\right)^{2p}\mathcal{J}^*_{-\nu,-p}(u)-\left|\frac{u}{2}\right|^{2\nu}\left(\frac{iu}{|u|}\right)^{-2p}\mathcal{J}^*_{\nu,p}(u)}{\sin\pi(\nu-p)},\\
\mathcal{J}^*_{\nu,p}(u) &= J^*_{\nu-p}(u)J^*_{\nu+p}(\bar{u}),\\
J^*_\mu(u) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(\mu+n+1)}\left(\frac{u}{2}\right)^{2m}.
\end{aligned}$$

PSL_2 **over totally real number field** F

$\Gamma = \Gamma_0(I)$, $I \neq \{0\}$ ideal in ring of integers $\mathcal{O}$ of F is discrete subgroup of $\prod_{j=1}^d \mathrm{PSL}_2(\mathbb{R})$, where $d = [F : \mathbb{Q}]$.

Test function $f(\nu) = \prod_{j=1}^d f_j(\nu_j)$, f_j holomorphic on $|\mathrm{Re}\,\nu| \leq \tau \in \left(\frac{1}{4}, \frac{1}{2}\right)$ and also defined on $\frac{1}{2} + \mathbb{Z}$, with suitable decay.

Fourier term orders $r, r' \in \mathcal{O}' \setminus \{0\}$.

$$\begin{aligned}
&\sum_{\varpi} \overline{c^r(\varpi)} c^{r'}(\varpi) f(\nu_\varpi) \\
&\quad + \sum_{\kappa} c_\kappa \sum_{\mu \in \Lambda_\kappa} \int_{-\infty}^{\infty} \overline{D^r(\kappa; iy, i\mu)} \\
&\quad \cdot D^{r'}(\kappa; iy + i\mu) f(iy + i\mu)\, dy \\
= \; & \frac{\alpha(r, r')\sqrt{|D_F|}}{(2\pi)^d} \int f(\nu)\, d\mathrm{Pl}(\nu) \\
&\quad + \sum_{c \in I \setminus \{0\}} \frac{S(r', r; c)}{|N(c)|} Bf\left(4\pi\sqrt{|rr'|}/c\right).
\end{aligned}$$

For SL_2 over a number field, and the discrete subgroup $\Gamma_0(I)$, the Fourier term orders r and r' are elements of $\mathcal{O}'$, the inverse fractional ideal of the different of the field F over $\mathbb{Q}$.

Note that the test functions have product form. The Bessel transformation is computed place by place. If r and r' have the same sign at an archimedean place, we use the formulas mentioned above. At places with an opposite sign, I-Bessel functions are used.

The spectral parameter ν_ϖ is now a vector of dimension d.

On the geometric side, α replaces δ. It is equal to 2 if r'/r is the square of a unit; it is zero otherwise.

Weighted density
For a totally real number field F:
$\lambda_{\varpi,j}$ eigenvalue of Casimir operator at real place j
$r \in \mathcal{O}' \setminus \{0\}$ Fourier term order
Choose a sequence (B_n) of boxes

$$B_n = \prod_j \left[a_j^{(n)}, b_j^{(n)}\right]$$

in $\mathbb{R}^d$, with the restriction: $a_j^{(n)}$ and $b_j^{(n)}$ not of the form $\frac{b}{2}\left(1-\frac{b}{2}\right)$ with $b \geq 2$ even
Define

$$N^r(B_n) = \sum_{\varpi,\, \lambda_\varpi \in B_b} |c^r(\varpi)|^2$$

Additional requirements: There is $\varepsilon > 0$ such that for all j and all n:

$$b_j^{(n)} - a_j^{(n)} > \varepsilon \left(\sqrt{|a_j^{(n)}|} + \sqrt{|b_j^{(n)}|}\right),$$

and

$$\lim_{n\to\infty} \max_j \max\left(\left|a_j^{(n)}\right|, \left|b_j^{(n)}\right|\right) = \infty.$$

Then

$$N^r\left(B_n\right) \sim \frac{2\sqrt{|D_F|}}{(2\pi)^d} Pl\left(B_n\right),$$

as $n \to \infty$, with the Plancherel measure:

$$Pl\left(B_n\right) = \prod_j Pl_j\left([a_j^{(n)}, b_j^{(n)}]\right),$$

$$Pl\left(B_n\right) = \int_{[a_j^{(n)}, b_j^{(n)}]\cap[1/4,\infty)} \tanh\left(\pi\sqrt{y-1/4}\right)\, dy + \sum_{\substack{b \geq 2,\ b \equiv 0(2) \\ a_j^{(n)} < \frac{b}{2}\left(1-\frac{b}{2}\right) < b_j^{(n)}}} (b-1).$$

Here we have not used the spectral parameter ν, but the eigenvalue $\lambda = \frac{1}{4} - \nu^2$.

These density results by Roberto Miatello and me are a bit stronger than those published a few years ago. We work on strengthening them.

Consequences

1) Keep $[a_j, b_j]$ fixed for all $j \neq j_0$, and let $[a_{j_0}^{(n)}, b_{j_0}^{(n)}]$ grow. The result shows that there are infinitely many ϖ with $a_j \leq \lambda_{\varpi,j} \leq b_j$ for $j \neq j_0$, provided $\prod_{j \neq j_0} [a_j, b_j]$ has positive Plancherel measure.

2) Take $a_j^{(n)} = n$ and $b_j^{(n)} = n + \sqrt{n}$ for all j. Then

$$N^r(B_n) \sim 2\sqrt{|D_F|} \left(\frac{\sqrt{n}}{2\pi} \right)^d$$

as $n \to \infty$.

In a family of boxes with the Plancherel measure tending to ∞, the eigenvalue count is proportional to the Plancherel measure. So there are infinitely many automorphic representations for which all but one of the eigenvalue coordinates are in a fixed interval with positive Plancherel measure.

If we prescribe at one coordinate an interval of exceptional eigenvalues, the Plancherel measure of the boxes is zero. So exceptional eigenvalues are rare.

In the second application we let all coordinates go off to ∞. The number of eigenvalue vectors in this family of boxes increases. Formulated in terms of the spectral parameters ν, the area of these moving boxes stays finite.

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