Work with M. Baruch

Let E be an elliptic curve over $\mathbf{Q}$.

Define $L_E(s, \chi) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \chi(n)$.

Question: Compute the values $L_E(1, \chi_D)$ for all fundamental discriminants D.

Conjecture 1 *(Conrey,Keating,Rubenstein,Snaith)*

$$\sum_{|D|<T, L_E(1,\chi_D)=0} 1 \sim c_E T^{3/4} (\log T)^{-5/8}$$

Gross: for f weight 2 level N a prime, compute $L(f, D, 1)$ for $D < 0$.
Construct a weight 3/2 form, then use Kohnen-Zagier formula.

Q: quaternion algebra ramified at ∞ and N
R: a maximal order of Q

I_i, $i = 1, \ldots, n$: representatives of right ideal classes of R
R_i, $i = 1, \ldots, n$: left orders of I_i
$2w_i$: the number of units in R_i^*

$M_{ij} = I_i I_j^{-1}$, $q = e^{2\pi i z}$

$f_{ij}(z) = \frac{1}{2w_j} \sum_{b \in M_{ij}} q^{Nb/NM_{ij}} = \frac{1}{2} \sum_{m \geq 0} B_{ij}(m) q^n.$

$B(m) = (B_{ij}(m))_{1 \leq i,j \leq n}$: Brandt matrices

$S(Q)$: functions on the ideal classes of R

e_i: basis with $e_i(I_j) = \delta_{ij}$

$< e_i, e_j >= \frac{1}{w_i} \delta_{ij}$: Height pairing

$e_{f_1}, \ldots, e_{f_n}$: eigenvectors of $B(m)^t$, corresponds to weight 2 forms f_i of level N

$S_i = \mathbf{Z} + 2R_i$: suborder of index 8 in R_i

S_i^0: trace 0 elements in S_i

$g_i(z) = \frac{1}{2}\sum_{b\in S_i^0} q^{Nb}$: theta series of weight 3/2

$G(z) = \sum_{i=1}^n g_i(z)e_i$: generating function

$g(z) =< G(z), e_f >= \sum_{m>0} c(m)q^m$: the desired weight 3/2 form

Theorem 1 *(Kohnen-Zagier)*

Let $w \in \{\pm 1\}$ be the eigenvalue of Atkin-Lerner involution acting on $f(z)$, let D be a fundamental discriminant with $D < 0$ and $(\frac{D}{N}) = w$, then

$$\frac{|c(|D|)|^2}{< g, g >} = \frac{2}{\pi}|D|^{1/2}\frac{L(f, D, 1)}{< f, f >}.$$

Let $N = p_1 \dots p_l$ be the product of distinct odd prime. Let f be of weight $2k$ and level N. Let S_N be the set of all prime divisors of N.

Theorem 2 *(Baruch, Mao)*

Assoicated to $f(z)$ is a set of weight $k+1/2$ forms $\{g_S(z) | S \subset S_N\}$. For the fundamental discriminants D satisfying $(\frac{D}{p}) = -w_p$ if and only if $p \in S$, we have: if $(-1)^{|S|+k} \neq sgn(D)$, then $L(f, D, k) = 0$; if $(-1)^{|S|+k} = sgn(D)$, then

$$\frac{|c(|D|)|^2}{< g_S, g_S >} = \frac{L(f, D, k)}{< f, f >} |D|^{k-1/2} \frac{(k-1)!}{\pi^k} 2^{l-t} \prod_{p \in S} \frac{p}{p+1}.$$

t: number of primes dividing both D and N

N prime: $g_+(z)$ for $D > 0$, $g_-(z)$ for $D < 0$

Level of $g_-(z)$: $4N$, Level of $g_+(z)$: $4N^2$

Construction of $g_\pm(z)$: use the Waldspurger correspondence between Q and $\overline{SL}_2$

Example: $N = 11$, unique elliptic curve over $\mathbf{Q}$ of conductor 11

$g_{\pm}(z) = g_{1,\pm}(z) - g_{2,\pm}(z)$

$g_{1,-} = \frac{1}{2}\sum_{a \equiv b \bmod 2} q^{a^2+11b^2+11c^2}$

$g_{2,-} = \frac{1}{2}\sum_{a \equiv b \bmod 3, b \equiv c \bmod 2} q^{(a^2+11b^2+33c^2)/3}$

Define a function $\mu : \mathbf{Z}/11 \times \mathbf{Z}/11 \mapsto \{1, -1, 0\}$:

$(a, b) \in \mathbf{Z}/11 \times \mathbf{Z}/11$, $c = a + bi$:

$c\bar{c} = 0$ or is not a square in $\mathbf{Z}/11$: $\mu(a, b) = 0$;

$c\bar{c} = 1$: there is $r = s + ti$ such that $c = r/\bar{r}$,

$\mu(a, b) = 1$: $r\bar{r}$ is a square, $\mu(a, b) = -1$ if not

$\mu(a, b) = \mu(ad, bd)$ for $d \in F_{11}^*$

$$g_{1,+} = \frac{1}{2} \sum_{c \equiv b \text{ mod } 2} \mu(a,b) q^{a^2+b^2+11c^2},$$

$$g_{2,+} = \frac{1}{2} \sum_{\substack{c \equiv -2a \text{ mod } 9 \\ b \equiv c \text{ mod } 2}} \mu(4a,b) q^{(a^2+11c^2)/9+b^2}.$$

$g_+ = q^1 + 3q^4 - 5q^5 - 2q^9 - 5q^{12} + 4q^{16} - 5q^{20} - 6q^{36} + 5q^{37} - 10q^{48} + 3q^{49} + 10q^{56} - 5q^{60} + 2q^{64} + 15q^{69} - q^{81} - 5q^{89} - 5q^{92} + 5q^{93} + \dots$

$3g_{1,+}(q) + 2g_{2,+}(q) = 3\sum_{n=1}^{\infty} (\frac{n}{11}) q^{n^2}$

Corollary 1 *Let $D > 0$ be a fundamental discriminant. $L_E(1, \chi_D) = 0$ if and only if*

$$\sum_{\substack{c \equiv b \text{ mod } 2 \\ a^2+b^2+11c^2 = D}} \mu(a,b) = 0.$$

Tunnell: Congruence number problem

π: irreducible cuspidal representation of PGL_2
$\varphi = \otimes_{v \in S} \varphi_v \otimes_{v \notin S} \varphi_{0,v}$: a vector in π

$W_\varphi(g) = \int_{\mathbf{A}/F} \varphi(n(u)g)\psi(-u)du$
L_v: local Whittaker functional

$W_\varphi(e) = c_1(\pi, S, \psi, \{L_v\}) \prod_{v \in S} L_v(\varphi_v)$

$(v, v') = \int_{F_v^*} L_v(\pi_v(\underline{a})v)\overline{L_v(\pi_v(\underline{a})v')}\frac{da}{|a|_v}$

$\|\varphi\| = c_2(\pi, S, \psi, \{L_v\}) \prod_{v \in S} \|\varphi_v\|$

$d_\pi(S, \psi) = |c_1(\pi, S, \psi, \{L_v\})/c_2(\pi, S, \psi, \{L_v\})|$
independent of L_v

$d_{\tilde{\pi}}(S, \psi^D)$ is defined similarly for $\tilde{\pi}$ of $\overline{SL}_2$

Theorem 3 *Let $\tilde{\pi}_D = \Theta(\pi, \psi^D)$. Let S be a large enough finite set of places.*

$$|d_\pi(S, \psi)|^2 L^S(\pi, 1/2) = |d_{\tilde{\pi}_D}(S, \psi^D)|^2$$

Apply to $\tilde{\pi}^D = \Theta(\pi \otimes \chi_D, \psi^D)$

$P_{0,v}$: set of discrete series represenations of $PGL_2(F_v)$.

$(\frac{D}{\pi_v}) = \chi_D(-1)\epsilon(\pi_v, 1/2)/\epsilon(\pi_v \otimes \chi_D, 1/2) \in \{\pm 1\}$

$F_v^* = F_v^+ \cup F_v^-$ where

$F_v^\pm(\pi_v) = \{D \in F_v^* | \, (\frac{D}{\pi_v}) = \pm 1\}$

Theorem 4 *(Waldspurger)*

When $\pi_v \notin P_{0,v}$, $F_v^+ = F_v^*$ and $\tilde{\pi}_v^D = \Theta(\pi_v, \psi_v)$.

When $\pi_v \in P_{0,v}$, there are two representations $\tilde{\pi}_v^+$ and $\tilde{\pi}_v^-$, such that $\tilde{\pi}^D = \tilde{\pi}_v^+ = \Theta(\pi_v, \psi_v)$ when $D \in F_v^+$, and $\tilde{\pi}_v^D = \tilde{\pi}_v^-$ when $D \in F_v^-$.

$\tilde{A}_{00}$: cuspidal representations of $\overline{SL}_2(\mathbf{A})$ excluding the theta representations.

$\bar{A}_{00}$: near equivalent class in $\tilde{A}_{00}$

$A_{0,i}$: cuspidal representations on $PGL_2(\mathbf{A})$, such that there is D with $L(\pi \otimes \chi_D, 1/2) \neq 0$.

$\Sigma = \Sigma(\pi)$: set of places v where $\pi_v \in P_{0,v}$.

$\epsilon(D, \pi) = (\frac{D_v}{\pi_v})_{v \in \Sigma}$.

$F^\epsilon(\pi)$: set of $D \in F^*$ with $\epsilon(D, \pi) = \epsilon$

Theorem 5 *(Waldspurger)*

There is a bijection between $\bar{A}_{00}$ and $A_{0,i}$: $\pi = S_\psi(\tilde{\pi})$; the near equivalence class of $\tilde{\pi}$ consists of all the nonzero $\tilde{\pi}^D$'s.

If $\prod_{v \in \Sigma} \epsilon_v \neq \epsilon(\pi, 1/2)$, then $\tilde{\pi}^D = 0$ for all $D \in F^\epsilon(\pi)$.

If $\prod_{v \in \Sigma} \epsilon_v = \epsilon(\pi, 1/2)$, then there is a unique $\tilde{\pi}^\epsilon$ such that for $D \in F^\epsilon(\pi)$, $\tilde{\pi}^D = \tilde{\pi}^\epsilon$ when $L(\pi \otimes \chi_D, 1/2) \neq 0$ and $\tilde{\pi}^D = 0$ otherwise.

Let D be a square free integer: D_v a unit or prime (D_2 could be a square or cube of prime)

Theorem 6 *Fix $\tilde{\pi}$. Let $\pi = S_\psi(\tilde{\pi})$ and $\tilde{\pi} = \tilde{\pi}^{\epsilon_0}$. If $D \in F^{\epsilon_0}(\pi)$ then*

$$|d_{\tilde{\pi}}(S, \psi^D)|^2 = |d_\pi(S, \psi)|^2 L^S(\pi \otimes \chi_D, 1/2) \prod_{v \in S} |D|_v^{-1}.$$

If $D \notin F^{\epsilon_0}(\pi)$, $d_{\tilde{\pi}}(S, \psi^D) = 0$.

Fix π. For a given ϵ and $D \in F^\epsilon(\pi)$,

$$|d_{\tilde{\pi}^\epsilon}(S, \psi^D)|^2 = |d_\pi(S, \psi)|^2 L^S(\pi \otimes \chi_D, 1/2) \prod_{v \in S} |D|_v^{-1}.$$

Application: Define

$$d_{\tilde{\pi}}(\tilde{\varphi}, S_\infty, \psi^D) = \frac{|W_{\tilde{\varphi}}^D(e)|}{\|\tilde{\varphi}\|} \prod_{v \in S_\infty} \frac{\|\tilde{\varphi}_v\|}{|L_v^D(\tilde{\varphi}_v)|}$$

Conjecture 2 *(Ramanujuan conjecture). Let $\tilde{\varphi}$ be a vector in $\tilde{\pi} \in \tilde{A}_{00}$. For D a square free integer in F^*, as $|D| \mapsto \infty$, for all $\alpha > 0$*

$$|d_{\tilde{\pi}}(\tilde{\varphi}, S_\infty, \psi^D)| << _{\tilde{\pi}, \tilde{\varphi}, \alpha} |D|_{S_\infty}^{\alpha - 1/2}$$

Lindelőf hypothesis: for D square free integer, as $|D| \mapsto \infty$, for all $\beta > 0$

$$|L^{S_\infty}(\pi \otimes \chi_D, 1/2)| <<_{\pi,\beta} |D|_{S_\infty}^{\beta}$$

Theorem 7 *RC for $\alpha \leftrightarrow$ LH for $\beta = 2\alpha$.*

Sketch of proof of Theorem 3:

Let $B(\varphi) = \int_{F^*\backslash \mathbf{A}^*} \varphi(\underline{a}) d^*a$ be a linear form on π, define a distribution

$$I_\pi(f) = (\pi(f)B, W), I_{\pi_v}(f_v) = (\pi_v(f_v)B_v, W_v)$$

Then $I_\pi(f) = |d_\pi(S,\psi)|^2 L^S(\pi, 1/2) \prod_{v\in S} I_{\pi_v}(f_v)$.

Similarly let

$$I_{\tilde{\pi}}(f') = (\tilde{\pi}(f')W^D, W^D), I_{\tilde{\pi}_v}(f'_v) = (\tilde{\pi}_v(f'_v)W_v^D, W_v^D)$$

Then $I_{\tilde{\pi}}(f') = |d_{\tilde{\pi}}(S, \psi^D)|^2 \prod_{v\in S} I_{\tilde{\pi}_v}(f'_v)$

Relative trace formula: when f and f' match (with matching orbital integrals)

$$I_\pi(f) = I_{\tilde{\pi}}(f')$$

Local analogue gives the theorem.

Generalization: with Lapid, Rallis

G and G': a dual pair in M
ω_ψ: minima representation of M

H be a closed subgroup of G, χ its character
$\omega_\psi(H,\chi)$: space spanned by $\omega_\psi(h)v - \chi(h)v$
$\omega_\psi[H,\chi] = \omega_\psi/\omega_\psi(H,\chi)$ the Jacquet module

N': the maximal unipotent subgroup of G'
χ_ψ: a nondegenerate character of N'.
Assume:

$$\omega_\psi[H,\chi^{-1}] \cong ind_{N'}^{G'}\chi_\psi, \omega_\psi[N',\chi_\psi^{-1}] \cong ind_{H'}^{G}\chi'$$

Then there should be a RTF giving: ($\tilde{\pi}$ of G')

$$(\tilde{\pi}(f')W,W) = I_{\tilde{\pi}}(f') = I_\pi(f) = (\pi(f)L_1,L_2).$$

L_1, L_2 are (H,χ) and (H',χ') equivariant linear forms on the space of π.

RTF checked in many cases (with Rallis).
Sketch of proof:
$I_{\tilde{\pi}}(f') = I_\pi(f) = L_2 \otimes W(\Phi)$, $\Phi \in \omega_\psi$
Lapid: direct proof without going through RTF.