A KERNEL FORMULA FOR THE ACTION OF THE WEYL ELEMENT, IN THE KIRILLOV MODEL OF $SL(2,\mathbb{C})$.

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Abstract. We give an explicit description of the action of the Wyel element on smooth functions with compact support in the Kirillov model of complementary series and non principal series irreducible representations of $SL(2,\mathbb{C})$ and $GL(2,\mathbb{C})$ generalizing a result of Motohashi. An important ingredient in the proof of the Kernel formula is a new "classical" formula for an integral involving Bessel functions.

Contents

1. Introduction

The action of the Weyl element in the Kirillov model of representations of $GL(2)$ over a local field was first studied by Gelfand, Graev and Piatetski Shapiro ([7]). The case of $GL(2,\mathbb{R})$ was further considered by Vilenkin, ([12]) Cogdell and Piatetski Shapiro ([4]) and Baruch and Mao ([2]). The kernel formula in the Kirillov model of $GL(2,\mathbb{C})$ was first studied by Motohashi ([11]). He obtained a kernel formula for K-finite vectors in the principal series representations. In this paper we extend the results of Motohashi to compactly supported functions in the Kirillov model of non-principal series representations. In certain applications of our formula, such as a recent Voronoi summation formula for Gaussian integers ([1]), it is crucial to use compactly supported functions. We believe that our technique will allow us to prove the kernel formula for compactly supported functions in the Kirillov model of principal series representations of $GL(2,\mathbb{C})$ and we will come back to it in a future publication.

The Kirillov model is a $SL(2,\mathbb{C})$ representation space of functions on $\mathbb{C}$ with a prescribed action of the Borel elements. To describe the representation it is enough

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to understand the action of the Weyl element w. We will prove that w acts as an integral trasform in the following way:

$$(w \cdot \psi)(b) = \int_{\mathbb{C}} \kappa_{p,\mu}(z,b)\psi(z)dz \quad .$$

Here ψ is a smooth function with compact support in the Kirillov model, p is an integer and μ is a complex number with $2 < Re(\mu)$. p and μ are the parameters of the irreducible representation we are considering.

The technique of the proof of the kernel formula that we use is similar to the one in [2]. However, since our integrals are over $\mathbb{C}$ and not over $\mathbb{R}$ we encounter difficulties which are not present in the real case. One difference is that our integrals are not given by integral tables but have to be computed using analytic methods. Another difference is a convergence issue which allows us to first treat only a partial set of representations and later use analytic continuation to extend the results.

The paper is organized in the following way. In section 2 we introduce our notations, give a brief explanation of the representations of $SL(2,\mathbb{C})$ that we use and state the main results. In section 3 we prove the existence of a kernel formula for a partial set of the representations and we obtain an expression of the kernel function in terms of an integral of a K-Bessel function. Section 4 deals with calculating these integrals and obtaining an explicit expression of the kernel function. In section 5, having already obtained an explicit kernel formula for a certain set of representations, we use its analytic properties in order to extend the formula for all the representations we are interested in and we give a similar formula for $GL(2,\mathbb{C})$.

2. Preliminaries and main results.

We denote the group $SL(2,\mathbb{C})$ by $\mathbf{G}$ and its Borel subgroup of upper triangular matrices by $\mathbf{B}$. (These notations will be slightly modified when we will consider the case of $GL(2,\mathbb{C})$). The characters of $\mathbf{B}$ are denoted by $\chi_{p,\mu}$ and are given by:

$$\chi_{p,\mu}\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = |a|^{\mu}\left(\frac{a}{|a|}\right)^{p} \qquad \mu \in \mathbb{C} \quad , \quad p \in \mathbb{Z} \ .$$

The Weyl element $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is denoted by w.

The representation space of the induced representation $Ind_{\mathbf{B}}^{\mathbf{G}}\chi_{p,\mu}$ is denoted by $V_{p,\mu}$ and is given by:

(2.1)
$$V_{p,\mu} = \left\{F : \mathbf{G} \to \mathbb{C} | \forall b \in \mathbf{B}, \forall g \in \mathbf{G}, F(bg) = \chi_{p,\mu}(b)F(g), F|_{SU(2)} \in L^2(SU(2))\right\}$$

$\mathbf{G}$ acts on $V_{p,\mu}$ by right translations. For $g, g_1 \in \mathbf{G}$ and $F \in V_{p,\mu}$ we define

$$(gF)(g_1) = F(g_1 g) \ .$$

The invariant subspace of smooth vectors in $V_{p,\mu}$ is the set of smooth functions $F \in V_{p,\mu}$ and it is denoted by $V_{p,\mu}^{\infty}$. By restricting the functions in $V_{p,\mu}^{\infty}$ to matrices of the form $w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ we get an isomorphic representation $\rho_{p,\mu}$ whose representation space is denoted by $\widetilde{V}_{p,\mu}^{\infty}$.

$$\widetilde{V}^{\infty}_{p,\mu} = \left\{ f_{_F} : \mathbb{C} \to \mathbb{C} \,\middle|\, f_{_F}(x) = F\left(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \ , \ F \in V^{\infty}_{p,\mu} \right\}.$$

The Bruhat decomposition of $\mathbf{G}$ implies that the mapping $F \to f_{_F}$ is one to one. If we take an element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathbf{G}$ then the corresponding action of g on $f_{_F} \in \widetilde{V}^{\infty}_{p,\mu}$ is given by:

$$\left(\rho_{p,\mu}(g)\big(f_{_F}\big) \right)(x) = |a+cx|^{-\mu} \left(\frac{a+cx}{|a+cx|} \right)^{-p} f_{_F}\left(\frac{b+dx}{a+cx} \right). \tag{2.2}$$

It is easy to show that the functions in the space $\widetilde{V}^{\infty}_{p,\mu}$ can be characterized by their behaviour at ∞. In fact, the space is given by:

$$\widetilde{V}^{\infty}_{p,\mu} = \left\{ \phi : \mathbb{C} \to \mathbb{C} \,\middle|\, \phi \in C^{\infty}(\mathbb{C}) \ , \ |x|^{-\mu} \left(\frac{x}{|x|} \right)^{-p} \phi\left(\frac{-1}{x} \right) \in C^{\infty}(\mathbb{C}) \right\}. \tag{2.3}$$

For μ with $2 < Re(\mu)$ we define a Whittaker functional $L : V^{\infty}_{p,\mu} \to \mathbb{C}$ by:

$$L(F) = \int_{\mathbb{C}} F\left(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) e^{-2\pi i Re(x)} dx \tag{2.4}$$

for $F \in V^{\infty}_{p,\mu}$. (The convergence of this integral follows from (2.3)). Using this Whittaker functional, we get from any function $F \in V^{\infty}_{p,\mu}$ a new function $W_F : G \to \mathbb{C}$ in the following way:

$$W_F(g) = L(g \cdot F) \ .$$

The space of all of these functions is the representation space of the Whittaker model and we denote it by:

$$\mathcal{W}_{p,\mu} = \left\{ W_F \,\middle|\, F \in V^{\infty}_{p,\mu} \right\}.$$

From each function $W_F \in \mathcal{W}_{p,\mu}$, we get a new function, $\psi_{_F} : \mathbb{C}^* \to \mathbb{C}$ by restricting W_F to $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$:

$$\psi_{_F}(a) = W_F \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

The representation space of the Kirillov model which we denote by $U_{p,\mu}$, is the set of all such functions:

$$U_{p,\mu} = \left\{ \psi_{_F} \,\middle|\, F \in V^{\infty}_{p,\mu} \right\}.$$

We have a simple connection between functions in $U_{p,\mu}$ and functions in $\widetilde{V}^{\infty}_{p,\mu}$ which is given by:

$$\psi_{_F}(a) = \chi_{p,\mu} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} |a|^4 \widehat{f}_{_F}(a^2) \ . \tag{2.5}$$

It will be convenient to work also with a representation that is slightly different from the Kirillov model. We denote by $\widehat{V}^{\infty}_{p,\mu}$ the following representation space:

$$\widehat{V}^{\infty}_{p,\mu} = \left\{ \widehat{f}_{_F} \,\middle|\, F \in V^{\infty}_{p,\mu} \right\}. \tag{2.6}$$

The corresponding representation which we denote $\widehat{\rho}_{p,\mu}$ is given by:

$$\widehat{\rho}_{p,\mu}(g)(\widehat{f}_{F}) = \widehat{f}_{gF}.$$

The main results that we prove in this paper are the following. We give an explicit description of the action of w on compactly supported smooth functions in $\widehat{V}^{\infty}_{p,\mu}$ as a kernel formula. The kernel formula with the explicit kernel function (which is given in terms of Bessel functions J_ν) is given by the following theorem:

Theorem 2.1. *Let $\mu \in \mathbb{C}$ be such that $2 < Re(\mu)$. Let $\widehat{f} \in \widehat{V}^{\infty}_{p,\mu}$ be a smooth function with compact support in $\mathbb{C} - \{0\}$. Then the action of w on $\widehat{f}$ is given by:*

$$(w \cdot \widehat{f})(b) = \int_{\mathbb{C}} \widehat{f}(z) k_{p,\mu}(z,b) dz$$

where:

$$k_{p,\mu}(z,b) = \frac{\pi}{2} \frac{l^{+}_{p,\mu}(z,b) - l^{-}_{p,\mu}(z,b)}{\sin \pi(2-\mu)}$$

and:

$$l^{+}_{p,\mu}(z,b) = 4\pi e^{\frac{ip\pi}{2}} \left[\cos\left(\frac{\pi}{2}(p-(\mu-2))\right)\right] \left|\frac{z}{b}\right|^{-\frac{\mu-2}{2}} \left(\frac{\frac{z}{|z|}}{\frac{b}{|b|}}\right)^{-\frac{p}{2}} J_{\frac{\mu-2-p}{2}}(2\pi\sqrt{\overline{zb}}) J_{\frac{\mu-2+p}{2}}(2\pi\sqrt{zb})$$

$$l^{-}_{p,4-\mu}(\overline{b},\overline{z}) = l^{+}_{p,\mu}(z,b)\ .$$

Remark 2.2. The Bessel function $J_\nu(z)$ can be defined by the power series $J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}$. Therefore it can be thought of as $z^\nu \mathcal{J}_\nu(z)$ where $\mathcal{J}_\nu(z)$ is an entire function. Moreover, all the powers in the power series that define $\mathcal{J}_\nu(z)$ are even and hence $\mathcal{J}_\nu(\sqrt{z})$ has a natural extension to an entire function. It can easily be verified this way that the function $k_{p,\mu}(z,b)$ in the theorem is naturally defined for any $z, b \in \mathbb{C}$.

Using 2.5 this theorem immediately implies an analog theorem for the Kirrilov model. The formula for the Kirrilov model is given by:

Theorem 2.3. *Let $\mu \in \mathbb{C}$ be such that $2 < Re(\mu)$. Let $\psi_F \in U_{p,\mu}$ be a smooth function with compact support in $\mathbb{C} - \{0\}$. Then the action of w on ψ_F is given by:*

$$(w \cdot \psi_F)(b) = W_F\left(\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} w\right) = \int_{\mathbb{C}} \kappa_{p,\mu}(z,b) W_F\left(\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}\right) \frac{dz}{|z|^2}$$

where:

$$\kappa_{p,\mu}(z,b) = 2\pi^2 |zb|^2 \frac{(-i)^p J_{\frac{\mu-2-p}{2}}(2\pi\overline{zb}) J_{\frac{\mu-2+p}{2}}(2\pi zb) - (i)^p J_{\frac{-(\mu-2)-p}{2}}(2\pi zb) J_{\frac{-(\mu-2)+p}{2}}(2\pi\overline{zb})}{\sin(\frac{\pi}{2}(2-\mu+p))}\ .$$

Note that $\kappa_{p,\mu}(z,b)$ is defined for any $z, b \in \mathbb{C}$ in the way explained in remark 2.2.

A key step in the proof of these kernel formulas is a calculation of a classical integral of Bessel functions. This is done in section 4 where we prove:

Theorem 2.4. *For any $\mu \in \mathbb{C}$ with $1 < Re(\mu)$ and for any $p \in \mathbb{Z}$ we have the following equality:*

$$\int_0^{2\pi} 2(e^{i\theta})^{-p} \frac{(\beta(z,\theta))^{-\frac{\mu-2}{2}}}{(\gamma(b,\theta))^{-\frac{\mu-2}{2}}} I_{\mu-2}(2(\beta(z,\theta))^{\frac{1}{2}}(\gamma(b,\theta))^{\frac{1}{2}})d\theta =$$

$$= 4\pi e^{\frac{ip\pi}{2}} \left[\cos\left(\frac{\pi}{2}(p-(\mu-2))\right)\right] \left|\frac{z}{b}\right|^{-\frac{\mu-2}{2}} \left(\frac{\frac{z}{|z|}}{\frac{b}{|b|}}\right)^{-\frac{p}{2}} J_{\frac{\mu-2-p}{2}}(2\pi\sqrt{zb}) J_{\frac{\mu-2+p}{2}}(2\pi\sqrt{zb})$$

where $z = z_1 + iz_2$, $b = b_1 + ib_2$ and

$$\beta(z,\theta) = 2\pi i(z_1 \cos\theta + z_2 \sin\theta)$$
$$\gamma(b,\theta) = 2\pi i(b_1 \cos\theta - b_2 \sin\theta)$$

Remark 2.5. In the last theorem we define z^μ to be $e^{\mu \log z}$ for $z \in \mathbb{C} \setminus (-\infty, 0]$ and any $\mu \in \mathbb{C}$, where $\log z$ is defined by $\log z = \log|z| + i\arg(z)$ with $|\arg z| < \pi$.

The proof of these results is done in several steps which we begin in the next section.

3. Existence of a kernel formula for $2 < Re(\mu) < 3$.

While our final goal is to obtain a kernel formula for all the representations $\widehat{\rho}_{p,\mu}$ with $2 < Re(\mu)$, we do it first only for $2 < Re(\mu) < 3$. The reason for working only with such μ's is that this condition guarantees the convergence of some of the integrals involved and thus allows us to do explicit calculations. Later on, we will use analytic continuation to extend this kernel formula to any μ with $2 < Re(\mu)$. In this section we will prove:

Theorem 3.1. *Let $\mu \in \mathbb{C}$ be such that $2 < Re(\mu) < 3$. Let $\widehat{f} \in \widehat{V}^\infty_{p,\mu}$ be a smooth function with compact support in $\mathbb{C} - \{0\}$. Then the action of w on $\widehat{f}$ is given by*

$$(w \cdot \widehat{f})(b) = \int_{\mathbb{C}} \widehat{f}(z) k_{p,\mu}(z,b) dz$$

where $k_{p,\mu}(z,b)$ is defined by:

$$k_{p,\mu}(z,b) = \int_0^{2\pi} 2(e^{i\theta})^{-p} \frac{(\beta(z,\theta))^{\frac{\mu-2}{2}}}{(\gamma(b,\theta))^{\frac{\mu-2}{2}}} K_{\mu-2}(2(\beta(z,\theta))^{\frac{1}{2}}(\gamma(b,\theta))^{\frac{1}{2}})d\theta$$

with

$$\beta(z,\theta) = 2\pi i(z_1 \cos\theta + z_2 \sin\theta)$$
$$\gamma(b,\theta) = 2\pi i(b_1 \cos\theta - b_2 \sin\theta)$$

and $K_\nu(z)$ is a modified Bessel function.

To prove theorem 3.1 we consider the following. Let $\widehat{f}(z) \in \widehat{V}^\infty_{p,\mu}$ be a smooth function with compact support and let $M \subseteq \mathbb{C} - \{0\}$ denote the support of $\widehat{f}(z)$. Let f be the corresponding function in $\widetilde{V}^\infty_{p,\mu}$. By 2.2 we get that $\big(w \cdot f\big)(y) = |y|^{-\mu}(\frac{y}{|y|})^{-p} f(\frac{-1}{y})$. We will calculate $(|y|^{-\mu}(\frac{y}{|y|})^{-p} f(\frac{-1}{y}))\,\widehat{}\,(b)$. To do this, we first observe that $|y|^{-\mu}(\frac{y}{|y|})^{-p} f(\frac{-1}{y})$ is in $L^1(\mathbb{C})$. This is true since $f \in \widetilde{V}^\infty_{p,\mu}$ and hence by 2.3 we know that $|y|^{-\mu}(\frac{y}{|y|})^{-p} f(\frac{-1}{y})$ is smooth. When $y \to \infty$ the absolute value behaves like $|y|^{-\mu}$ (or smaller if $f(0) = 0$) so indeed this function is in $L^1(\mathbb{C})$. We note

that, in fact, we do not need to use 2.3 to see that $\big(w \cdot f\big)(y) \in L^1(\mathbb{C})$. It is enough to observe that f is a Schwartz function (since $\widehat{f}$ is smooth with compact support and hence a Schwartz function) in order to see that $\big(w \cdot f\big)(y)$ decays rapidly when $y \to 0$. Therefore we have:
(3.1)
$$(w \cdot \widehat{f})(b) = \left(|y|^{-\mu}\left(\frac{y}{|y|}\right)^{-p} f\left(\frac{-1}{y}\right)\right)\widehat{\ }(b) = \int_{\mathbb{C}} e^{-2\pi i(b\cdot y)}|y|^{-\mu}\left(\frac{y}{|y|}\right)^{-p} f\left(\frac{-1}{y}\right)dy$$
(We use here the notation $b \cdot y = Re(by)$). We proceed by calculating $f(-\frac{1}{y})$ in terms of $\widehat{f}$. Using inverse Fourier transform of $\widehat{f}(z)$ we get:
$$f\left(-\frac{1}{y}\right) = \int_{\mathbb{C}} e^{2\pi i(z\cdot(-\frac{1}{y}))}\widehat{f}(z)dz \ . \tag{3.2}$$
Here we have no problems of convergence since $\widehat{f}(z)$ has compact support. Using (3.2) in (3.1) we get:
$$(w \cdot \widehat{f})(b) = \int_{\mathbb{C}} e^{-2\pi i(b\cdot y)}|y|^{-\mu}\left(\frac{y}{|y|}\right)^{-p}\left(\int_{\mathbb{C}} e^{2\pi i(z\cdot(-\frac{1}{y}))}\widehat{f}(z)dz\right)dy \ . \tag{3.3}$$
We would like to switch the order of integration in (3.3) To do that we insert a convergence factor of the form $e^{-\delta(|y|+|y|^{-1})}$ into our integral. By dominated convergence we have:
$$(w \cdot \widehat{f})(b) = \int_{\mathbb{C}} e^{-2\pi i(b\cdot y)}|y|^{-\mu}\left(\frac{y}{|y|}\right)^{-p}\left(\int_{\mathbb{C}} e^{2\pi i(z\cdot(-\frac{1}{y}))}\widehat{f}(z)dz\right)dy =$$
$$= \lim_{\delta\to 0^+}\int_{\mathbb{C}} e^{-2\pi i(b\cdot y)-\delta(|y|+|y|^{-1})}|y|^{-\mu}\left(\frac{y}{|y|}\right)^{-p}\left(\int_{\mathbb{C}} e^{2\pi i(z\cdot(-\frac{1}{y}))}\widehat{f}(z)dz\right)dy \ . \tag{3.4}$$
Now we can use Fubini's theorem and switch the order of integration. We get:
$$(w \cdot \widehat{f})(b) = \lim_{\delta\to 0^+}\int_{\mathbb{C}}\left(\widehat{f}(z)\int_{\mathbb{C}} e^{-2\pi i((b\cdot y)-(z\cdot(-\frac{1}{y})))-\delta(|y|+|y|^{-1})}|y|^{-\mu}\left(\frac{y}{|y|}\right)^{-p}dy\right)dz \ .$$
Defining $k_{p,\mu,\delta}(z,b)$ to be:
$$k_{p,\mu,\delta}(z,b) = \int_{\mathbb{C}} e^{-2\pi i((b\cdot y)-(z\cdot(-\frac{1}{y})))-\delta(|y|+|y|^{-1})}|y|^{-\mu}\left(\frac{y}{|y|}\right)^{-p}dy \tag{3.5}$$
we can write the above equality as:
$$(w \cdot \widehat{f})(b) = \lim_{\delta\to 0^+}\int_{\mathbb{C}}\widehat{f}(z)k_{p,\mu,\delta}(z,b)dz \ .$$
In order to obtain a kernel formula, it is now sufficient to show that there exists a function $k_{p,\mu}(z,b)$ such that:
$$\lim_{\delta\to 0^+} k_{p,\mu,\delta}(z,b) = k_{p,\mu}(z,b)$$
and that:
$$\lim_{\delta\to 0^+}\int_{\mathbb{C}}\widehat{f}(z)k_{p,\mu,\delta}(z,b)dz = \int_{\mathbb{C}}\widehat{f}(z)(\lim_{\delta\to 0^+} k_{p,\mu,\delta}(z,b))dz = \int_{\mathbb{C}}\widehat{f}(z)k_{p,\mu}(z,b)dz \ .$$

We begin by showing the existence of $k_{p,\mu}(z,b)$. Writing: $b = b_1 + ib_2$, $y = y_1 + iy_2$, $z = z_1 + iz_2$, $b \cdot y = Re(by)$ and $z \cdot \frac{-1}{y} = Re(\frac{-z}{y})$, we can write (3.5) as:

$$k_{p,\mu,\delta}(z,b) = \int_{\mathbb{C}} e^{-2\pi i\left((b_1y_1-b_2y_2)-(\frac{-z_1y_1-z_2y_2}{(y_1)^2+(y_2)^2})\right)-\delta(|y|+|y|^{-1})}|y|^{-\mu}\left(\frac{y}{|y|}\right)^{-p} dy \ .$$

Switching to polar coordinates by setting $y = re^{i\theta}$ the integral becomes:

$$\int_0^{2\pi}\left((e^{i\theta})^{-p}\int_0^{\infty} e^{-r(2\pi i(b_1\cos\theta-b_2\sin\theta)+\delta)-\frac{1}{r}(2\pi i(z_1\cos\theta+z_2\sin\theta)+\delta)}r^{-\mu+1}dr\right)d\theta \ . \tag{3.6}$$

Next, we use the following formula ([6] ch.3.47 p.340):

$$\int_0^{\infty} x^{\nu-1}e^{-\frac{\beta}{x}-\gamma x}dx = 2\left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}}K_\nu(2\sqrt{\beta\gamma}) \tag{3.7}$$

for $Re(\beta) > 0$ and $Re(\gamma) > 0$. Here K_ν is a Bessel function of imaginary argument called Macdonald's function and defined by (see [10] p.108):

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin\nu\pi}, \qquad |arg(z)| < \pi, \qquad \nu \neq 0, \pm1, \pm2, \ldots$$

where I_ν is:

$$I_\nu(z) = \sum_{k=0}^{\infty}\frac{(z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}, \qquad |z| < \infty, \qquad |arg(z)| < \pi \ .$$

Using (3.7) and (3.6) we get that:

$$k_{p,\mu,\delta}(z,b) = \int_0^{2\pi} 2(e^{i\theta})^{-p}\left(\frac{\beta_\delta(z,\theta)}{\gamma_\delta(b,\theta)}\right)^{\frac{2-\mu}{2}} K_{2-\mu}(2\sqrt{\beta_\delta(z,\theta)\gamma_\delta(b,\theta)}\ )d\theta \tag{3.8}$$

Where:

$$\begin{aligned}\beta_\delta(z,\theta) &= (2\pi i(z_1\cos\theta + z_2\sin\theta) + \delta)\\ \gamma_\delta(b,\theta) &= (2\pi i(b_1\cos\theta - b_2\sin\theta) + \delta) \ .\end{aligned} \tag{3.9}$$

We now want to use dominated convergence to define $k_{p,\mu}(z,b)$ as the limit of $k_{p,\mu,\delta}(z,b)$ when $\delta \to 0$. In order to do this, we need to bound the integrand in 3.8 by some function that is in $L^1([0,2\pi])$. For this purpose, it is useful to notice (as in remark 2.2) that by the definition of the function $I_\nu(z)$ it can be written as $I_\nu(z) = (\frac{z}{2})^\nu\mathcal{I}_\nu(z)$ where $\mathcal{I}_\nu(z)$ is an entire function. Moreover $\mathcal{I}_\nu(\sqrt{z})$ is entire since all the powers in the power series of $\mathcal{I}_\nu(z)$ are even. Thus the integrand in (3.8) is:

$$\frac{\pi e^{-i\theta p}}{\sin(2-\mu)\pi}\left[(\gamma_\delta(b,\theta))^{\mu-2}\mathcal{I}_{\mu-2}\big(2\sqrt{\beta_\delta(z,\theta)\gamma_\delta(b,\theta)}\ \big)-(\beta_\delta(z,\theta))^{2-\mu}\mathcal{I}_{2-\mu}\big(2\sqrt{\beta_\delta(z,\theta)\gamma_\delta(b,\theta)}\ \big)\right] \tag{3.10}$$

Now, since $\mathcal{I}_{\mu-2}(\sqrt{z}), \mathcal{I}_{2-\mu}(\sqrt{z})$ are entire functions, there exists constants C_1, C_2 such that for any $\theta \in [0,2\pi]$, $\delta \in [0,1]$ and z in M (the support of $\widehat{f}$) we have:

$$|\mathcal{I}_{\mu-2}\big(2\sqrt{\beta_\delta(z,\theta)\gamma_\delta(b,\theta)}\ \big)| \leq C_1 \quad \text{and} \quad |\mathcal{I}_{2-\mu}\big(2\sqrt{\beta_\delta(z,\theta)\gamma_\delta(b,\theta)}\ \big)| \leq C_2$$

Denoting the integrand in (3.8) by $h_\delta(z,b,\theta)$ we can see that for any $\theta \in [0,2\pi]$ such that $\beta_\delta(z,\theta) \neq 0$ any $\delta \in [0,1]$ and any z in M we have:

$$|h_\delta(z,b,\theta)| \leq \frac{\pi}{\sin{(2-\mu)\pi}} \left[|\gamma_\delta(b,\theta)|^{Re(\mu)-2} C_1 + \frac{C_2}{|\beta_\delta(z,\theta)|^{Re(\mu)-2}} \right].$$

Our choice of $2 < Re(\mu)$ implies that $|\gamma_\delta(b,\theta)|^{Re(\mu)-2}$ is a continuous function of the variables δ and θ and since both are in compact sets, it is bounded there. Hence we can change the constant C_1 to get that for $z \in M$ and $\delta \in [0,1]$ we have:

$$|h_\delta(z,b,\theta)| \leq C_1 + \frac{C_2}{|\beta_\delta(z,\theta)|^{Re(\mu)-2}}.$$

$$\leq C_1 + \frac{C_2}{|2\pi(z_1\cos\theta + z_2\sin\theta)|^{Re(\mu)-2}} = C_1 + \frac{C_2}{(2\pi r \sin{(\theta+\alpha)})^{Re(\mu)-2}} \tag{3.11}$$

where $\alpha = \frac{\pi}{2} - \omega$ and $z = re^{i\omega}$ (Notice that $z \in M$ implies that $z \neq 0$). Since $Re(\mu) < 3$, (3.11) implies that the integrand $h_\delta(z,b,\theta)$ is dominated by an $L^1([0,2\pi])$ function. Fixing $z \neq 0$ and θ such that $z_1\cos\theta + z_2\sin\theta \neq 0$ we have:

$$2(e^{i\theta})^{-p}\left(\frac{\beta_\delta(z,\theta)}{\gamma_\delta(b,\theta)}\right)^{\frac{2-\mu}{2}} K_{2-\mu}(2\sqrt{\beta_\delta(z,\theta)\gamma_\delta(b,\theta)}\ ) \longrightarrow$$

$$\longrightarrow 2(e^{i\theta})^{-p}\frac{(\beta(z,\theta))^{\frac{2-\mu}{2}}}{(\gamma(b,\theta))^{\frac{2-\mu}{2}}} K_{2-\mu}(2(\beta(z,\theta))^{\frac{1}{2}}(\gamma(b,\theta))^{\frac{1}{2}}\ )$$

as $\delta \to 0^+$ and thus we can use dominated convergence to obtain that :

$$\lim_{\delta\to 0^+} k_{p,\mu,\delta}(z,b) = \int_0^{2\pi} 2(e^{i\theta})^{-p}\frac{(\beta(z,\theta))^{\frac{2-\mu}{2}}}{(\gamma(b,\theta))^{\frac{2-\mu}{2}}} K_{2-\mu}(2(\beta(z,\theta))^{\frac{1}{2}}(\gamma(b,\theta))^{\frac{1}{2}})d\theta.$$

Defining :

$$k_{p,\mu}(z,b) = \int_0^{2\pi} 2(e^{i\theta})^{-p}\frac{(\beta(z,\theta))^{\frac{2-\mu}{2}}}{(\gamma(b,\theta))^{\frac{2-\mu}{2}}} K_{2-\mu}(2(\beta(z,\theta))^{\frac{1}{2}}(\gamma(b,\theta))^{\frac{1}{2}})d\theta \tag{3.12}$$

we have shown that for $2 < Re(\mu) < 3$

$$\lim_{\delta\to 0^+} k_{p,\mu,\delta}(z,b) = k_{p,\mu}(z,b)$$

Next, we need to show that:

$$\lim_{\delta\to 0^+} \int_{\mathbb{C}} \widehat{f}(z) k_{p,\mu,\delta}(z,b)dz = \int_{\mathbb{C}} \widehat{f}(z) k_{p,\mu}(z,b)dz\ .$$

To justify this we want to use dominated convergence again. Remembering that $\widehat{f}(z)$ is continuous and supported on the compact set M it follows that $|\widehat{f}(z)|$ is bounded. Therefore we only need to deal with $k_{p,\mu,\delta}(z,b)$. Since $k_{p,\mu,\delta}(z,b) = \int_0^{2\pi} h_\delta(z,b,\theta)d\theta$ it is enough to bound $\int_0^{2\pi} |h_\delta(z,b,\theta)|d\theta$. Since $\widehat{f}$ is supported away from zero we can find $r_0 > 0$ such that $r_0 \leq |z|$ for every z in M. Hence by (3.11) we have:

$$|k_{p,\mu,\delta}(z,b)| \leq \int_0^{2\pi} |h_\delta(z,b,\theta)|d\theta \leq \int_0^{2\pi} C_1 + \frac{C_2}{(2\pi r_0 \sin{(\theta+\alpha)})^{Re(\mu)-2}} d\theta. \tag{3.13}$$

Since the last integral is independent of α we have that $|k_{p,\mu,\delta}(z,b)|$ is bounded by a constant which is independent of $\delta \in [0,1]$ and $z \in M$ hence we can use the dominated convergence to obtain the limit above. This proves theorem 3.1.

4. An explicit expression for $k_{p,\mu}(z,b)$.

In theorem 3.1 the kernel function $k_{p,\mu}(z,b)$ is expressed as an integral of a K-Bessel function. In this part we will compute $k_{p,\mu}(z,b)$ explicitly. By the definition of $k_{p,\mu}(z,b)$ (3.12) and the definition of $K_\nu(z)$ it is enough to calculate the following two integrals:

$$l^+_{p,\mu}(z,b) = \int_0^{2\pi} 2(e^{i\theta})^{-p} \frac{(\beta(z,\theta))^{\frac{2-\mu}{2}}}{(\gamma(b,\theta))^{\frac{2-\mu}{2}}} I_{\mu-2}(2(\beta(z,\theta))^{\frac{1}{2}}(\gamma(b,\theta))^{\frac{1}{2}})d\theta$$

and

$$l^-_{p,\mu}(z,b) = \int_0^{2\pi} 2(e^{i\theta})^{-p} \frac{(\beta(z,\theta))^{\frac{2-\mu}{2}}}{(\gamma(b,\theta))^{\frac{2-\mu}{2}}} I_{2-\mu}(2(\beta(z,\theta))^{\frac{1}{2}}(\gamma(b,\theta))^{\frac{1}{2}})d\theta$$

It is clear that under the restriction $2 < Re(\mu) < 3$ both of these integrals converge. More accurately, $l^+_{p,\mu}(z,b)$ converges for $1 < Re(\mu)$ and $l^-_{p,\mu}(z,b)$ converges for $Re(\mu) < 3$. It is easy to see that when both integrals converge we have the following connection between $l^+_{p,\mu}(z,b)$ and $l^-_{p,\mu}(z,b)$:

$$l^-_{p,4-\mu}(\bar{b},\bar{z}) = l^+_{p,\mu}(z,b) \qquad \text{for} \quad 1 < Re(\mu) < 3 \tag{4.1}$$

and thus it is enough to calculate $l^+_{p,\mu}(z,b)$ for $1 < Re(\mu)$ since if $\mu' = 4-\mu$ and $1 < Re(\mu)$ then $Re(\mu') = Re(4-\mu) < 3$ and (4.1) gives us $l^-_{p,\mu'}$. Theorem 2.4 which we will prove in this section, together with (4.1), clearly give us an explicit expression for $k_{p,\mu}(z,b)$ (for $2 < Re(\mu) < 3$).

The proof of theorem 2.4 will follow from a few lemmas that we will prove and it will involve a definition of some entire function related to $l^+_{p,\mu}(z,b)$. We begin with the first lemma which allows us to calculate $l^+_{p,\mu}(z,b)$ in points (z,b) that satisfy $\arg(z) = -\arg(b)$.

Lemma 4.1. *Let $z,b \in \mathbb{C} - \{0\}$ be such that $\arg(z) = -\arg(b)$. For any μ with $1 < Re(\mu)$ and any $p \in \mathbb{Z}$ we have:*

$$l^+_{p,\mu}(z,b) = 4\pi e^{ip\alpha}\left[\cos\left(\frac{\pi}{2}(p-(\mu-2))\right)\right]\left(\frac{|z|}{|b|}\right)^{\frac{2-\mu}{2}} J_{\frac{\mu-2-p}{2}}(2\pi\sqrt{|z||b|})J_{\frac{\mu-2+p}{2}}(2\pi\sqrt{|z||b|})$$

where $\alpha = \frac{\pi}{2} - \arg(z)$.

Proof. We denote $z = r_1 e^{i\omega}$ and $b = r_2 e^{-i\omega}$. By definition we have:

$$\beta(z,\theta) = 2\pi i r_1(\cos\omega\cos\theta + \sin\omega\sin\theta)$$

$$\gamma(b,\theta) = 2\pi i r_2(\cos(-\omega)\cos\theta - \sin(-\omega)\sin\theta) = 2\pi i r_2(\cos\omega\cos\theta + \sin\omega\sin\theta).$$

Substituting this into the definition of $l^+_{p,\mu}(z,b)$ we get:

$$l^+_{p,\mu}(z,b) = \int_0^{2\pi} 2(e^{i\theta})^{-p}\left(\frac{r_1}{r_2}\right)^{\frac{2-\mu}{2}} I_{\mu-2}(4\pi i\sqrt{r_1 r_2}(\cos\omega\cos\theta + \sin\omega\sin\theta))d\theta.$$

Since:

$$(\cos\omega\cos\theta + \sin\omega\sin\theta) = \cos(\omega-\theta) = \sin(\theta+\alpha)$$

where $\alpha = \frac{\pi}{2} - \omega$, we need to calculate:

$$\int_0^{2\pi} 2(e^{i\theta})^{-p} I_{\mu-2}(4\pi i\sqrt{r_1 r_2}\sin(\theta+\alpha))d\theta \tag{4.2}$$

some simple changes of variables will now give us that (4.2) equals:

$$(-1)^p e^{ip\alpha}\Bigg[\int_0^{\pi} 2\cos(p\theta)I_{\mu-2}(-4\pi i\sqrt{r_1r_2}\sin\theta)d\theta + \int_0^{\pi} 2\cos(p\theta)I_{\mu-2}(4\pi i\sqrt{r_1r_2}\sin\theta)d\theta$$
$$-i\int_0^{\pi} 2\sin(p\theta)I_{\mu-2}(-4\pi i\sqrt{r_1r_2}\sin\theta)d\theta + i\int_0^{\pi} 2\sin(p\theta)I_{\mu-2}(4\pi i\sqrt{r_1r_2}\sin\theta)d\theta\Bigg] \tag{4.3}$$

Using the following identities ([10]):

$$I_\mu(z) = e^{\frac{-\mu\pi i}{2}}J_\mu(iz) \qquad -\pi < arg(z) < \frac{\pi}{2}$$
$$I_\mu(z) = e^{\frac{\mu\pi i}{2}}J_\mu(-iz) \qquad \frac{-\pi}{2} < arg(z) < \pi$$

we get that this equals:

$$(-1)^p e^{ip\alpha}\times$$
$$\Bigg[e^{\frac{-(\mu-2)\pi i}{2}}\int_0^{\pi} 2\cos(p\theta)J_{\mu-2}(4\pi\sqrt{r_1r_2}\sin\theta)d\theta + e^{\frac{(\mu-2)\pi i}{2}}\int_0^{\pi} 2\cos(p\theta)J_{\mu-2}(4\pi\sqrt{r_1r_2}\sin\theta)d\theta$$
$$-ie^{\frac{-(\mu-2)\pi i}{2}}\int_0^{\pi} 2\sin(p\theta)J_{\mu-2}(4\pi\sqrt{r_1r_2}\sin\theta)d\theta + ie^{\frac{(\mu-2)\pi i}{2}}\int_0^{\pi} 2\sin(p\theta)J_{\mu-2}(4\pi\sqrt{r_1r_2}\sin\theta)d\theta\Bigg] \tag{4.4}$$

Finally, in order to finish the proof of the lemma, we use the following identities ([6] ch.6.68 p.739):

$$\int_0^{\pi} \sin(2\eta x)J_{2\lambda}(2a\sin x)dx = \pi\sin(\eta\pi)J_{\lambda-\eta}(a)J_{\lambda+\eta}(a) \qquad Re(\lambda) > -1$$
$$\int_0^{\pi} \cos(2\eta x)J_{2\lambda}(2a\sin x)dx = \pi\cos(\eta\pi)J_{\lambda-\eta}(a)J_{\lambda+\eta}(a) \qquad Re(\lambda) > -\frac{1}{2}$$

(We can use them since we took $1 < Re(\mu)$). By applying these identities to (4.4) and some simple trigonometric identities, we get exactly what we stated in the lemma. □

In the last lemma, our choice to calculate $l^+_{p,\mu}(z,b)$ in z and b such that $Arg(z) = -Arg(b)$ allowed us to reduce the calculation to known integrals and thus to get an explicit expression. From this reason, we want to think of z as $r_1e^{i(\Omega+\omega)}$ and of b as $r_2e^{-i\Omega}$. We can then write:

$$\beta(z,\theta) = 2\pi i(r_1\cos(\Omega+\omega)\cos\theta + r_1\sin(\Omega+\omega)\sin\theta)$$

and

$$\gamma(b,\theta) = 2\pi i(r_2\cos(-\Omega)\cos\theta - r_2\sin(-\Omega)\sin\theta)$$

and thus:

$$\beta(z,\theta) = 2\pi ir_1\cos((\Omega+\omega)-\theta) \quad and \quad \gamma(b,\theta) = 2\pi ir_2\cos(-\Omega+\theta)$$

Using these notations in $l^+_{p,\mu}(z,b)$, motivates us to define the function $s^+_{\mu,p,r_1,r_2,\Omega}(\omega)$ in the following way:

Definition 4.2. For $0<r_1,r_2,\ \Omega\in\mathbb{R}$, $p\in\mathbb{Z}$ and $\mu\in\mathbb{C}$ with $1<Re(\mu)$ we define a function $s^+_{\mu,p,r_1,r_2,\Omega}(\omega)$ of complex variable by:
$s^+_{\mu,p,r_1,r_2,\Omega}(\omega)=$

$$\int_0^{2\pi}2e^{-ip\theta}\frac{(2\pi ir_1\cos(\Omega+\omega-\theta))^{\frac{2-\mu}{2}}}{(2\pi ir_2\cos(-\Omega+\theta))^{\frac{2-\mu}{2}}}I_{\mu-2}(2(2\pi ir_1\cos(\Omega+\omega-\theta))^{\frac{1}{2}}(2\pi ir_2\cos(-\Omega+\theta))^{\frac{1}{2}})d\theta.$$

Notice that when $\omega\in\mathbb{R}$ the integral defining $s^+_{\mu,p,r_1,r_2,\Omega}(\omega)$ is given by the integral defining $l^+_{p,\mu}(z,b)$ and we have:

$$s^+_{\mu,p,r_1,r_2,\Omega}(\omega)=l^+_{p,\mu}(r_1e^{i(\Omega+\omega)},r_2e^{-i\Omega})\tag{4.5}$$

Our main trick in this section is to think of ω as a complex variable. Notice that for $\omega\notin\mathbb{R}$ the integrand defining $s^+_{\mu,p,r_1,r_2,\Omega}(\omega)$ can not be realized as the integrand of $l^+_{p,\mu}(z,b)$. We are only interested in evaluating $s^+_{\mu,p,r_1,r_2,\Omega}(\omega)$ for real ω but it will be more convenient to evaluate $s^+_{\mu,p,r_1,r_2,\Omega}(\omega)$ for any $\omega\in\mathbb{C}$ and thus to get our formula for real ω. Our first step is to show that $s^+_{\mu,p,r_1,r_2,\Omega}(\omega)$ is defined for any $\omega\in\mathbb{C}$.

Lemma 4.3. *For $0<r_1,r_2,\ \Omega\in\mathbb{R}$, $p\in\mathbb{Z}$, $\mu\in\mathbb{C}$ with $1<Re(\mu)$ and any $\omega\in\mathbb{C}$ the integral defining $s^+_{\mu,p,r_1,r_2,\Omega}(\omega)$ converges.*

Proof. Since $I_\nu(z)$ equals $\left(\frac{z}{2}\right)^\nu\mathcal{I}_\nu(z)$ where $\mathcal{I}_\nu(z)$ is an entire function (as explained in the previous section) the integrand in definition 4.2 is:

$$2e^{-ip\theta}(2\pi ir_2\cos(-\Omega+\theta))^{\mu-2}\mathcal{I}_{\mu-2}(2(2\pi ir_1\cos(\Omega+\omega-\theta))^{\frac{1}{2}}(2\pi ir_2\cos(-\Omega+\theta))^{\frac{1}{2}})$$

Since $\mathcal{I}_{\mu-2}(\sqrt{z})$ is also entire and $Re(\mu)>1$ the convergence of the integral is obvious. $\square$

Lemma 4.4. *we have:*

$$s^+_{\mu,p,r_1,r_2,\Omega}(0)=4\pi e^{ip(\frac{\pi}{2}-\Omega)}\left[\cos\left(\frac{\pi}{2}(p-(\mu-2))\right)\right]\left(\frac{r_1}{r_2}\right)^{\frac{2-\mu}{2}}J_{\frac{\mu-2-p}{2}}(2\pi\sqrt{r_1r_2})J_{\frac{\mu-2+p}{2}}(2\pi\sqrt{r_1r_2})$$

Proof. This is immediate from lemma 4.1. $\square$

Lemma 4.5. *For any μ with $1<Re(\mu)$ the function $s^+_{\mu,p,r_1,r_2,\Omega}(\omega)$ is an entire function of ω which satisfies:*

$$\frac{d}{d\omega}s^+_{\mu,p,r_1,r_2,\Omega}(\omega)=-\pi r_1e^{i(\Omega+\omega)}s^+_{\mu+1,p+1,r_1,r_2,\Omega}(\omega)+\pi r_1e^{-i(\Omega+\omega)}s^+_{\mu+1,p-1,r_1,r_2,\Omega}(\omega)\tag{4.6}$$

Proof. The lemma will follow from the differentiation theorem in ([3] p.224 theorem 17.9). In order to apply this theorem to $s^+_{\mu,p,r_1,r_2,\Omega}(\omega)$ we need to show that the integrand in definition 4.2 of $s^+_{\mu,p,r_1,r_2,\Omega}(\omega)$ is continuous with respect to θ and analytic (entire) with respect to ω. To do so, we write the integrand as in the proof of lemma 4.3:

$$2e^{-ip\theta}(2\pi ir_2\cos(-\Omega+\theta))^{\mu-2}\mathcal{I}_{\mu-2}(2(2\pi ir_1\cos(\Omega+\omega-\theta))^{\frac{1}{2}}(2\pi ir_2\cos(-\Omega+\theta))^{\frac{1}{2}})\tag{4.7}$$

Since $\mathcal{I}_{\mu-2}(\sqrt{z})$ is entire, it is obvious that the integrand in (4.7) is entire with respect to ω. If we consider μ's such that $2<Re(\mu)$ then it is obvious that the integrand in

(4.7) is continuous with respect to θ and therefore, we can use the theorem to get that $s^+_{\mu,p,r_1,r_2,\Omega}(\omega)$ is an entire function of ω for $2 < Re(\mu)$. In order to deal with $1 < Re(\mu)$ we need to consider the continuity of the integrand. The problem is that there are θ's in which $(2\pi i r_2 \cos(-\Omega+\theta)) = 0$. Fixing the continuity problem of the integrand in (4.7), is a matter of subtracting an expression of the form $(2\pi i r_2 \cos(-\Omega+\theta))^{\mu-2}$. More accurately, if we denote the integrand in (4.7) by $h_{\mu,p,r_1,r_2,\Omega}(\theta,\omega)$ then we can write (4.7) as:

$$\left[h_{\mu,p,r_1,r_2,\Omega}(\theta,\omega) - \mathcal{I}_{\mu-2}(0)(2\pi i r_2 \cos(-\Omega+\theta))^{\mu-2}\right]+$$

$$+\mathcal{I}_{\mu-2}(0)(2\pi i r_2 \cos(-\Omega+\theta))^{\mu-2} \tag{4.8}$$

Now it is clear that the first part of 4.8 is continuous with respect to θ (when ω is fixed) and the integral of the second part converges. Hence, by the theorem, $s^+_{\mu,p,r_1,r_2,\Omega}(\omega)$ is an analytic function of ω for $1 < Re(\mu)$. Verifying the recursive formula (4.6) for $\frac{d}{d\omega}s^+_{\mu,p,r_1,r_2,\Omega}(\omega)$, that is stated in the lemma is simply a matter of differentiating under the integral sign. It is done using the following formulas for derivatives of Bessel functions (see [10]):

$$\frac{d}{dz}(z^{-\nu}I_\nu(z)) = z^{-\nu}I_{\nu+1}(z) \tag{4.9}$$

□

Having proved these lemmas, we are now ready to compute $s^+_{\mu,p,r_1,r_2,\Omega}(\omega)$.

Theorem 4.6. *For any $0 < r_1, r_2$, $\Omega \in \mathbb{R}$, $p \in \mathbb{Z}$ and $\mu \in \mathbb{C}$ with $1 < Re(\mu)$, we have:*
$s^+_{\mu,p,r_1,r_2,\Omega}(\omega) =$

$$4\pi e^{\frac{ip(\pi-\omega-2\Omega)}{2}}\left[\cos\left(\frac{\pi}{2}(p-(\mu-2))\right)\right]\left(\sqrt{\frac{r_1}{r_2}}\right)^{(2-\mu)} J_{\frac{\mu-2-p}{2}}(2\pi\sqrt{r_1r_2}e^{-\frac{i\omega}{2}})J_{\frac{\mu-2+p}{2}}(2\pi\sqrt{r_1r_2}e^{\frac{i\omega}{2}})$$

Proof. If we denote the right hand side of the equation in theorem 4.6 by $\widetilde{s}^+_{\mu,p,r_1,r_2,\Omega}(\omega)$, then one can verify that $\widetilde{s}^+_{\mu,p,r_1,r_2,\Omega}(\omega)$ is an entire function of ω (for any μ). Using the formulas ([10] p.103):

$$\frac{d}{dz}(z^{\nu}J_\nu(z)) = z^{\nu}J_{\nu-1}(z) \quad , \quad \frac{d}{dz}(z^{-\nu}J_\nu(z)) = -z^{-\nu}J_{\nu+1}(z) \tag{4.10}$$

one can verify that the derivative $\frac{d}{d\omega}\widetilde{s}^+_{\mu,p,r_1,r_2,\Omega}(\omega)$ satisfies a recursive formula that is identical to the recursive formula (4.6) that we proved for $\frac{d}{d\omega}s^+_{\mu,p,r_1,r_2,\Omega}(\omega)$. For $\omega = 0$, the equality in theorem 4.6 is an immediate result of lemma 4.4. It follows that for any $n \in \mathbb{N}$ we have $(s^+_{\mu,p,r_1,r_2,\Omega})^{(n)}(0) = (\widetilde{s}^+_{\mu,p,r_1,r_2,\Omega})^{(n)}(0)$. This proves the theorem since by lemma (4.5), we know that $s^+_{\mu,p,r_1,r_2,\Omega}(\omega)$ is entire for $1 < Re(\mu)$). □

Having computed $s^+_{\mu,p,r_1,r_2,\Omega}(\omega)$ we can now easily prove Theorem 2.4 using equation 4.5.

Proof. (of theorem 2.4) To recover $l^+_{p,\mu}(z,b)$ from $s^+_{\mu,p,r_1,r_2,\Omega}(\omega)$ and get the expression in theorem (2.4) we recall that by (4.5) we have for $\omega \in \mathbb{R}$:

$$s^+_{\mu,p,r_1,r_2,\Omega}(\omega) = l^+_{p,\mu}(r_1e^{i(\Omega+\omega)}, r_2e^{-i\Omega})$$

which gives us $l^+_{p,\mu}(z,b)$ by choosing appropriate r_1, r_2, Ω, ω.

$\square$

5. Kernel formula for $3 \leq Re(\mu)$.

So far we proved that for $2 < Re(\mu) < 3$ and $\widehat{f} \in \widehat{V}^{\infty}_{p,\mu}$ with compact support we have:

$$(w \cdot \widehat{f})(b) = \int_{\mathbb{C}} \widehat{f}(z) k_{p,\mu}(z,b) dz. \tag{5.1}$$

Since:

$$k_{p,\mu}(z,b) = \frac{\pi}{2} \frac{l^+_{p,\mu}(z,b) - l^-_{p,\mu}(z,b)}{\sin \pi(2-\mu)}$$

where:

$$l^+_{p,\mu}(z,b) = 4\pi e^{\frac{ip\pi}{2}} \left[\cos\left(\frac{\pi}{2}(p-(\mu-2))\right)\right] \left|\frac{z}{b}\right|^{-\frac{\mu-2}{2}} \left(\frac{\frac{z}{|z|}}{\frac{b}{|b|}}\right)^{-\frac{p}{2}} J_{\frac{\mu-2-p}{2}}(2\pi\sqrt{zb}) J_{\frac{\mu-2+p}{2}}(2\pi\overline{\sqrt{zb}}) \tag{5.2}$$

(and the connection between $l^+_{p,\mu}(z,b)$ and $l^-_{p,\mu}(z,b)$ is given by (4.1)) we can extend the definition of $k_{p,\mu}(z,b)$ to $3 \leq Re(\mu)$ in a natural way.

In this part we will prove theorem 2.1, which extends the kernel formula (5.1) to any μ with $2 < Re(\mu)$.

Proof. We would like to use analytic continuation to extend 5.1 to $3 \leq Re(\mu)$. To do this we write equation 5.1 in the form:

$$\left(|y|^{-\mu}\left(\frac{y}{|y|}\right)^{-p} f\left(\frac{-1}{y}\right)\right)^{\wedge}(b) = \int_{\mathbb{C}} \widehat{f}(z) k_{p,\mu}(z,b) dz. \tag{5.3}$$

This formula is valid for $2 < Re(\mu) < 3$ for any smooth function $\widehat{f}$ with compact support. Fix such $\widehat{f}$ and b then both sides of (5.3) are analytic in μ by the differentiation lemma ([9] p.409). $\square$

We recall that so far, we have been working in a representation that is slightly different from the Kirillov model. However, it is now very easy to get a kernel formula in the Kirillov model. This is Theorem 2.3 which we now prove.

Proof. Let ψ_F be as in theorem 2.3. By definition we have $w \cdot \psi_F(b) = \psi_{w \cdot F}(b)$. Using (2.5) we get that $\psi_{w \cdot F}(b) = \chi_{p,\mu}\begin{pmatrix} b^{-1} & 0 \\ 0 & b \end{pmatrix} |b|^4 \widehat{f}_{w \cdot F}(b^2)$. Now we can use the kernel formula which we already proved to calculate $\widehat{f}_{w \cdot F}(b^2)$. Using (2.5) again and a simple change of variables now gives us a kernel formula with a kernel function $\kappa_{p,\mu}(z,b)$ that equals:

$$\kappa_{p,\mu}(z,b) = 2|b|^4 \chi_{p,\mu}(\begin{pmatrix} b^{-1} & 0 \\ 0 & b \end{pmatrix}) \chi_{p,\mu}(\begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}) k_{p,\mu}(z^2, b^2) \tag{5.4}$$

From (5.4) it is easy to get the explicit expression for $\kappa_{p,\mu}(z,b)$ that is stated in the theorem. $\square$

In order to get an analog formula for $GL(2,\mathbb{C})$ we first introduce some notations. We will now use $\mathbf{G}$ to denote $GL(2,\mathbb{C})$ and $\mathbf{B}$ its Borel subgroup. We denote by $\chi_{p_1,p_2,\mu_1,\mu_2}$ the following character of $\mathbf{B}$:

$$\chi_{p_1,p_2,\mu_1,\mu_2}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = |a|^{\mu_1}\left(\frac{a}{|a|}\right)^{p_1}|d|^{\mu_2}\left(\frac{d}{|d|}\right)^{p_2} \tag{5.5}$$

The induced space is now:

(5.6)

$$V_{p_1,p_2,\mu_1,\mu_2} = \left\{F : \mathbf{G} \to \mathbb{C} | \forall b \in \mathbf{B}, \forall g \in \mathbf{G}, F(bg) = \chi_{p_1,p_2,\mu_1,\mu_2}(b)F(g), F|_{SU(2)} \in L^2(SU(2))\right\}$$

Clearly our method of proof, of the kernel formula for $SL(2,\mathbb{C})$ is valid also for $GL(2,\mathbb{C})$ and we get the same kernel formula $k_{p,\mu}(z,b)$ with $p = p_1 - p_2$ and $\mu = \mu_1 - \mu_2$. In the Kirillov model we will have an analog formula to the one in theorem 2.3:

Theorem 5.1. *Let F be a smooth function in V_{p_1,p_2,μ_1,μ_2} such that $W_F\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$ has compact support and assume that μ_1, μ_2 are such that $2 < Re(\mu_1 - \mu_2)$. Then:*

$$W_F\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} w\right) = \int_{\mathbb{C}} \kappa_{p_1,p_2,\mu_1,\mu_2}(z,b) W_F\left(\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\right)\frac{dz}{|z|^2}$$

with:

$$\kappa_{p_1,p_2,\mu_1,\mu_2}(z,b) = \pi^2 \left|\frac{b}{z}\right|^{\frac{\mu_1+\mu_2}{2}} \left(\frac{\frac{b}{|b|}}{\frac{z}{|z|}}\right)^{\frac{p_1+p_2}{2}} |zb| \times$$

$$\times \frac{(-i)^p J_{\frac{\mu-2-p}{2}}(2\pi\sqrt{zb}) J_{\frac{\mu-2+p}{2}}(2\pi\sqrt{zb}) - (i)^p J_{\frac{-(\mu-2)-p}{2}}(2\pi\sqrt{zb}) J_{\frac{-(\mu-2)+p}{2}}(2\pi\sqrt{zb})}{\sin(\frac{\pi}{2}(2-\mu+p))}$$

References

[1] E. M. Baruch and D. Bump, A Voronoi Summation for Gaussian Integers, Preprint.

[2] E. M. Baruch and Z. Mao, Bessel identities in the waldspurger correspondence over the real numbers, *Israel J. Math.* **145** (2005),1-81.

[3] J. Bak and D. J. Newman, *Complex Analysis - Second Edition* , Springer-Verlag New York, Inc.

[4] James W. Cogdell and Ilya Piatetski-Shapiro, The arithmetic and spectral analysis of Poincare series, Perspectives in Mathematics, vol. 13, Academic Press Inc., Boston, MA, 1990.

[5] G. B. Folland, *Fourier analysis and its applications* (1992), Wadsworth, Inc.

[6] I. S. Gradshteyn and I. M. Ryzhik , *Table of Integrals, Series and Products , Fourth Edition* (1965), Academic Press Inc.

[7] I. M. Gelfand, M. I. Graev, and I. I. Pyatetskii-Shapiro, Representation theory and automorphic functions, Generalized Functions, vol. 6, Academic Press Inc., Boston, MA, 1990, Translated from the Russian by K. A. Hirsch, Reprint of the 1969 edition.

[8] A. W. Knapp, *Representation theory of semisimple groups* (1986), Princton University Press .

[9] S. Lang, *Complex Analysis - Fourth Edition*, Spriger-Verlag, Inc.

[10] N. N. Lebedev, *Special functions and their applications* (1965), PRENTICE-HALL, INC.

[11] Y. Motohashi, Mean values of zeta-functions via representation theory, *Proc. Symp. Pure Math., AMS* **75** (2006)

[12] N. Ja. Vilenkin, Special functions and the theory of group representations, Translated from Russian by V.N Singh. Translations of Mathematical Monographes, Vol.22, American Mathematical Society, Providence, R.I., 1968.

[13] G. N. Watson, *A treatise on the theory of Bessel functions* (1948), Cambridge University Press.

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