THE CLASSICAL HANKEL TRANSFORM IN THE KIRILLOV MODEL OF THE DISCRETE SERIES

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Abstract. We give a new and simple proof of the Hankel inversion formula for the classical Hankel transform of index ν which holds for $\mathrm{Re}(\nu) > -1$. Using the proof of this formula we obtain the full description of the Kirillov model for discrete series representations of $SL(2,\mathbb{R})$ and $GL(2,\mathbb{R})$.

1. Introduction

It has been noticed by [3] (See also [6],[17],[13], [2]) that the action of the Weyl element in the Kirillov model of an irreducible unitary representation of $GL(2,\mathbb{R})$ is given by a certain integral transform and that in the case of the discrete series this is a classical Hankel transform of integer order. It was proved in [4] that the classical Hankel transform of real order ν with $\nu > -1$ is an isomorphism of order two of a ceratin "Schwartz" space. In this paper we give a simple and elementary proof of this fact using "representation theoretic" ideas and extend the result to complex ν with $\mathrm{Re}(\nu) > -1$. Using these methods for the Hankel transform of integral order we determine the smooth space of the discrete series in the Kirillov model, thus giving for the first time an explicit model which is suitable for various applications. In a future paper we will describe the Kirillov model for the principal series and the complementary series and give an inversion formula for a generalized Hankel transform.

The inversion formula for the Hankel transform which states that the Hankel transform is self reciprocal was studied by many authors starting from Hankel in [9]. This formula is classicaly stated as follows ([20] p.453). Let $J_\nu(z)$ be the classical J-Bessel function. Let f be a complex valued function defined on the positive real line. Then under certain assumptions on f and ν (See [20] or Theorem 3.2) we have

$$f(z) = \int_0^\infty \int_0^\infty f(x) J_\nu(xy) J_\nu(yz) xy dx dy \tag{1.1}$$

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In more modern notation we define the Hankel transform of order ν of f to be

$$h_\nu(f)(y) = \int_0^\infty f(x)J_\nu(xy)xdx$$

Then under certain assumptions on f and ν the Hankel transform is self reciprocal, that is, $h_\nu^2 = Id$. A general discussion of the history of this result and various attempts at a proof can be found in ([20], p. 454). It is mentioned in [20] that Hankel was the first to give the formula in 1869 and that Weyl ([21] p.324) was the first to give a complete proof when f is in a certain space of twice differential functions and ν is real, $\nu > -1$. Watson gives a complete proof when ν is real, $\nu > -1/2$ and $\phi \in L^1((0,\infty), \sqrt{x}dx)$ and is of bounded variation in an interval around the point z in equation 1.1. MacRobert [12], proves the result for $\mathrm{Re}(\nu) > -1$ under the additional assumption that ϕ is analytic. In more recent results it was proved by [22] that the Hankel transform preserves a certain "Schwartz" space. Duran [4] gives a very elegant proof of the inversion formula on the "Schwartz" space when $\nu > -1$. In this paper we will extend the inversion formula on a "Schwartz" space to complex ν such that $\mathrm{Re}(\nu) > -1$. Our method is to "move" the Hankel transform from the Schwartz space on which it acts to a different space using Fourier transform which we think of as an "intertwining operator" . The main part of the proof is to compute the effect of this "intertwining operator" on the Hankel transform. The operator is built in such a way that the Hankel transform is replaced with an operator which sends a function ϕ on the real line to the function $|x|^{-\nu-1}e^{\mathrm{sgn}(x)\pi i(\nu+1)/2}\phi(-1/x)$. It is trivial to see that this operator is self reciprocal and from this follows the same result for the Hankel transform.

In the case where ν is a positive integer, the operator above is giving the action of the Weyl element of $SL(2,\mathbb{R})$ or $GL(2,\mathbb{R})$ on the discrete series representations. The operator which moves us from the action above to the Hankel transform is an honest intertwining operator for a discrete series representations which moves us from a model for the induced space to the Kirillov model. Using the results above we can determine completely the "smooth" space of this Kirillov model and give an explicit action in this model.

Our paper is divided as follows. In section 2 we describe the Schwartz space for the general Hankel transform of order ν with $\mathrm{Re}(\nu) > -1$ and prove that the Hankel transform preserves this space. In section 3 we define our " intertwining operator" and compute its composition with the Hankel transform. Using that we prove the inversion formula for the Hankel transform. In sections 4 to 7 we turn our attention to integer order and to the discrete series representations. We compute the full image of the induced space into the "Kirillov space" and find two invariant subspaces. We use the Fréchet topology to show that the invariant subspaces that we found are precisely the smooth space of the various discrete series representations of $SL(2,\mathbb{R})$. In section 8 we describe the Kirillov representation of $GL(2,\mathbb{R})$.

It is our purpose to make the presentation as elementary as possible and suitable for non representation theory specialists.

2. A Schwartz space for the classical Hankel transform

In this section we describe the Schwartz space ([1],[4],[22]) for the Hankel transform and show that this space is invariant under the Hankel transform.

Assume that $\mathrm{Re}(\nu) > -1$ and $x > 0$ and let $J_\nu(x)$ be the classical J-Bessel function defined by

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}$$

Let f be a complex valued function on $[0,\infty)$. Define the Hankel transform of order ν by

$$\mathcal{H}_\nu(f)(y) = \int_0^\infty f(x)\sqrt{xy}J_\nu(2\sqrt{xy})\frac{dx}{x}.$$

It is possible by a simple change of variable to go from this Hankel transform to the "classical" Hankel transform used in [20]. Let $S([0,\infty))$ be the Schwartz space of functions on $[0,\infty)$. That is, $f : [0,\infty) \to \mathbb{C}$ is in $S([0,\infty))$ if f is smooth on $[0,\infty)$ and f and all its derivatives are rapidly decreasing at ∞. Let

$$S_\nu([0,\infty)) = \{f : [0,\infty) \to \mathbb{C} | f(x) = x^{1/2+\nu/2} f_1(x) \text{ and } f_1 \in S([0,\infty))\}$$

Since $J_\nu(z)$ is of order z^ν when z is near zero, it follows that the integral defining $\mathcal{H}_\nu(f)$ above converges absolutely when $f \in S_\nu([0,\infty))$ and $\mathrm{Re}(\nu) > -1$. It follows from [4] that when ν is real, $\nu > -1$ then H_ν is a linear isomorphism of $S_\nu([0,\infty))$ satisfying $H_\nu^2 = Id$. Moreover, H_ν is an $L_2([0,\infty), dx/x)$ isometry. We will give a simple proof that H_ν is a linear isomorphism of $S_\nu([0,\infty))$ satisfying $H_\nu^2 = Id$ when $\mathrm{Re}(\nu) > -1$. We start by showing that H_ν preserves $S_\nu([0,\infty))$.

Let D be the differential operator

$$D = x\frac{d}{dx}.$$

Proposition 2.1. *Assume that Re(ν) > -1. Then*
(a) D maps $S_\nu([0,\infty))$ into $S_\nu([0,\infty))$
(b) $\mathcal{H}_\nu(f)(y)$ is rapidly decreasing in y.
(c) $\mathcal{H}_\nu(Df)(y) = -D(\mathcal{H}_\nu(f))(y)$.
(d) $y^{-1/2-\nu/2}\mathcal{H}_\nu(f)(y)$ is smooth on $[0,\infty)$.

Proof. (a) is immediate. (b), (c) and (d) are standard using differentation under the integral and integration by parts. We will prove (b).

We will need the following formula:

$$\frac{d}{dx}\left(y^{-1/2}x^{(\nu+1)/2}J_{\nu+1}(2\sqrt{xy})\right) = x^{\nu/2}J_\nu(2\sqrt{xy})$$

which follows from ([11], (5.3.5)). To prove (b) we shall use the formula to do a repeated integration by parts on the integral defining H_ν. Let $f(x) = x^{\nu/2+1/2} f_1(x)$ where $f_1 \in S([0,\infty))$. Then

$$\begin{aligned}\mathcal{H}_\nu(f)(y) &= y^{1/2}\int_0^\infty f_1(x)\left(x^{\nu/2}J_\nu(2\sqrt{xy})\right)dx \\ &= \int_0^\infty \frac{d}{dx}\left(f_1(x)\right)x^{(\nu+1)/2}J_{\nu+1}(2\sqrt{xy})dx\end{aligned} \tag{2.1}$$

Repeating this process n times will give

$$\mathcal{H}_\nu(f)(y) = y^{1-n}\int_0^\infty \frac{d^n}{(dx)^n}\left(f_1(x)\right)(xy)^{(\nu+n)/2}J_{\nu+n}(2\sqrt{xy})dx$$

Since the function $z^\mu J_\mu(z)$ is bounded for $Re(\mu) > 0$ on $[0,\infty)$ and the derivatives of f_1 are rapidly decreasing it follows that the integral above is bounded hence $|\mathcal{H}_\nu(f)(y)| << y^{1-n}$ and we have that (b) holds. □

Corollary 2.2. *Assume that $Re(\nu) > -1$. Then $\mathcal{H}_\nu$ maps $S_\nu([0,\infty))$ into itself.*

Proof. Let $f \in S_\nu([0,\infty))$. It follows from (d) that $\mathcal{H}_\nu(f)(y)$ is smooth on $(0,\infty)$ and from (a),(b) and (c) that $D^n(\mathcal{H}_\nu(f))(y)$ is rapidly decreasing for $n = 0,1,2,\ldots$ It follows immediately that $\mathcal{H}_\nu(f))(y)$ and all its derivatives are rapidly decreasing. By (d), $g(y) = y^{-1/2-\nu/2}\mathcal{H}_\nu(f))(y)$ is smooth on $[0,\infty)$ hence we can write $\mathcal{H}_\nu(f)(y) = y^{1/2+\nu/2}g(y)$ for $g \in S([0,\infty))$. □

3. The inversion formula for the Hankel transform

We now turn to prove the self reciprocity of the Hankel transform. The crucial idea is to define an operator ("intertwining operator") T_ν from the Schwartz space above to another space of smooth functions (but not Schwartz in the usual sense) which we will call I_ν and to compute the composition of the Hankel transform with this operator. We will find an operator $\mathcal{W}_\nu$ on I_ν so that the following diagram commutes:

$$\begin{array}{ccc} S_\nu & \xrightarrow{\mathcal{H}_\nu} & S_\nu \\ {\scriptstyle T_\nu}\downarrow & & \downarrow{\scriptstyle T_\nu} \\ I_\nu & \xrightarrow{\mathcal{W}_\nu} & I_\nu \end{array}$$

We now define the above operators. Let $f \in S_\nu([0,\infty))$. Extend f to $(-\infty,\infty)$ by setting it to be zero on the negative reals. We will denote this extension again by f. Assume $\mathrm{Re}(\nu) > -1$. Let $T_\nu(f) = (x^{-1/2+\nu/2}f)^\vee$ where $\check{f}$ is the inverse Fourier transform. That is, for $z \in \mathbb{R}$ we let

$$T_\nu(f)(z) = (2\pi)^{-1/2}\int_0^\infty x^{-1/2+\nu/2}f(x)e^{ixz}dx \tag{3.1}$$

Since f is rapidly decreasing at ∞ and of order $x^{1/2+\nu/2}$ at $x = 0$, it follows that the integral is absolutely convergent. From the Plancherel theorem and the fact that f is continuous on $(0,\infty)$ it follows that T_ν is one to one. We will show that T_ν maps $S_\nu([0,\infty))$ into a space of function I_ν which will be defined below. First we define the operator $\mathcal{W}_\nu$: Let $\phi:\mathbb{R}^*\to\mathbb{C}$. Define

$$\mathcal{W}_\nu(\phi)(x)=|x|^{-\nu-1}e^{\mathrm{sgn}(x)\pi i(\nu+1)/2}\phi(-1/x).$$

Our main theorem of this section is the following:

Theorem 3.1. *Let $f\in S_\nu([0,\infty))$ then $T_\nu\circ\mathcal{H}_\nu(f)=\mathcal{W}_\nu\circ T_\nu(f)$.*

Notice that this theorem is precisely the statement that the above diagram is commutative. We regard it as our main theorem since it allows us to move from the complicated Hankel transform $\mathcal{H}_\nu$ to the simple operator $\mathcal{W}_\nu$.

Proof. The proof is based on the Weber integral ([11], p.132):

$$\int_0^\infty u^{\nu+1}e^{-\alpha u^2}J_\nu(\beta u)du=\frac{\beta^\mu}{(2\alpha)^{\mu+1}}e^{-\frac{\beta^2}{4\alpha}} \tag{3.2}$$

where $\mathrm{Re}(\mu)>-1$, $\mathrm{Re}(\alpha)>0$ and $\mathrm{Re}(\beta)>0$. Assume that $\mathrm{Re}(\nu)>-1$. It follows from the dominated convergence theorem that if $f\in S_\nu([0,\infty))$ then

$$T_\nu(f)(z)=\lim_{\epsilon\to0^+}(2\pi)^{-1/2}\int_0^\infty y^{-1/2+\nu/2}f(y)e^{iyz}e^{-\epsilon y}dy$$

Hence

(3.3)

$$\begin{aligned}
&T_\nu\circ\mathcal{H}_\nu(f)(z)=\\
&=\lim_{\epsilon\to0^+}(2\pi)^{-1/2}\int_0^\infty y^{-1/2+\nu/2}e^{iyz}e^{-\epsilon y}\int_0^\infty f(x)\sqrt{xy}J_\nu(2\sqrt{xy})\frac{dx}{x}\,dy\\
&=\lim_{\epsilon\to0^+}(2\pi)^{-1/2}\int_0^\infty f(x)x^{-1/2}\int_0^\infty y^{\nu/2}e^{-(-iz+\epsilon)y}J_\nu(2\sqrt{xy})dydx\\
&=\lim_{\epsilon\to0^+}2(2\pi)^{-1/2}\int_0^\infty f(x)x^{-1/2}\left(\int_0^\infty u^{\nu+1}e^{-(-iz+\epsilon)u^2}J_\nu(2\sqrt{x}u)du\right)dx\\
&\overset{(3.2)}{=}\lim_{\epsilon\to0^+}(2\pi)^{-1/2}\int_0^\infty f(x)x^{-1/2+\nu/2}(-iz+\epsilon)^{-\nu-1}e^{-\frac{x}{\epsilon-iz}}dx\\
&=(2\pi)^{-1/2}|z|^{-\nu-1}e^{\mathrm{sgn}(z)\pi i(\nu+1)/2}\int_0^\infty f(x)x^{-1/2+\nu/2}e^{\frac{ix}{-z}}dx\\
&=|z|^{-\nu-1}e^{\mathrm{sgn}(z)\pi i(\nu+1)/2}T_\nu(f)(-1/z).
\end{aligned}$$

∎

As a Corollary we obtain the inversion formula of the Hankel transform on the Schwartz space. For the inversion formula on a bigger space see [20].

Theorem 3.2. *Assume $Re(\nu)>-1$ and $f\in S_\nu([0,\infty))$. Then*

$$\mathcal{H}_\nu\circ\mathcal{H}_\nu(f)=f$$

Proof. This follows immediately from Theorem 3.1 and the self reciprocity of $\mathcal{W}_\nu$,

$$\mathcal{W}_\nu \circ \mathcal{W}_\nu = Id$$

which is easy to check. The argument is the following. Let $f \in S_\nu([0,\infty))$. Then

$$T_\nu \circ \mathcal{H}_\nu \circ \mathcal{H}_\nu(f) = \mathcal{W}_\nu \circ T_\nu \circ \mathcal{H}_\nu(f) = \mathcal{W}_\nu \circ \mathcal{W}_\nu \circ T_\nu(f) = T_\nu(f).$$

Since T_ν is one to one, it follows that $\mathcal{H}_\nu \circ \mathcal{H}_\nu(f) = f$. □

We can also define the space I_ν (which we think of as an induced representation space).

(3.4) $\quad I_\nu = \{\phi : \mathbb{R} \to \mathbb{C} \mid \phi \text{ is smooth on } \mathbb{R} \text{ and } \mathcal{W}_\nu(\phi) \text{ is smooth on } \mathbb{R}\}.$

Proposition 3.3. *T_ν maps $S_\nu([0,\infty))$ into I_ν.*

Proof. Let $f \in S_\nu([0,\infty))$. We will show that $T_\nu(f) = \phi \in I_\nu$. Let $g(x) = x^{-1/2+\nu/2} f(x)$. Then $T_\nu(f) = \check{g}$. Since $x^n g(x)$ is absolutely integrable for every positive integer n, it follows from standard Fourier analysis that $\phi = \check{g}$ is smooth. Now apply T_ν to $\mathcal{H}_\nu(f)$. Since $\mathcal{H}_\nu(f) \in S_\nu([0,\infty))$ it follows from the same argument that $T_\nu(\mathcal{H}_\nu(f))$ is smooth. But from Theorem 3.1 this is precisely $\mathcal{W}_\nu(T_\nu(f)) = \mathcal{W}_\nu(\phi)$. □

3.1. L^2 **Isoemtry of the real Hankel transform.** The L^2 isometry of the Hankel transform is well known and follows from the Plancherel formula for the Hankel transform. (See for example [1]).

Let ν be real and $\nu > -1$. For $f_1, f_2 \in S_\nu([0,\infty))$ we let

$$< f_1, f_2 >= \int_0^\infty f_1(x)\overline{f_2(x)}\frac{dx}{x}$$

Proposition 3.4. *Assume ν is real and $\nu > -1$. Let $f_1, f_2 \in S_\nu([0,\infty))$. Then*

$$< \mathcal{H}_\nu(f_1)\, ,\, \mathcal{H}_\nu(f_2) > = < f_1\, ,\, f_2) > .$$

The proof is immediate using Fubini and the fact that $J_\nu(x)$ takes real values when ν and x are real.

4. The Induced space for the Discrete Series

In this section we define an action of the group $G = SL(2,\mathbb{R})$ on the space I_d where d is a positive integer. This is the smooth part of the full induced space of the Discrete series representation. Using the results in the first section we will compute the image of this space under the inverse of the intertwining map T_d which was defined in (3.1). The image gives us a "Kirillov space" for the full induced. It is well known that the full induced is reducible and has two closed invariant irreducible subspaces. We will later show that the image of one of them is the space $S_d([0,\infty))$. In this section we will show that the image of $S_d([0,\infty))$ under the operator T_d is invariant.

Let $G = SL(2,\mathbb{R})$. Let N and A be subgroups of G defined by

$$N = \left\{ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\} \quad A = \left\{ s(z) = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in \mathbb{R}^* \right\}.$$

Let

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad r(\theta) = \begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix}.$$

4.1. **The asymptotics description of the space I_d.** Let d be a positive integer. The space I_d is the space of smooth functions ϕ on $\mathbb{R}$ such that

$$\mathcal{W}_d(\phi)(x) = (i)^{d+1} x^{-d-1} \phi(-1/x)$$

is smooth. This is also the space of smooth functions ϕ such that

$$w(\phi)(x) = x^{-d-1} \phi(-1/x)$$

is smooth. We first prove some basic properties of this space:

Lemma 4.1. *The differential operators d/dx and $D = x(d/dx)$ map I_d into itself.*

Proof. Let $\phi \in I_d$. Then $w(\phi)(x)$ is smooth hence $w(\phi)'(x) = -(d+1)x^{-d-2}\phi(-1/x) + x^{-d-3}\phi'(-1/x)$ is smooth. Hence $xw(\phi)'(x) = -(d+1)x^{-d-1}\phi(-1/x) + x^{-d-2}\phi'(-1/x)$ is smooth. Since $x^{-d-1}\phi(-1/x)$ is smooth it follows that $x^{-d-2}\phi'(-1/x)$ is smooth. It is clear that $\phi'(x) = d/dx\,\phi$ and $D\phi$ are smooth. To finish the proof we need to show that $w(\phi')$ and $w(D\phi)$ are smooth. But $w(D\phi)(x) = -x^{-d-2}\phi'(-1/x)$ which we showed was smooth and $w(\phi')(x) = -xw(D\phi)(x)$ hence is also smooth. □

We shall now give another description of I_d.

4.1.1. *Asymptotic expansions.* We recall some results from the theory of Asymptotic Expansions. For the proofs see ([5], [15], [16],[14]). Let $\phi : \mathbb{R} \to \mathbb{C}$. Let $a_0, a_1, \ldots$ be complex numbers. We say that ϕ has the asymptotic expansion

$$\phi(x) \approx \sum_{m=0}^{\infty} a_m x^{-m}$$

at infinity if $\phi(x) - \sum_{m=0}^{N} a_m x^{-m} = O(|x|^{-N-1})$ when $|x| \to \infty$ for all non negative integers N where the implied constant is dependent on N. (Notice that we have grouped together ∞ and $-\infty$ which is not needed in a more general definition.) It is easy to see that the constants a_m are determined uniquely by ϕ although ϕ is not determined uniquely by $\{a_m\}$. (For example, $\phi(x) + e^{-|x|}$ will have the same asymptotic expansion as ϕ.). The following results are well known ([16]):

Lemma 4.2. *Let N be a non negative integer and c be a real constant. Then $\phi(x) = (x+c)^{-N}$ has an asymptotic expansion of the form*

$$(x+c)^{-N} \approx x^{-N} - Ncx^{-N-1} + \ldots$$

Corollary 4.3. *If $\phi(x)$ has an asymptotic expansion then $\phi(x+c)$ has an asymptotic expansion for every real constant c.*

Proposition 4.4. *Assume that ϕ is differentiable and both ϕ and ϕ' have asymptotic expansions. Then the asymptotic expansion of ϕ' is the derivative term by term of the asymptotic expansion of ϕ. That is*

$$\phi'(x) \approx \sum_{m=0}^{\infty} -m a_m x^{-m-1}$$

Theorem 4.5. *Assume that ϕ is smooth on $\mathbb{R}$ and let $\gamma(x) = \phi(1/x)$. Then ϕ and all its derivatives have asymptotic expansions on $\mathbb{R}$ if and only if γ is smooth at $x = 0$. (hence if and only if γ is smooth on $\mathbb{R}$).*

Proof. If γ is smooth at $x = 0$ then the asymptotic expansions of ϕ and all its derivative follow from the Taylor expansion for γ and its derivatives. In the other direction, the smoothness of γ at $x = 0$ follows from the proof of (Theorem 3, [16]). There it is proved from an estimate on $\gamma^{(m)}(x)$. (See [16] (14) with $l = 0$.) □

Proposition 4.6. *The space I_d is the space of smooth functions ϕ such that ϕ and all its derivatives have asymptotic expansions and such that the asymptotic expansion for ϕ is of the form*

$$\phi(x) \approx a_{d+1} x^{-d-1} + \dots \tag{4.1}$$

Proof. Let $\phi \in I_d$ and let $\gamma = w\phi$. Since γ is smooth and since $\phi(x) = x^{-d-1}\gamma(-1/x)$ is smooth, it follows from Theorem 4.5 that ϕ has the required asymptotic expansion. Now assume that ϕ (and its derivatives) has the asymptotic exapnsion (4.1). Then $\alpha(x) = x^{d+1}\phi(x)$ is smooth and has an asymptotic exapansion. It follows from Theorem 4.5 that $\gamma(x) = \alpha(-1/x) = x^{-d-1}\phi(-1/x) = (w\phi)(x)$ is smooth hence $\phi \in I_d$. □

We now define a representation π_d of $G = SL(2,\mathbb{R})$ on I_d in the following way: Let $\phi \in I_d$.

$$(w\phi)(x) = (i)^{d+1}\mathcal{W}_d(\phi)(x) = x^{-d-1}\phi(-1/x) \tag{4.2}$$

$$(n(y)\phi)(x) = \phi(x+y) \tag{4.3}$$

$$(s(z)\phi)(x) = z^{-d-1}\phi(z^{-2}x) \tag{4.4}$$

The fact that $\phi(x+y) \in I_d$ follows from Corollary 4.3. It is easy to see that these maps give isomorphisms of I_d. More generally we have

$$\left(\pi_d \begin{pmatrix} p & q \\ r & s \end{pmatrix} \phi\right)(x) = (rx+p)^{-d-1}\phi(\frac{sx+q}{rx+p}) \tag{4.5}$$

where the matrix above is in G. It is easy to check that this gives a representation of G, that is $\pi_d(g_1 g_2) = \pi_k(g_1)\pi_d(g_2)$ for all $g_1, g_2 \in G$.

We shall now define an “intertwining operator” M_d on I_d and compute its image. This operator is the inverse of the operator T_d and will take us back to the "Kirillov model". For $\phi \in I_d$ we define

$$M_d(\phi)(y) = |y|^{(-d+1)/2}\hat{\phi}(y) = |y|^{(-d+1)/2}(2\pi)^{-1/2}\int_{-\infty}^{\infty}\phi(x)e^{-ixy}dx.$$

We view $M_d(\phi)(y)$ as a function on $\mathbb{R} - \{0\}$. Its behaviour at zero is central to the description of this mapping. Let $\tilde{S}_d = M_d(I_d)$.

Lemma 4.7. *Let $f \in \tilde{S}_d$. Then f is differentiable for every $y \neq 0$. The operator $D = y(d/dy)$ maps $\tilde{S}_d$ into itself.*

Proof. We first assume that $d > 1$. The case $d = 1$ will be proved later. Let $\phi \in I_d$. Then $x\phi(x) \in L^1(\mathbb{R})$ hence $\hat{\phi}$ is differentiable. We have $(D\phi)^\wedge(y) = -\hat{\phi}(y) - D\hat{\phi}(y)$ and

$$M_d(D(\phi))(y) = |y|^{(-d+1)/2}(D\phi)^\wedge(y) = -|y|^{(-d+1)/2}\hat{\phi}(y) - |y|^{(-d+1)/2}D\hat{\phi}(y).$$

Since $D\phi \in I_d$ it follows that $|y|^{(-d+1)/2}D(\hat{\phi})(y) \in \tilde{S}_d$. Now $D(|y|^{(-d+1)/2}\hat{\phi}(y)) = \frac{-d+1}{2}|y|^{(-d+1)/2}\hat{\phi}(y) + |y|^{(-d+1)/2}D\hat{\phi}(y)$ Since both summands belong to $\tilde{S}_d$ we get our conclusion. □

Corollary 4.8. *If $f \in \tilde{S}_d$ then f is smooth at every $y \neq 0$.*

Proof. We prove by induction on m that f is m times differentiable. For $m = 1$ this follows from the Lemma. Also by the Lemma $D^m f \in \tilde{S}_d$ hence it is differentiable. But $D^m f(y)$ equals $y^m f^{(m)}(y)$ plus a sum involving lower order derivatives of f. Since these summands are all differentiable from the induction assumption, it follows that $y^m f^{(m)}(y)$ is differentiable hence $f^{(m)}(y)$ is differentiable for $y \neq 0$. □

Lemma 4.9. *Let $f \in \tilde{S}_d$. Then f is a Schwartz function, that is, f and all its derivatives are rapidly decreasing at ∞ and $-\infty$.*

Proof. Let $\phi \in I_d$ be such that $f = M_d(\phi)$. Since ϕ and all its derivatives are in $L^1(\mathbb{R})$ it follows that $\hat{\phi}$ is rapidly decreasing hence $f(y) = |y|^{(-d+1)/2}\hat{\phi}(y)$ is rapidly decreasing. Since $D(f) \in \tilde{S}_d$ it follows that $D(f)$ is rapidly decreasing hence f' is rapidly decreasing. Using induction and the fact that $D^m(f)$ is rapidly decreasing we get that $f^{(m)}$ is rapidly decreasing. □

We shall now describe the behaviour at zero of the functions in our space $\tilde{S}_d$. We will also complete the case $d = 1$ that was left out in the proof of Lemma 4.7. We let

$$\phi_0(x) = (1 + x^2)^{(-d-1)/2}$$

and $\phi_j(x) = \phi_0^{(j)}$, the jth derivative of ϕ_0. It is easy to check that $\phi_0 \in I_d$ hence by Lemma 4.1, $\phi_i \in I_d$ for every positive integer i. We shall compute the functions $f_j = (\hat{\phi}_j)$. From ([8] 3.771 (2)) it follows that

$$f_0(y) = \frac{2^{(d+1)/2}}{\Gamma((d-1)/2)} |y|^{d/2} K_{d/2}(|y|) \tag{4.6}$$

$$f_j(y) = (-i)^j y^j f_0(y) = \frac{2^{(d+1)/2}}{\Gamma((d-1)/2)} (-i)^j y^j |y|^{1/2} K_{d/2}(|y|) \tag{4.7}$$

Lemma 4.10. *$f_i(y)$ are smooth on $y \neq 0$. They are smooth on the right and smooth on the left at $y = 0$. $f_0(y)$ is continuous at $y = 0$. $f_j(y)$ is j times differentiable at $x = 0$ and satisfies $f_j^{(r)}(0) = 0$ for $r = 0, 1, \ldots, j-1$ and $f_j^{(j)}(0) \neq 0$*

Proof. The first part of the lemma follows from the fact that $f_0(y)$ is a linear conbination of functions of the form $|y|^t e^{-|y|}$ where t is a non-negative integer. This follows from the fact that f_0 is even and is given by ([8] 3.737 (1)). Since f_j is a multiple of f_0 by a constant times an integer power of y the smoothness of f_j also follows. The continuity of f_0 follows from the same reason or from the fact that it is a Fourier transform of an L^1 function. Since f_0 is smooth for $y \neq 0$ and continuous at $y = 0$ it follows that the function $y^j f_0(y)$ is j times differentiable and that the mth derivative vanihes at $y = 0$ when $m < j$. Since $f_0(0) \neq 0$ as it is an integral of a positive function, it follows that $f_j^{(j)}(0) \neq 0$. □

Proof of Lemma 4.7 for $d = 1$

We need to prove that $M_1(\phi)(y)$ is differentiable at $y \neq 0$. The rest of the proof will follow the same lines as in the proof of Lemma 4.7. Let $\phi \in I_1$. Then ϕ has an asymptotic expansion of the form $\phi(x) \approx a_2 x^{-2} + \ldots$ Let $\alpha(x) = \phi(x) - a_2\phi_0(x) = \phi(x) - a_2(1 + x^2)^{-1}$. Then $\alpha(x)$ and $x\alpha(x)$ are in L^1 hence $\hat{\alpha}$ is differentiable. Now $\hat{\phi}(y) = \hat{\alpha}(y) + a_2\hat{\phi}_0(y) = \hat{\alpha}(y) + 2a_2 e^{-|y|}$ is differentiable. The same argument will be used in the proof of the following corollary.

Corollary 4.11. *Let $f \in \tilde{S}_d$. Then $|y|^{-d+1)/2} f(y)$ is smooth on the right and on the left at $y = 0$.*

Proof. There exist $\phi \in I_d$ such that $|y|^{(d-1)/2} f(y) = \hat{\phi}(y)$. We need to show that $\hat{\phi}$ is smooth on the left and on the right at $y = 0$. For each integer $m > 0$ we can write $\phi = \alpha_m + \sum_{j=0}^{m} a_{j+d-1}\phi_j$ so that α_m satisfies that $x^r \alpha_m(x) \in L^1(\mathbb{R})$ for $r = 0, 1, \ldots, m+d-1$. It follows that $\hat{\alpha_m}$ is $m+d-1$ times differentiable on $\mathbb{R}$. But $\hat{\phi} = \hat{\alpha}_m + \sum_{j=0}^{m} a_{j+d-1} f_j$ and since f_j are smooth on the right and on the left at $y = 0$ it follows that $\hat{\phi}$ is $m + d - 1$ smooth on the right and on the left at $y = 0$. Since this is true for all m we get the conclusion. □

Lemma 4.12. *Let $f \in \tilde{S}_d$. Then $|y|^{(d-1)/2} f(y)$ has continuous $d-1$ derivatives at $y = 0$.*

Proof. There exist $\phi \in I_d$ such that $|y|^{d-1/2} f(y) = \hat{\phi}(y)$. From the asymptotics of ϕ it follows that $\phi(x), x\phi(x), \ldots, x^{d-1}\phi(x)$ are all in $L^1(\mathbb{R})$. The conclusion follows. □

We will show that Lemma 4.9, Corollary 4.11 and Lemma 4.12 give a complete description of $\tilde{S}_d$. To do that we define the inverse map T_d by $T_d(f) = (|y|^{(d-1)/2} f)^\vee$. That is,

$$T_d(f)(z) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} |y|^{(d-1)/2} f(y) e^{iyz} dy$$

Notice that if f vanishes on $(-\infty, 0)$ then this is the same as T_d that was defined in (3.1).

Proposition 4.13. *Let f be such that $|y|^{(-d-1)/2} f(y)$ is smooth on $[0, \infty)$ and on $(-\infty, 0]$ (but not necessarily smooth at $y = 0$) and such that f and all its derivatives are rapidly decreasing. Then $f \in \tilde{S}_d$.*

Proof. It is enough to show that $T_d(f) \in I_d$ since $f = M_d(T_d(f))$ hence is in $\tilde{S}_d$. To do that we write $f = f_+ + f_-$ where f_+ vanishes on $(0, -\infty)$ and is smooth on $[0, \infty)$ and f_- vanishes on $(0, \infty)$ and is smooth on $(-\infty, 0]$. (Notice that $f(0) = 0$.) Now $f_+ \in S_d([0, \infty))$ and by Proposition 3.3 $T_d(f_+) \in I_d$. Similar arguments as in the proof of Proposition 3.3 will show that $T_d(f_-) \in I_d$ hence $T_d(f) \in I_d$. □

Theorem 4.14. *$\tilde{S}_d$ is the set of smooth functions f on $\mathbb{R} - \{0\}$ such that f and all its derivatives are rapidly decreasing at $\pm\infty$ and such that $|y|^{(d-1)/2} f(y)$ is smooth on the right and on the left at $y = 0$ and has $d - 1$ continuous derivatives at $y = 0$. (When $d = 1$, this means that $|y|^{d-1} f(y) = f(y)$ is continuous.)*

Proof. By Lemma 4.9, Corollary 4.11 and Lemma 4.12 $\tilde{S}_d$ is contained in the space of functions satisfying the above conditions. Now assume that f is as above and we will show that it is in $\tilde{S}_d$, that is, it is the image under M_d of a function in I_d. Let $\tilde{f} = |y|^{(d-1)/2} f(y)$. By the properties of $\{f_k\}$ in Lemma 4.10 we can find using a triangulation argument constants $c_0, \ldots, c_{d-1}$ so that

$$\tilde{h} = \tilde{f} - \sum_{j=0}^{d-1} c_j \tilde{f}_j$$

vanishes at zero and all its first $d - 1$ derivatives vanish at zero. It follows that $h(y) = |y|^{(-d+1)/2} \tilde{h}$ satisfies the conditions of Proposition 4.13 hence is in $\tilde{S}_d$. Hence $f = h + \sum_{j=0}^{d-1} c_j f_j$ is in $\tilde{S}_d$. □

Since I_d is isomorphic to $\tilde{S}_d$ under the isomorphism M_d it follows that the action of G on I_d induces an action of G on $\tilde{S}_d$. Let I_d^+ be the subspace of I_d which is given by

$$I_d^+ = T_d(S_d([0, \infty))). \tag{4.8}$$

By Corollary 2.2 $\mathcal{H}_d$ preserves the space $S_d([0,\infty))$. Hence by Theorem 4.14, w preserves the space I_d^+. It is easy to see using (4.3) and(4.4) that the induced action of $n(y)$ and $s(z)$ on $\tilde{S}_d$ stabilize the space $S_d([0,\infty))$. Hence it follows that $n(y)$ and $s(z)$ stabilize the space I_d^+.Since w, $n(y)$ and $s(z)$ generate G (when y and z vary) we get the following corollary:

Corollary 4.15. *I_d^+ is a G invariant subspace of I_d.*

Let
(4.9)
$$S_d((-\infty,0]) = \{f : (-\infty,0], \to \mathbb{C} \mid f(x) = x^{(d+1)/2} f_1(x) \text{ and } f_1 \in S((-\infty,0])\}$$

For $y \in (-\infty,0]$ and $f \in S_d((-\infty,0])$ we define

$$\mathcal{H}_d(f)(y) = \int_{-\infty}^{0} f(x)\sqrt{|xy|}J_d(2\sqrt{|xy|})\frac{dx}{|x|}.$$

It is clear that $S_d((-\infty,0])$ is a subspace of $\tilde{S}_d$. It follows from Corollary 2.2 that $\mathcal{H}_d$ stabilizes $S_d((-\infty,0])$ and that the induced action of $n(y)$ and $s(z)$ given by

$$(n(y)f)(x) = e^{iyx} f(x), \quad (s(z)f)(x) = f(z^2x) \tag{4.10}$$

also stabilize $S_d((-\infty,0])$. Define

$$I_d^- = T_d(S_d((-\infty,0])). \tag{4.11}$$

Then I_d^- is a G invariant subspace of I_d. We will show in Section 6 that $I_d^{\pm}$ is a closed subspace under the Frechet topology on I_d hence (from the theory of $(\mathfrak{g},K)$ modules) $I_d^+ \oplus I_d^-$ is the unique maximal closed invariant subspace of I_d.

5. The Kirillov Model of the discrete series

The Kirillov model is a particular realization of the representations of $SL(2,\mathbb{R})$ or $GL(2,\mathbb{R})$ (or more generally, representations of $GL(n,F)$ where F is a local field) with a prescribed action of the Borel subgroup. In this section we will describe the Kirillov model of the discrete series representations of $G = SL(2,\mathbb{R})$. Our main theorem of this section and this paper is a description of the smooth space of the Kirillov model. This Theorem will be stated in this section and proved in the next two sections.

5.1. **The action of the group in the Kirillov model.** Let d be a positive integer. Using the map T_d and its inverse M_d we can move the action of G from the space I_d^+ to the space S_d. We shall denote this action by R_d^+. That is, if $g \in G$ then the action of g on $f \in S_d$ is given by

$$R_d^+(g)(f) = M_d(gT_d(f)).$$

Thus we obtain the following formulas. For each $f \in S_d([0,\infty))$ we have

(5.1) $$(R_d^+(n(y))f)(x) = e^{iyx} f(x)$$

(5.2) $$(R_d^+(s(z))f)(x) = f(z^2 x)$$

(5.3) $$(R_d^+(w)f)(x) = i^{-(d+1)} \mathcal{H}_d(f)(x)$$

Let

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

be a basis for the Lie Algebra. The action of the Lie Algebra on the Kirillov model above is given by

(5.4)

$$(R_d^+(X)f)(x) = ixf(x) \quad (R_d^+(H)f)(x) = 2xf'(x) \quad (R_d^+(Y)f)(x) = -i\frac{d^2-1}{4x} f(x) + ixf''(x)$$

Another (non-isomorphic) action can be obtained by considering the action of G on $S_d((-\infty,0])$ (see (4.9) and the remark below it) and moving this action to $S_d([0,\infty))$. The action is given by

(5.5)

$$(R_d^-(n(y))f)(x) = e^{-iyx} f(x) \quad (R_d^-(s(z))f)(x) = f(z^2 x) \quad (R_d^-(w)f)(x) = i^{d+1} \mathcal{H}_d(f)(x)$$

It is easy to see that the action of $n(y)$ and $s(z)$ extend to the space $L^2((0,\infty), dx/x)$. It follows from Corollary 3.4 that the action of w also extends to $L^2((0,\infty), dx/x)$. We denote again by $R_d^\pm$ the representation on the space $\mathcal{H}_d = L^2((0,\infty), dx/x)$ obtained in such a way. This is called the Kirillov Hilbert representation of G.

Proposition 5.1. *$R_d^\pm$ is a strongly continous unitary representation on $L^2((0,\infty), dx/x)$.*

Proof. It is enough to show that the map $g \to R_d^\pm(g)f$ is continuous at $g = w$ for every $f \in L^2((0,\infty), dx/x)$.([10] p.11) This is easy to show for a characteristic function of an interval using the formulas in (5.1) and by approximation for a general function. □

Proposition 5.2. *$R_d^\pm$ is an irreducible representation.*

Proof. This representation is already irreducible when restricted to the Borel subgroup. (See [10] proof of Proposition 2.6) □

Our main theorem of this section is the following.

Theorem 5.3. *The space of smooth vectors of $\mathcal{H}_d = L^2((0,\infty), dx/x)$ under the action of $R_d^\pm$ is $S_d([0,\infty))$. That is, $\mathcal{H}_d^\infty = S_d([0,\infty))$.*

To prove Theorem 5.3 we will need to compare the Fréchet topologies on the induced space and on the Kirillov space.

6. The Fréchet topology

In this section we describe the Fréchet topology on the different smooth spaces that we are considering. This topology will play a role in determining the smooth space of the Kirillov model.

Let $G = SL(2,\mathbb{R})$. Let d be a positive integer and let

(6.1)
$$Ind_d = \{F : G \to \mathbb{C}|\ F \text{ is smooth and } \ F(n(x)s(a)g) = a^{d+1}F(g), \ \text{ for all } x \in \mathbb{R}, a \in \mathbb{R}^*\}$$

G acts on this space by right translations and $\mathfrak{g} = \mathrm{Lie}(G)$ acts by left invariant differential operators and induces an action of the enveloping algebra $U(\mathfrak{g})$ on Ind_d. There is an L^2 norm defined on this space by

$$||F||^2 = \frac{1}{2\pi}\int_0^{2\pi} |F(r(\theta))|^2 d\theta \tag{6.2}$$

The space Ind_d is given the Fréchet topology defined by the seminorms

$$||F||_D = ||D(F)||, \quad D \in U(\mathfrak{g}).$$

We will now show that the space Ind_d is isomorphic as a vector space to the space I_d defined in (3.4). For each $F \in Ind_d$ define

$$\phi_F(x) = F(wn(x)) \ . \tag{6.3}$$

The following proposition is well known:

Proposition 6.1. *The mapping $F \to \phi_F$ is an isomorphism of vector spaces.*

Proof. Let $F \in Ind_d$. Since F is smooth on G it follows that ϕ_F is smooth on $\mathbb{R}$. It is easy to check that $\phi_{wF} = w\phi_F$ where $w\phi_F(x) = x^{-d-1}\phi_F(-1/x)$ hence $w\phi_F$ is smooth and we have that the mapping sends Ind_d into I_d. Since the set of elements of the form $n(y)s(a)wn(x)$ is dense in G, it follows that a function F in Ind_d is determined by its restriction to $wn(x)$ hence the mapping $F \to \phi_F$ is one to one. It remains to show that this mapping is onto. We have

$$wn(x) = s((1+x^2)^{-1/2})n(-x)r(\theta_x) \tag{6.4}$$

where $\theta_x \in (0,\pi)$ is determined by the equalities $sin(\theta_x) = (1+x^2)^{-1/2}$, $cos(\theta_x) = -x(1+x^2)^{-1/2}$. Given $\phi \in I_d$ we define

$$F_\phi(r(\theta_x)) = (1+x^2)^{(d+1)/2}\phi(x) \tag{6.5}$$

for $\theta_x \in (0,\pi)$ and define

$$F_\phi(r(\pi)) = F_\phi(-I) = \lim_{x\to\infty}(1+x^2)^{(d+1)/2}\phi(x) = (w\phi)(0).$$

F_ϕ is defined on all of G via the equivariance property

$$F_\phi(n(y)s(a)r(\theta)) = sgn(a)a^{d+1}F_\phi(r(\theta)) \tag{6.6}$$

and the fact that each $g \in G$ can be written uniquely in the form $g = n(y)s(a)r(\theta)$ for $\theta \in (0,\pi]$. Since $F_\phi(r(\theta-\pi)) = F_\phi(r(-\pi)r(\theta)) = (-1)^{d+1}F_\phi(r(\theta))$

it follows that F_ϕ satisfies (6.6) for all θ. It remains to show that $F_\phi(r(\theta))$ is smooth as a function of θ (which will show that F_ϕ is smooth on G.) This is obvious by (6.5) when $\theta \in (0, \pi)$. For $\theta = \pi$ we notice the general equality $F_\phi(r(\theta)) = F_{w\phi}(r(\theta - \pi/2))$ hence the smoothness of F_ϕ at $\theta = \pi$ is equivalent to the smoothness of $F_{w\phi}$ at $\theta = \pi/2$ which was already established. It is also immediate that $\phi_{F_\phi} = \phi$. □

The isomorphism $F \to \phi_F$ defined above induces a Fréchet topology on the space I_d. To give this topology explicetely we need:

Proposition 6.2. *Let $F \in Ind_d$ and $\phi_F : \mathbb{R} \to \mathbb{C}$ as defined above. Then*

$$||F||^2 = \frac{1}{\pi} \int_{-\infty}^{\infty} (1 + x^2)^d |\phi_F(x)|^2 dx$$

Proof. This is immediate using (6.3)and (6.4). □

Hence we define a Fréchet topology on I_d using the seminorms

$$||\phi||_{d,D} = \int_{-\infty}^{\infty} (1 + x^2)^d |(D\phi)(x)|^2 dx, \quad D \in U(\mathfrak{g}).$$

It is useful to replace the seminorms given in the Fréchet topology of Ind_d with an equivalent set of seminorms which gives the topology of uniform convergence of functions and derivatives. That is, For $F \in Ind_d$ define $||F||_\infty = \max_\theta |F(\theta)|$ be the L-infinity norm on Ind_d. It is well known ([18] Theorem 1.8) that the set of seminorms

$$||F||_{\infty,D} = ||D(F)||_\infty, D \in U(\mathfrak{g})$$

is an equivalent set of seminorms to the set above and gives the same Fréchet topology. Moreover, let

$$\partial = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ and } \quad \partial(F)(r(\theta)) = \frac{\partial}{\partial\theta} F((r(\theta))).$$

Then it is enough to have the seminorms

$$||F||_{\infty,n} = ||\partial^n(F)||_\infty.$$

Hence we get the following Corollary:

Corollary 6.3. *The Fréchet topology on I_d is given by the set of seminorms*

$$||f||_{\infty,d,D} = |(1 + x^2)^{(d+1)/2}(D\phi)(x)|_\infty, \quad D \in U(\mathfrak{g})$$

or by the subset of seminorms

$$||f||_{\infty,d,\partial^n} = |(1 + x^2)^{(d+1)/2}(\partial\phi)(x)|_\infty, \quad n = 0, 1, ...$$

Proposition 6.4. *The functional $l_{\lambda,n}$ on I_d defined by*

$$l_{\lambda,n}(\phi) = \int_{-\infty}^{\infty} t^n \phi(t) e^{-i\lambda t} dt$$

is continous for $n = 0, ..., d - 1$ and $\lambda \in \mathbb{R}$.

Proof. Assume that $\phi_m \in I_d$ satisfy $\lim_{m\to\infty} \phi_m = 0$. in the Fréchet topology. Then $||(1+t^2)^{(d+1)/2}\phi_m)||_\infty \to 0$ hence there exist a sequence of positive constants c_m such that

$$|\phi_m(t)| \le c_m(1+t^2)^{-(d+1)/2}, \quad \text{for all } t \in \mathbb{R}.$$

and $\lim_{m\to\infty} c_m = 0$. It follows that

$$|l_{\lambda,n}(\phi_m)| \le c_m \int_{-\infty}^{\infty} |t|^n (1+t^2)^{-(d+1)/2} dt.$$

The integral above converges by the assumption on n hence we have that $\lim_{m\to\infty} l_{\lambda,n}(\phi_m) = 0$. □

Let I_d^+ and I_d^- be the subspaces of I_d defined in (4.8), (4.11). Then it is easy to see that

$$I_d^+ = \{\phi \in I_d \mid l_{\lambda,n}(\phi) = 0, \text{ for all } 0 \le n \le d-1 \text{ and } \lambda \le 0\}$$

and

$$I_d^- = \{\phi \in I_d \mid l_{\lambda,n}(\phi) = 0, \text{ for all } 0 \le n \le d-1 \text{ and } \lambda \ge 0\}$$

Corollary 6.5. *I_d^+ and I_d^- are closed invariant subspaces of I_d. They are irreducible as Fréchet representations.*

Proof. By Proposition 6.4 they are closed as intersection of closed subspaces. By Corollary 4.15 and the discussion below it they are G invariant subspaces. By the theory of $(\mathfrak{g}, K)$ modules (See for example [7] p. 2.8 Theorem 2) I_d has exactly three nontrivial closed invariant subspaces: two irreducible (lowest weight and highest weight) subspaces and their sum. It follows that I_d^+ and I_d^- are irreducible. □

Our next goal is to show that $S_d([0,\infty))$ is a space of smooth vectors in $\mathcal{H}_d = L^2((0,\infty), dx/x)$. That is, our first step in showing that $\mathcal{H}_d^\infty = S_d([0,\infty))$ is to show that $S_d([0,\infty)) \subseteq \mathcal{H}_d^\infty$. To do that we will need to compare the L^2 inner product on $S_d([0,\infty))$ with the standard (compact) inner product on Ind_d. Combining the isomorphism T_d from S_d to I_d and the isomorphism $\phi \to F_\phi$ defined in (6.5),(6.6) from I_d to Ind_d we get a one to one mapping from $S_d([0,\infty))$ to Ind_d and a subspace of Ind_d. This infinite dimensional subspace has two different inner products. The standard inner product on the induced space (6.2) and the inner product induced from the L^2 inner product on S_d. Following ([7], (288)) we will give an explicit formula for this induced inner product and compare the two.

We shall define an invariant "norm" on Ind_d. (It is in fact a norm on the invariant subspaces)

$$||f||_I^2 = \frac{1}{2\pi}\int_0^{2\pi} A(f)(r(\theta))\overline{f(r(\theta))}d\theta$$

where $A(f)$ is the intertwining operator defined by

$$A(f)(g) = \int_{-\infty}^{\infty} f(w^{-1}n(x)g)dx, \quad g \in G, f \in Ind_d.$$

Remark 6.6. Since $f(r(\theta - \pi)) = (-1)^d f(r(\theta))$ for every θ and since the same holds for $A(f)$ we have that

$$||f||_I = \frac{1}{\pi}\int_0^{\pi} A(f)(r(\theta))\overline{f(r(\theta))}d\theta$$

For $x \in \mathbb{R}$ we define $\theta_x \in (0, \pi)$ by

$$\theta_x = \cot^{-1}(-x)$$

We have

$$wn(x) = s((1 + x^2)^{-1/2})n(-x)r(\theta_x) \tag{6.7}$$

and

$$w^{-1}n(x) = s(-(1 + x^2)^{-1/2})n(-x)r(\theta_x) \tag{6.8}$$

Hence using the change of variable $\theta = \cot^{-1}(x)$ we have

$$A(f)(g) = \int_0^{\pi} f(r(\theta)g)(sin(\theta))^{d-1}d\theta$$

and in particular $|A(f)(r(\theta))| \leq \pi \, ||f||_{\infty}$. Hence we have proved:

Lemma 6.7.

$$||f||_I \leq \pi \, ||f||_{\infty}$$

Proposition 6.8. *Let $f \in I_d^+$. Then*

$$\frac{(-2\pi i)^d}{2((d-1)!)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} sgn(y)y^{d-1}f(x - y)\overline{f(x)}dydx = ||M(f)||_2^2 \tag{6.9}$$

where $||M(f)||_2$ is the L^2 norm of $M(f)$ in the space $L^2\left((0, \infty), dx/x)\right)$.

Proof. The proof is the same as in ([7] p. 1.62 - 1.64). We will use the Tate identity

$$\int_{-\infty}^{\infty} f(y)sgn(y)y^{d-1}dy = \frac{2((d-1)!)}{(-2\pi i)^d}\int_{-\infty}^{\infty} \hat{f}(x)x^{-d}dx$$

which is valid for $f \in I_d$. It follows from this identity that the left hand side of (6.9) is equal to

$$\begin{aligned}
&\int_{-\infty}^{\infty} \overline{f(x)}e^{2\pi ixy}\int_{-\infty}^{\infty} \hat{f}(y)y^{-d}dydx \\
&= \int_{-\infty}^{\infty} \overline{\hat{f}(y)}\hat{f}(y)y^{-d}dy \\
&= \int_{-\infty}^{\infty} \overline{\hat{f}(y)y^{(-d+1)/2}}\hat{f}(y)y^{-(d+1)/2}dy/y \\
&= ||M(f)||_2
\end{aligned}$$

□

For $f \in S_d$ we attach $F \in Ind_d$ by $F = F_{T_d(f)}$ where F_ϕ is defined in (6.5),(6.6).

Proposition 6.9. *Let $f \in S_d$. Then*

$$||f||_2 = \frac{2((d-1)!)}{(-2\pi i)^d}||F||_I$$

Proof.

$$\begin{aligned} ||f||_I^2 &= \frac{1}{\pi}\int_0^\pi A(f)(r(\theta))\overline{f(r(\theta))}d\theta \\ &= \frac{1}{\pi}\int_0^\pi A(f)(s(\sin(\theta))n(\cot(\theta))r(\theta)))\overline{f(s(\sin(\theta))n(\cot(\theta))r(\theta))}d\theta \\ &= \int_{-\infty}^\infty A(f)\overline{f(wn(x))}dx \\ &= \int_{-\infty}^\infty\int_{-\infty}^\infty f(w^{-1}n(y)wn(x)\overline{f(wn(x))}dydx \end{aligned}$$

□

Corollary 6.10. *Let f and F be as above then there exist a constant c (depending on d but not on f) such that $||f||_2 \le c||F||_\infty$.*

Corollary 6.11. *S_d is a space of smooth functions in the Hilbert representations space $\mathcal{H}_d$ of R_d.*

Proof. By ([18], Theorem 1.8) the smooth vectors in the Hilbert representation associated with Ind_d are the smooth functions on G satisfying (6.1). (That is, the space Ind_d is the space of smooth vectors in the appropriate L^2 space) Hence, the map f to F given above is a G invariant map sending the space $S_d([0,\infty))$ to a space of smooth vectors in Ind_d. Using Corollary 6.10 and the definition of smooth vectors we get our result. □

7. $(\mathfrak{g}, K)$ modules and Fréchet spaces isomorphism

In this section we will show that $S_d([0,\infty))$ is the smooth space of the representation $R_d^\pm$ on the space $\mathcal{H}_d = L^2((0,\infty), dx/x)$. To do that we will show that the operator M_d from I_d^+ to $S_d([0,\infty))$ is an isomorphism of the $(\mathfrak{g}, K)$ modules of K-finite vectors and also an isomorphism between the spaces of smooth vectors.

7.1. K-finite vectors.

Using the map M_d we can find K finite vectors in $\mathcal{H}_d = L^2((0,\infty), dy/y)$.

Lemma 7.1. *Every function of the form $y^{(d+1)/2}p(y)e^{-y}$ where $p(y)$ is a polynomial is a K-finite vector in $\mathcal{H}_d$.*

Proof. We let $F_0 \in Ind_d$ be defined by $F_0(r(\theta)) = e^{-(d+1)\theta}$ and extended to G as in (6.6). Then $\phi(x) = \phi_{F_0}(x) = (1+x^2)^{-(d+1)/2} e^{i(d+1)tan^{-1}(x)}$ and by ([8] 3.944 (5),(6)) we have that $M_d(\phi) = \frac{\sqrt{2\pi}}{d!} y^{(d+1)/2} e^{-y}$. Now the n-th derivative $\phi(x)$ is also a K-finite vector since it is the application n times of the differential operator $X \in \mathfrak{g}$. Since $M_d(\phi^{(n)}) = \lambda y^n y^{(d+1)/2} e^{-y}$ for a nonzero constant λ it follows that $p(y) y^{(d+1)/2} e^{-y}$ is a K finite vector in $\mathcal{H}_d$ for every polynomial $p(y)$ and we have proved the Lemma. □

The Leguerre orthogonal polynomials $L_n^d(x)$ are defined by the formula

$$L_n^d(x) = e^x \frac{x^{-d}}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+d})$$

It is well known (see [11] (4.21.1) and 4.23 Theorem 3) that for a fixed integer $d \geq 0$, the set of functions $\phi_n(x) = \left(\frac{n!}{(n+d)!}\right)^{1/2} x^{d/2} L_n^d(x)$, $n = 0, 1, ...$ is a complete orthonormal system for $L^2((0,\infty), dx)$. Hence we get

Proposition 7.2. *Let d be a positive integer. The set of functions* $e_n(x) = \left(\frac{n!}{(n+d)! 2^{d+1}}\right)^{1/2} x^{(d+1)/2} e^{-x}$ $n = 0, 1, ...$ *is a complete orthonormal system of K finite vectors (in fact, K eigenfunctions) for* $\mathcal{H}_d = L^2((0,\infty), dx/x)$.

Proof. By Lemma 7.1 the functions $e_n(x)$ are all K-finite. By the remark on the Laguerre polynomials it follows that the set of functions $x^{-1/2} e_n(x)$, $n = 0, 1, ...$ is a complete orthonormal set for $L^2((0,\infty), dx)$. If $f(x) \in L^2((0,\infty), dx/x)$ is orthogonal to all functions $e_n(x)$, $n = 0, 1, ...$ then $x^{-1/2} f(x) \in L^2((0,\infty), dx)$ is orthogonal to $x^{-1/2} e_n(x)$, $n = 0, 1, ...$ hence is the zero function. □

Corollary 7.3. *The set of all K finite vectors in $\mathcal{H}_d$ is the set of functions $x^{(d+1)/2} e^{-x} p(x)$ where $p(x)$ is a polynomial.*

Proof. Otherwise we would be able to find a K finite eigenfunction which is orthogonal to all the e_n which is a contradiction. □

Corollary 7.4. *The $(\mathfrak{g}, K)$ module of K-finite vectors in I_d^+ is isomorphic to the $(\mathfrak{g}, K)$ module of K-finite vectors in $\mathcal{H}_d$ and both are irrecducible.*

Proof. The operator T_d is a one to one intertwining operator between the $(\mathfrak{g}, K)$ of the K-finite vectors in I_d^+ and the $(\mathfrak{g}, K)$ module of the K-finite vectors in $\mathcal{H}_d$. By the above lemma it is onto. □

Our main result of this paper is the following:

Theorem 7.5. *The space $S_d([0,\infty))$ is the space of smooth vectors in $\mathcal{H}_d$, that is, $S_d([0,\infty)) = \mathcal{H}_d^\infty$*

Proof. By Corollary 6.10 the operator M_d is a smooth intertwining operator between I_d^+ and $\mathcal{H}_d^\infty$ whose image is $S_d([0,\infty))$. By Corollary 7.4 M_d restricts to a $(\mathfrak{g}, K)$ isomorphism of the spaces of K-finite vectors. By a theorem of Casselman and Wallach ([19], Theorem 11.6.7) there is a unique continuous extension of such isomrphism to an isomorphism of the smooth spaces of each representation. It follows that M_d is that extension and that M_d is onto $\mathcal{H}_d^\infty$ hence $S_d([0,\infty)) = \mathcal{H}_d^\infty$. □

8. The Kirillov model for the discrete series representations of $GL(2,\mathbb{R})$

In this section we use are previous results to describe the Kirillov model and in particular the smooth space of the Kirillov model of the discrete series representations of $GL(2,\mathbb{R})$.

The discrete series representations of $GL(2,\mathbb{R})$ are parametrized by two real characters $\chi_1(t) = |t|^{s_1}\mathrm{sgn}(t)^{m_1}$, $\chi_2(t) = |t|^{s_2}\mathrm{sgn}(t)^{m_2}$ where $s_1, s_2 \in \mathbb{C}$ and $m_1, m_2 \in \{0,1\}$ are such that

$$|t|^{s_1-s_2}\mathrm{sgn}(t)^{m_1-m_2} = t^d\mathrm{sgn}(t) \tag{8.1}$$

for some positive integer d. The smooth space of the discrete series is a subspace of the induced representation $\mathrm{Ind}(\chi_1,\chi_2)$ which is given by the space of smooth functions $F : GL(2,\mathbb{R}) \to \mathbb{C}$ satisfying

$$F\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g\right) = \chi_1(a)\chi_2(b)|a/b|^{1/2}F(g)$$

for every $g \in GL(2,\mathbb{R})$. The action of $GL(2,\mathbb{R})$ on this space is by right translations. We denote the central character of this representation by ω where

$$\omega(b) = |b|^{s_1+s_2+1}\mathrm{sgn}(b)^{m_1+m_2}.$$

It is easy to see that the mapping $F \to \phi_F$ in (6.3) gives an isomorphism between $\mathrm{Ind}(\chi_1,\chi_2)$ and I_d where d is the positive integer given by (8.1). Moreover, the induced action of the subgroup $SL(2,\mathbb{R})$ on I_d is given by (4.5). It follows from Corollary 6.5 and the theory of $(\mathfrak{g}, K)$ modules ([7] p. 2.7, 2.8) that the space $I_d^+ \oplus I_d^-$ is an irreducible closed subspace of I_d under the action of $GL(2,\mathbb{R})$. To obtain the smooth space of the Kirillov model we define a mapping M_{χ_1,χ_2} from I_d by

$$M_{\chi_1,\chi_2}(\phi)(y) = |y|^{(d+1)/2}\mathrm{sgn}(y)^{m_2}\hat{\phi}(y)$$

Remark 8.1. The mapping $F \to M_{\chi_1,\chi_2}(\phi_F)(y)$ from the subspace of the induced representation $\mathrm{Ind}(\chi_1,\chi_2)$ is identical to the map

$$F \to W_F\begin{pmatrix} |y|^{1/2}\mathrm{sgn}(y) & 0 \\ 0 & |y|^{-1/2} \end{pmatrix}$$

defined in ([7], (75)) where W_F is the Whittaker function associated with F. This space of functions can be called a "normalized" Kirillov model. It is related to the "standard" Kirillov model,

$$F \to W_F\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$$

via the relation in ([7], (75)) which is:

$$W_F \begin{pmatrix} |y|^{1/2}\mathrm{sgn}(y) & 0 \\ 0 & |y|^{-1/2} \end{pmatrix} = \omega(|y|^{1/2}) W_F \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$$

It is easy to see that $M_{\chi_1,\chi_2}(\phi)(y) = \mathrm{sgn}(y)^{m_2} M_d(\phi)(y)$. Since M_d maps the space I_d^+ onto $S_d([0,\infty))$ and the sapce I_d^- onto $S_d((-\infty,0])$ it follows that M_{χ_1,χ_2} does the same. Hence M_{χ_1,χ_2} sends the space $I_d^+ \oplus I_d^-$ onto the space $\mathcal{K}_d$ which can be described as follows: $\mathcal{K}_d$ is the space of smooth functions $f : (\mathbb{R} - \{0\}) \to \mathbb{C}$ such that f and all its derivatives are rapidly decreasing at $\pm\infty$ and such that the function $g(x) = |x|^{(d+1)/2} f(x)$ is smooth on the right and left at $x = 0$. An example for such a function is the function $|x|^{(d+1)/2} e^{-|x|}$.

The proof for these assertions is the same as the proof of Theorem 3.1. The idea is to define an irreducible representation of $GL(2,\mathbb{R})$ on an L^2 space and to show using the various Frechet topologies that the smooth space of this representation is $\mathcal{K}_d$. We now describe the Hilbert space and the action of $GL(2,\mathbb{R})$ on this space. The details of the proofs are left to the reader.

We define $\mathcal{V}_d = L^2(\mathbb{R}, dx/|x|)$. We define a representation R_{χ_1,χ_2} on $\mathcal{V}_d$ by

$$\left(R_{\chi_1,\chi_2}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} f\right)(x) = e^{iyx} f(x)$$

$$\left(R_{\chi_1,\chi_2}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} f\right)(x) = |a|^{(s_1+s_2+1)/2} f(ax) = \omega(|a|^{1/2}) f(ax)$$

$$\left(R_{\chi_1,\chi_2}\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} f\right)(x) = |b|^{s_1+s_2+1}\mathrm{sgn}(b)^{m_1+m_2} f(x) = \omega(b) f(x)$$

$$\left(R_{\chi_1,\chi_2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} f\right)(y) = \mathrm{sgn}(y)^{m_2+d+1} \int_{-\infty}^{\infty} f(x) j_d(xy) dx/|x|$$

where

$$j_d(x) = \begin{cases} (i)^{-(d+1)}\sqrt{x} J_d(2\sqrt{x}) & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} \tag{8.2}$$

and the integral is defined as an L^2 extension of a unitary operator on the space $\mathcal{K}_d$.

Theorem 8.2. *The representation R_{χ_1,χ_2} on the space $L^2(\mathbb{R}, dx/|x|)$ is strongly continuous and irreducible. It is unitary if the central character $\omega = \chi_1\chi_2$ is unitary.*

Theorem 8.3. *The smooth space of the representation R_{χ_1,χ_2} on $L^2(\mathbb{R}, dx/|x|)$ is the space $\mathcal{K}_d$.*

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