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1. Gelfand pairs and Rankin-Selberg type identities

1.1. **Preliminaries.**

1.1.1. *Automorphic representations.* Let G be a real group and $\Gamma \subset G$ a lattice. To set the notations, we assume for simplicity that the automorphic space $X = \Gamma \setminus G$ is compact and is endowed with the invariant measure of mass 1. We fix (in order not to deal with multiplicities) a decomposition $L^2(X) = \sum_\kappa (\pi_\kappa^{aut}, L_\kappa^{aut})$ into irreducible unitary representations of G and denote by $V_\kappa^{aut} \subset L_\kappa^{aut}$ the space of smooth vectors. We denote by $pr_\kappa : L^2(X) \to L_\kappa$ the corresponding projection.

Let (π, L, V) be an irreducible unitary representation of G, and the corresponding space of smooth vectors. We assume that π is realized in some "explicit" model (e.g., in the space of sections of a vector bundle over a G-manifold).

We call a tuple $(\pi, L, \nu_\kappa, \pi_\kappa^{aut}, L_\kappa^{aut})$ of the corresponding representations together with a G-equivariant isometry $\nu_\kappa : L \to L_\kappa$ an automorphic representation. Note that ν_κ is determined by κ up to a constant of the absolute value one, and that $\nu_\kappa : V \to C^\infty(X)$. Where it does not cause a confusion, we will denote $(\pi_\kappa^{aut}, L_\kappa^{aut}, V_\kappa^{aut})$ by (π, L, V). We have $\sum_\kappa \nu_\kappa^{-1} \circ pr_\kappa = Id \in End(L^2(X))$.

1.1.2. *Gelfand pairs.* We will call a pair (A, B), of a group A and a subgroup B, a *strong* Gelfand pair if for any smooth irreducible representations V of A and W of B, the condition $\dim \mathrm{Mor}_B(V, W) \leq 1$ is satisfied. We always will work with the spaces of smooth vectors in unitary representations.

We will use the notion of (strong) Gelfand pairs repeatedly in the following standard situation. Let $X_A = \Gamma_A \setminus A$ be an automorphic space of A and $X_B \subset X_A$ a closed B-orbit. We choose invariant measures on X_A and on X_B. Let $V^{aut} \subset L^2(X_A)$ and $W^{aut} \subset L^2(X_B)$ be spaces of smooth vectors in two automorphic unitary irreducible representations. Denote by ν_V and ν_W the corresponding isometric imbeddings. The restriction to X_B and then projection to W^{aut} of functions in V^{aut}, together with identifications ν_V and ν_W, define B-equivariant map $T_{X_B}^{aut} : V \to W$. Assuming that (A, B) is a strong Gelfand pair, the space of such maps is at most one-dimensional. This implies that if we choose in the *models* of abstract representations V and W (which are unrelated to their automorphic realizations) a model equivariant map $T^{mod} : V \to W$ then there exists the constant of proportionality a_{X_B, ν_V, ν_W} such that $T_{X_B}^{aut} = a_{X_B, \nu_V, \nu_W} \cdot T^{mod}$. We would like to study these constants. Of course, these constants depend, among other things, on the choice of model maps. Eventually, we hope to find a way to canonically normalize norms of these maps in the adelic setting. We now explain how in certain situations one can obtain spectral identities for the coefficients a_{X_B, ν_V, ν_W}.

1.2. **The construction.** Our main observation is that we can write a Rankin-Selberg type formula once we have two *different* triples of strong Gelfand pairs inside the ambient group, and the corresponding arrangement of closed automorphic orbits.

Let $\mathcal{G}$ be a (real reductive) group and $\mathcal{F} \subset \mathcal{H}_i \subset \mathcal{G}$, $i = 1, 2$ be a collection of subgroups such that in the following commutative diagram each imbedding is a strong Gelfand pair (i.e., $(\mathcal{G}, \mathcal{H}_i)$ and $(\mathcal{H}_i, \mathcal{F})$ are strong Gelfand pairs)

$$\begin{tikzcd} & \mathcal{G} & \\ \mathcal{H}_1 \arrow[ur, hook, "j_1"] & & \mathcal{H}_2 \arrow[ul, hook', "j_2"'] \\ & \mathcal{F} \arrow[ul, hook', "i_1"] \arrow[ur, hook, "i_2"'] & \end{tikzcd} \tag{1.1}$$

We call such a collection $(\mathcal{G}, \mathcal{H}_1, \mathcal{H}_2, \mathcal{F})$ a strong Gelfand formation (or a strong Gelfand pattern).

Let $\Gamma \subset \mathcal{G}$ be a lattice and denote by $X_{\mathcal{G}} = \Gamma \setminus \mathcal{G}$ the corresponding automorphic space. Let $\mathcal{O}_i \subset X_{\mathcal{G}}$ and $\mathcal{O}_{\mathcal{F}} \subset X_{\mathcal{G}}$ be closed orbits of $\mathcal{H}_i$ and $\mathcal{F}$ respectively, satisfying the following commutative diagram of imbeddings

$$\begin{tikzcd} & X_{\mathcal{G}} & \\ \mathcal{O}_1 \arrow[ur, hook, "j'_1"] & & \mathcal{O}_2 \arrow[ul, hook', "j'_2"'] \\ & \mathcal{O}_{\mathcal{F}} \arrow[ul, hook', "i'_1"] \arrow[ur, hook, "i'_2"'] & \end{tikzcd} \tag{1.2}$$

assumed to be compatible with the diagram (1.1). We endow each orbit (as well as $X_{\mathcal{G}}$) with a measure invariant under the corresponding subgroup (for simplicity, we assume that all orbits are compact, and hence, these measures could be normalized to have mass one).

Let $\mathcal{V} \subset C^\infty(X_{\mathcal{G}})$ be the space of smooth vectors in an irreducible automorphic representation of $\mathcal{G}$. The integration over the orbit $\mathcal{O}_{\mathcal{F}} \subset X_{\mathcal{G}}$ defines an $\mathcal{F}$-invariant functional $I_{\mathcal{O}_{\mathcal{F}}} : \mathcal{V} \to \mathbb{C}$. This is our main object of study.

In general, an $\mathcal{F}$-invariant functional on V does not satisfy the uniqueness property, as $(\mathcal{G}, \mathcal{F})$ is not a Gelfand pair. Instead, we will write two different spectral expansions for $I_{\mathcal{O}_{\mathcal{F}}}$ using two intermediate groups $\mathcal{H}_1$ and $\mathcal{H}_2$.

Namely, for any $v \in \mathcal{V}$, we have two different ways to compute the value of the functional $I_{\mathcal{O}_{\mathcal{F}}}$ by restricting the function $v \in C^\infty(X_{\mathcal{G}})$ to the orbit $\mathcal{O}_1$ and then integrating over $\mathcal{O}_{\mathcal{F}}$ or, alternatively, to restricting v to $\mathcal{O}_2$ and then integrating over $\mathcal{O}_{\mathcal{F}}$. Hence we have the identity

$$\int_{\mathcal{O}_{\mathcal{F}}} res_{\mathcal{O}_1}(v) d\mu_{\mathcal{O}_{\mathcal{F}}} = I_{\mathcal{O}_{\mathcal{F}}}(v) = \int_{\mathcal{O}_{\mathcal{F}}} res_{\mathcal{O}_2}(v) d\mu_{\mathcal{O}_{\mathcal{F}}} \ .$$

The restriction $res_{\mathcal{O}_1}$ has the spectral expansion $res_{\mathcal{O}_1} = \sum_{W_j \subset L^2(\mathcal{O}_1)} pr_{W_j}(res_{\mathcal{O}_1})$ induced by the decomposition of $L^2(\mathcal{O}_1)$ into irreducible representations of $\mathcal{H}_1$ (and similarly $res_{\mathcal{O}_2} = \sum_{U_k \subset L^2(\mathcal{O}_2)} pr_{U_k}(res_{\mathcal{O}_2})$ for the group $\mathcal{H}_2$).

The integration over the orbit $\mathcal{O}_{\mathcal{F}} \subset \mathcal{O}_1$ defines an $\mathcal{F}$-invariant functional $I_{\mathcal{O}_{\mathcal{F}},j} : W_j \to \mathbb{C}$ on the smooth part W_j of each irreducible unitary representation of $\mathcal{H}_1$ appearing in the decomposition of $L^2(\mathcal{O}_1)$ (and correspondingly an $\mathcal{F}$-invariant functional $J_{\mathcal{O}_{\mathcal{F}},k} : U_k \to \mathbb{C}$ on representations U_k of $\mathcal{H}_2$). This time such functionals do satisfy the uniqueness property due to the assumption that pairs $(\mathcal{H}_i, \mathcal{F})$ are strong Gelfand pairs.

Hence we obtain two spectral decompositions for the functional $I_{\mathcal{O}_{\mathcal{F}}}$:

$$\sum_{W_j \subset L^2(\mathcal{O}_1)} I_{\mathcal{O}_{\mathcal{F}},j}\left(pr_{W_j}(res_{\mathcal{O}_1}(v))\right) = I_{\mathcal{O}_{\mathcal{F}}}(v) = \sum_{U_k \subset L^2(\mathcal{O}_2)} J_{\mathcal{O}_{\mathcal{F}},k}\left(pr_{U_k}(res_{\mathcal{O}_2}(v))\right) \tag{1.3}$$

for any $v \in \mathcal{V}$. Note that the summation on the left is over the set of irreducible representations of $\mathcal{H}_1$ occurring in $L^2(\mathcal{O}_1)$ and the summation on the right is over the set of irreducible representations of $\mathcal{H}_2$ occurring in $L^2(\mathcal{O}_2)$. As groups $\mathcal{H}_1$ and $\mathcal{H}_2$ might be quite different, the identity (1.3) is nontrivial.

The identity (1.3) is the origin of our Rankin-Selberg type identities. We show how one can transform it to a more familiar form. To this end we use the standard device of model invariant functionals. Our main observation is that the functionals $I_{\mathcal{O}_{\mathcal{F}},j}$, $J_{\mathcal{O}_{\mathcal{F}},k}$ and the maps $pr_{W_j}(res_{\mathcal{O}_1}) : \mathcal{V} \to W_j$ and $pr_{U_k}(res_{\mathcal{O}_2}) : \mathcal{V} \to U_k$ satisfy the uniqueness property due to the assumption that the pairs $(\mathcal{H}_i, \mathcal{F})$ and $(\mathcal{G}, \mathcal{H}_i)$ are strong Gelfand pairs (in fact, it is enough for $(\mathcal{H}_i, \mathcal{F})$ to be the usual Gelfand pairs).

Hence, by choosing explicit "models" $\mathcal{V}^{mod}$, W_j^{mod}, U_k^{mod} for the corresponding automorphic representations, we can construct model invariant functionals $I_j^{mod} = I_{W_j}^{mod}$, $J_k^{mod} = J_{U_k}^{mod}$ and the model equivariant maps $T_j^{mod} : \mathcal{V}^{mod} \to W_j^{mod}$ and $S_k^{mod} : \mathcal{V}^{mod} \to U_k^{mod}$. The model functionals and maps could be constructed regardless of the automorphic picture and we define them for *any* irreducible representations of $\mathcal{G}$ and $\mathcal{H}_i$. The uniqueness principle then implies the existence of coefficients of proportionality $a_j,\ b_j,\ c_k,\ d_k$ such that

$$I_{\mathcal{O}_{\mathcal{F}},j} = a_j \cdot I_j^{mod}\ , \quad pr_{W_j}(res_{\mathcal{O}_1}) = b_j \cdot T_j^{mod} \ \text{ for any } j,$$

and similarly

$$J_{\mathcal{O}_{\mathcal{F}},k} = c_k \cdot J_k^{mod}\ , \quad pr_{U_k}(res_{\mathcal{O}_2}) = d_k \cdot S_k^{mod} \ \text{ for any } k.$$

This allows us to rewrite the relation (1.3) in the form

$$\sum_{\{W_j\}} \alpha_j \cdot h_j(v) = \sum_{\{U_k\}} \beta_k \cdot g_k(v) \tag{1.4}$$

for any $v \in \mathcal{V}^{mod}$. Where we denoted by $\alpha_j = a_j b_j$, $\beta_k = c_k d_k$, $h_j(v) = I_j^{mod}(T_j^{mod}(v))$ and $g_k(v) = J_k^{mod}(S_k^{mod}(v))$.

This is what we call Rankin-Selberg type spectral identity associated to the diagram (1.2).

Remark. We note that one can associate a non-trivial spectral identity of a kind we described above to a pair of different *filtrations* of a group by subgroups forming strong Gelfand pairs. Namely, we associate a spectral identity to two filtrations $\mathcal{F} = G_0 \subset G_1 \subset \cdots \subset G_n = \mathcal{G}$ and $\mathcal{F} = H_0 \subset H_1 \subset \cdots \subset H_m = \mathcal{G}$ of subgroups in the same group $\mathcal{G}$ such that all pairs (G_{i+1}, G_i) and (H_{j+1}, H_j) are strong Gelfand pairs having the same intersection $\mathcal{F}$. One also can "twist" such an identity by a nontrivial character or an irreducible representation of the group $\mathcal{F}$.

1.2.1. *Bounds for coefficients.* The Rankin-Selberg type formulas can be used in order to obtain bounds for coefficients α_j or β_k. To this end one has to study properties of the integral transforms $h_W = I^{mod}(T_W^{mod}) : \mathcal{V}^{model} \to C(\hat{\mathcal{H}}_1)$, $v \mapsto h_W(v) = I_W^{mod}(T_W^{mod}(v))$ induced by the corresponding *model* functionals and maps (here $\hat{\mathcal{H}}_1$ is the unitary dual of $\mathcal{H}_1$ and $\mathcal{V}^{mod}$ an explicit model of the representation $\mathcal{V}$); similarly for the triple $(\mathcal{G}, \mathcal{H}_2, \mathcal{F})$. This is a problem in harmonic analysis which has nothing to do with the automorphic picture. We study the corresponding transforms, in the particular cases under the consideration in [R], and prove some instance of what might be called an "uncertainty principle" for the pair of such transforms is established. The idea behind the proof of the corresponding bounds for the coefficients α_i or β_k is quite standard (and in this context was learned by us from papers of A. Good), once we have the appropriate Rankin-Selberg type identity and the necessary information about corresponding integral transforms. Namely, we find a family of test vectors $v_T \in \mathcal{V}$, $T \geq 1$ such that when substituted in the Rankin-Selberg type identity (1.4) it will pick up the (weighted) sum of coefficients α_j for j in certain "short" interval around T (i.e., the transform $h_j(v)$ has essentially small support in $\hat{\mathcal{H}}_1$). We show then that the integral transform $g_k(v)$ of such a vector is a slowly changing function on $\hat{\mathcal{H}}_2$. This allows us to bound the right hand side in (1.4) using Cauchy-Schwartz inequality and the *mean value* bound for the coefficients β_k. The simple way to obtain these mean value bounds was explained by us in [BR2].

We note that in order to obtain bounds for the coefficients in (1.4) one needs to have a kind of positivity which is not always easy to achieve. In our examples we consider representations of the type $\mathcal{V} = V \otimes \bar{V}$ for the group $\mathcal{G} = G \times G$ and V an irreducible representation of G. For such representations the necessary positivity is automatic.

1.3. **Examples.** In [R] we implement the above strategy in two cases: for the unipotent subgroup N of $G = PGL_2(\mathbb{R})$ and a compact connected subgroup $K \subset G$ (i.e., the identity connected component of $PO(2)$). The first case corresponds to the classical unipotent Fourier coefficients and the second one corresponds to the spherical Fourier coefficients.

1. Let $\mathcal{G} = G \times G$ and $\mathcal{V} = V \otimes \bar{V}$, where V is an irreducible automorphic representation of G. We set $\mathcal{H}_2 = \Delta G \overset{j_2}{\hookrightarrow} G \times G$ and $\mathcal{H}_1 = N \times N$, $\mathcal{F} = \Delta N \overset{i_1}{\hookrightarrow} N \times N \overset{j_1}{\hookrightarrow} G \times G$ for the case of unipotent Fourier coefficients. Let ϕ be a Maass form and $V \subset C^\infty(X)$ the space of smooth vectors for the corresponding automorphic representations. Let $a_n(\phi)$ be the Fourier coefficients of the cusp form ϕ. We assume, for simplicity, that the so-called residual spectrum is trivial (i.e., the Eisenstein series $E(s,z)$ are holomorphic for $s \in (0,1)$). The identity (1.4) then amounts to the classical Rankin-Selberg formula

$$\sum_n |a_n(\phi)|^2 \hat{\alpha}(n) = \alpha(0) + \frac{1}{2\pi i} \int\limits_{Re(s)=\frac{1}{2}} D(s,\phi,\bar{\phi}) M(\alpha)(s) ds \ , \tag{1.5}$$

where $\alpha \in C^\infty(\mathbb{R})$ is an appropriate test function with the Fourier transform $\hat{\alpha}$ and the Mellin transform $M(\alpha)(s)$,

$$D(s,\phi,\bar{\phi}) = \Gamma(s,\tau) \cdot < \phi\bar{\phi}, E(s) >_{L^2(Y)} \ , \tag{1.6}$$

where $E(z,s)$ is an appropriate non-holomorphic Eisenstein series and $\Gamma(s,\tau)$ is explicitly given in terms of the Euler Γ-function and depends on a choice of model functionals (this is routine an is explained in detail in [R]). Strictly speaking, for the subgroup N the uniqueness principle is not satisfied, but the theory of the constant term of the Eisenstein series provides necessary remedy in the automorphic setting.

2. In the second example we consider let $\mathcal{G}$, $\mathcal{V}$, $\mathcal{H}_2$ be as before and $\mathcal{H}_1 = K \times K$, $\mathcal{F} = \Delta K \hookrightarrow K \times K \hookrightarrow G \times G$. This leads to an identity involving spherical Fourier coefficients $b_n(\phi)$ of Maass forms (e.g., periods with grossencharacters at a CM point). We assume for simplicity that Γ is co-compact. Let $x_0 \in X = \Gamma \setminus G$ be a point and $\mathcal{K} = x_0 \cdot K \simeq S^1$ the corresponding orbit. One have the Fourier expansion along the orbit $\mathcal{K}$ defined in the same way as the classical Fourier expansion of cusp forms (where one considers the expansion along the N-orbit $\Gamma \cap N \setminus N \subset X$). This leads to the spherical Fourier coefficients $b_n(\phi)$ for a Maass form ϕ (or for a holomorphic form; these were introduced by H. Petersson long time ago). We have then the following

Theorem 1.1. *Let $\{\phi_{\lambda_i}\}$ be an orthonormal basis of $L^2(Y)$ consisting of Maass forms. Let ϕ_τ be a fixed Maass form.*

There exists an explicit integral transform $^\sharp : C^\infty(S^1) \to C^\infty(\mathbb{C})$, $u(\theta) \mapsto u^\sharp_\tau(\lambda)$, such that for all $u \in C^\infty(S^1)$, the following relation holds

$$\sum_n |b_n(\phi_\tau)|^2 \hat{u}(n) = u(1) + \sum_{\lambda_i \neq 1} \mathcal{L}_{x_0}(\phi_{\lambda_i}) \cdot u^\sharp_\tau(\lambda_i) \ , \tag{1.7}$$

with some explicit coefficients $\mathcal{L}_{x_0}(\phi_{\lambda_i}) \in \mathbb{C}$ which are independent of u.

Here $\hat{u}(n) = \frac{1}{2\pi} \int\limits_{S^1} u(\theta) e^{-in\theta} d\theta$ and $u(1)$ is the value at $1 \in S^1$.

To us, it looks very similar to the classical Rankin-Selberg formula (1.5).

The definition of the integral transform $^\sharp$ is based on the uniqueness of invariant trilinear functionals on irreducible unitary representations of G. The main point of the relation (1.7) is that the transform $u^\sharp_\tau(\lambda_i)$ depends only on the parameters λ_i and τ, but not on the choice of Maass forms ϕ_{λ_i} and ϕ_τ.

The coefficients $\mathcal{L}_{x_0}(\phi_{\lambda_i})$ are essentially given by the product of the triple product coefficients $< \phi^2_\tau, \phi_{\lambda_i} >_{L^2(Y)}$ times the values of the Maass form ϕ_{λ_i} at the point x_0. In the special cases (squares of) both types of these coefficients are related to L-functions via results of Waldspurger, Jacquet and T. Watson. Coefficients $|b_n(\phi_\tau)|^2$ are related to special values of the L-function for the base change, twisted by a grossencharacter (Waldspurger, Jacquet).

3. We would like to note that the method described above also lies behind the proof of the subconvexity for the triple L-function given in [BR1] (but not understood at the time). In that case, $\mathcal{G} = G \times G \times G \times G$, $\mathcal{F} = \Delta G$ and $\mathcal{H}_i = G \times G$ with two different imbeddings for $i = 1$ and 2. This is related to the triple L-function via the formula of T. Watson .

Yet another intriguing example exists in the Hilbert-Blumenthal case for a quadratic extension E/F: one considers the period with respect to $GL_2(F) \subset GL_2(E)$. This leads to two Gelfand pair flirtations inside of $GL_2(E) \times GL_2(E)$ by $\mathcal{H}_1 = \Delta GL_2(E)$ and by $\mathcal{H}_2 = GL_2(F) \times GL_2(F)$ having common intersection $\mathcal{F} = \Delta GL_2(F)$. This is related to the Gross-Prasad period. In fact, there are many other strong Gelfand formations related to the Gross-Prasad period, including ones satisfying the positivity condition. It is not yet clear what analytic information would be possible to extract from the corresponding spectral identities.

References

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