Lifting of cusp forms from $\widetilde{SL}_2$ to $GSpin(1,4)$

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Saito-Kurokawa Lift :

$$\left(\begin{array}{c}\text{Cusp forms on } \widetilde{SL}_2 \\ \text{of weight } k+1/2 \text{ for } \Gamma_0(4)\end{array}\right) \longrightarrow \left(\begin{array}{c}\text{Siegel modular forms} \\ \text{on } Sp_4 \text{ of weight } 2+k\end{array}\right)$$

$$g(z) = \sum_{n>0} a(n) q^n \longrightarrow G(Z) = \sum_{T>0} A(T) e^{2\pi i Tr(TZ)}$$

where

$$A(T) = A\left(\begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix}\right) = \sum_{d|(m,n,r)} a\left(\frac{det(2T)}{d^2}\right) d^k$$

Proof of automorphy of the Saito-Kurokawa lift can be given using Jacobi forms. In 1996, Duke and Imamoglu gave a proof using Imai's Converse Theorem.

Ikeda Lift (2000) :

$$\left(\begin{array}{c}\text{Cusp forms on } \widetilde{SL}_2 \\ \text{of weight } k+1/2\end{array}\right) \xrightarrow{\text{Ikeda Lift}} \left(\begin{array}{c}\text{Siegel modular forms} \\ \text{on } Sp_{4n} \text{ of weight } n+k\end{array}\right)$$

The lift is defined by giving explicit formula for the Fourier coefficient and automorphy is proved by the method of Jacobi forms.

Why $Spin(1,4)$ **?**

- For every prime $p \neq 2$ we have

$$Sp_4(\mathbb{Q}_p) \simeq Spin(1,4)(\mathbb{Q}_p)$$

- Analogue of Ikeda lift possible in the Spin case but we have to be careful with $p = 2$

- Maaß Converse Theorem for $Spin(1,n), n \geq 2$

$Spin(1,4)$ **and Vahlen matrices**

Let $C_n(\mathbb{R}) := \langle i_1, \cdots, i_n \rangle_{\mathbb{R}}, i_j^2 = -1, i_j i_k = -i_k i_j$ be the Clifford algebra with n generators. Note that $C_2(\mathbb{R})$ is the Hamiltonian quaternion algebra.

$\mathbf{SL_2}$	$\mathbf{Spin(1,4)}$
$\mathbb{H}_1=\{x+iy:y>0\}$	$\mathbb{H}_3=\{x_0+x_1i_1+x_2i_2+x_3i_3:x_3>0\}\subset C_3(\mathbb{R})$
$\frac{dxdy}{y^2}$	$\frac{dx_0dx_1dx_2dx_3}{x_3^2}$
$\Omega = y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$	$\Omega_3 = x_3^2\sum_{j=0}^{3}\frac{\partial^2}{\partial x_j^2}-(3-1)x_3\frac{\partial}{\partial x_3}$
$SL_2(\mathbb{R})=\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}: ad-bc=1\}$	$SV_2(\mathbb{R})=\{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\in Mat_2(C_2(\mathbb{R})): \alpha\delta^*-\beta\gamma^*=1,\ \alpha\beta^*,\delta\gamma^*\in V_2(\mathbb{R})\}\cong Spin(1{,}4)$
$g\in SL_2(\mathbb{R}), z\in\mathbb{H}_1$, $g.z=\frac{az+b}{cz+d}$	$g\in SV_2(\mathbb{R}), x\in\mathbb{H}_3$, $g.x=(\alpha x+\beta)(\gamma x+\delta)^{-1}$
$SL_2(\mathbb{Z})=$ integer points of $SL_2(\mathbb{R})=$ $=\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$	$SV_2(\mathbb{Z})=$ integer points of $SV_2(\mathbb{R})=$ $=\left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}_{j=1,i_1,i_2} \right\rangle$

$F : \mathbb{H}_3 \to \mathbb{C}$ is called an **Automorphic function** with respect to $SV_2(\mathbb{Z})$ if :

1. $\Omega_3 F = \lambda F$ for some $\lambda \in \mathbb{C}$
2. For certain positive constants κ_1 and κ_2 we have $F(x) = O(x_3^{\kappa_1})$ for $x_3 \to \infty$, $F(x) = O(x_3^{-\kappa_2})$ for $x_3 \to 0$ uniformly on x_0, x_1, x_2,
3. $F(\gamma x) = F(x)$ for all $\gamma \in SV_2(\mathbb{Z}), x \in \mathbb{H}_3$.

This is a higher dimensional analogue of weight 0 Maaß form.

F has the Fourier expansion

$$F(x) = \sum_{\beta \in V_2(\mathbb{Z})} A(\beta) x_3^{\frac{3}{2}} K_{ir}(2\pi |\beta| x_3) e^{2\pi i Re(\beta x)}$$

F is called a cuspidal automorphic function if $A(0) = 0$.

Input Data

We start with weight $\frac{1}{2}$ cuspidal Maaß Hecke eigenform f given by the Fourier expansion

$$f(z) = \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} c(n) W_{\frac{sign(n)}{4}, \frac{ir}{2}}(4\pi|n|y) e^{2\pi i n x}$$

satisfying

$$c(n) = 0 \text{ if } n \equiv 2, 3 \pmod 4.$$

The space of these functions is the non-holomorphic analogue of the Kohnen Plus space for half-integer weight holomorphic forms.

Katok-Sarnak (1993) gave the Shimura correspondence between weight 0 and weight $\frac{1}{2}$ Maaß forms. In particular the space of functions defined above is infinite dimensional. Denote this space by $S^{+}_{1/2}(4)$.

Definition of the Lift :

Let $\beta = \beta_0 + \beta_1 i_1 + \beta_2 i_2 \in V_2(\mathbb{Z})$ and write $gcd(\beta_0, \beta_1, \beta_2) = 2^u d$ with $u \geq 0$ and d odd. We define

$$A(\beta) := 2^{3/4} |\beta| \sum_{t=0}^{u} \left(\sum_{n|d} c\left(\frac{-|\beta|^2}{(2^t n)^2} \right) n^{-1/2} \right) (-1)^t 2^{t/2}.$$

Theorem 1 **:** With $A(\beta)$ defined as above $F(x)$ is a cuspidal automorphic function with respect to $SV_2(\mathbb{Z})$.

Comparison to Saito-Kurokawa :

$$A(T) = A \begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix} = \sum_{d|(m,n,r)} a\left(\frac{det(2T)}{d^2} \right) d^k$$

For every non-negative integer l fix a basis $\{P_{l,\nu}\}$ of spherical harmonic polynomials of degree l in 3 variables. Maaß proves the following Converse Theorem

Maaß Converse Theorem (1949) **:**

The following two statements are equivalent

1. $F(x) = \sum_{\substack{\beta \in V_2(\mathbb{Z}) \\ \beta \neq 0}} A(\beta) x_3{}^{\frac{3}{2}} K_{ir}(2\pi|\beta|x_3) e^{2\pi i Re(\beta x)}$ is a cuspidal automorphic function with respect to $SV_2(\mathbb{Z})$.
2. For all l, $P_{l,\nu}$, the Dirichlet series $\xi(s, P_{l,\nu}) := \pi^{-2s}\Gamma(s + \frac{ir}{2})\Gamma(s - \frac{ir}{2}) \sum_{\substack{\beta \in V_2(\mathbb{Z}) \\ \beta \neq 0}} \frac{A(\beta)P_{l,\nu}(\beta)}{(|\beta|^2)^s}$

   satisfy

   (a) $\xi(s, P_{l,\nu})$ have analytic continuation to the complex plane,

   (b) $\xi(s, P_{l,\nu})$ are bounded on vertical strips and

   (c) $\xi(s, P_{l,\nu})$ satisfy the functional equation

   $$\xi(\frac{3}{2} + l - s, P_{l,\nu}) = (-1)^l \xi(s, P'_{l,\nu})$$

   where $P'_{l,\nu}(\beta) := P_{l,\nu}(\beta')$ for $\beta \in V_2(\mathbb{Z})$.

Proof of Theorem 1 :

We follow the idea of Duke-Imamoglu (1996)

Step 1 :

$$\xi(s+\frac{l}{2}+\frac{1}{4},P_{l,\nu}) = \pi^{-2s-l-\frac{1}{2}}2^{\frac{3}{4}}\left[\frac{2^s-2^{-s}}{2^s+2^{1-s}}\right]\times$$
$$\Gamma(s+\frac{l}{2}+\frac{1}{4}+\frac{ir}{2})\Gamma(s+\frac{l}{2}+\frac{1}{4}-\frac{ir}{2})\zeta(2s)\sum_{m=1}^{\infty}\frac{c(-m)b(m)}{m^{s+\frac{l}{2}-\frac{1}{4}}}$$

where $b(m) := \sum_{|\beta|^2=m} P_{l,\nu}(\beta)$.

Step 2 :

- $\Theta_{l,\nu}(z) := \sum_{\beta\in\mathbb{Z}^3} P_{l,\nu}(\beta)e^{2\pi i|\beta|^2 z} = \sum_{m=1}^{\infty} b(m)e^{2\pi imz}$
  weight $l+3/2$ automorphic form with respect to $\Gamma_0(4)$.
- $\widetilde{E}_\infty(z,s)$ normalized Eisenstein series of weight $-(l+2)$ with respect to $\Gamma_0(4)$
- $I(s) := \int_{\Gamma_0(4)\backslash\mathbb{H}_1} f(z)\Theta_{l,\nu}(z)\widetilde{E}_\infty(z,s)y^{\frac{l+2}{2}-\frac{1}{4}}\frac{dxdy}{y^2}$

Rankin Integral Formula :

$$\xi(s+\frac{l}{2}+\frac{1}{4},P_{l,\nu})=\pi^{-l-\frac{3}{4}}2^{\frac{5}{4}}\left[\frac{2^{3s}-2^{s}}{2^{s}+2^{1-s}}\right]I(s)$$

This gives the analytic continuation and boundedness on vertical strips.

Step 3 :

To get the functional equation we need to show that

$$(2^{3-3s}-2^{1-s})I(1-s)=(2^{3s}-2^{s})I(s)$$

This follows from the following functional equation of the Eisenstein series

$$\widetilde{E}_{\infty}(z,1-s) = \frac{2^{4s-3}}{1-2^{2s-2}}\widetilde{E}_{\infty}(z,s)+\frac{2^{2s-2}(1-2^{2s-1})}{1-2^{2s-2}}\times \times\left[\widetilde{E}_{0}(z,s)+\widetilde{E}_{\frac{1}{2}}(z,s)\right]$$

Here $\widetilde{E}_0$ and $\widetilde{E}_{\frac{1}{2}}$ are obtained from $\widetilde{E}_\infty$ by a change of variable that takes the cusp ∞ to the cusps 0 and 1/2 of $\Gamma_0(4)$.

Non-Vanishing of Lift :

Theorem 2 : $F(x)$ defined in Theorem 1 is non-zero, i.e., the map $f \mapsto F$ is injective.

Step 1 **:**

$A(\beta) = 0$ for all $\beta \in V_2(\mathbb{Z})$ if and only if $c(-n) = 0$ for all $n > 0$ and not of the form $n = 4^u(8k+7)$.

Step 2 **:** **Waldspurger formula**

Using Baruch-Mao we show that for a fundamental discriminant D we have

$$|c(-|D|)|^2 = CL^{\{\infty,2\}}(\frac{1}{2}, \sigma \otimes \chi_D)$$

where $C \neq 0$ and σ is the automorphic representation of $SL_2(\mathbb{A})$ obtained by Shimura lift from f.

Step 3 :

Friedberg-Hoffstein (1995) gives us the non-vanishing of special values of infinitely many twisted L-functions.

Hecke Theory :

One defines the Hecke algebra for the group $Spin(1,4)$ in the standard way using double cosets $\Gamma\backslash GSpin^+(1,4)/\Gamma$. Here $\Gamma = SV_2(\mathbb{Z})$. For an odd prime p the Hecke algebra H_p has 2 generators T_p and T_{p^2}.

$$(T_pF)_{(\beta)} = p^{3/2}A(p\beta) + p^{3/2}A(\beta/p) + p\sum_{\alpha} A\left(\frac{\alpha'\beta\overline{\alpha}}{p}\right)$$

$$\begin{aligned}(T_{p^2}F)_{(\beta)} \;&=\; p^{3/2}\sum_{\alpha}\left(A(\alpha'\beta\bar{\alpha}) + A\left(\frac{\alpha'\beta\bar{\alpha}}{p^2}\right)\right) + \\ &\quad + \left(\sum_{v} e^{2\pi i Re(\frac{\beta v}{p})}\right) A(\beta)\end{aligned}$$

Theorem 3 **:** If f is a Hecke eigenform with p^{th} eigenvalue λ_p, for an odd prime p, then the lift F defined in Theorem 1 is a Hecke eigenform such that

$$T_pF = (p^{3/2}\lambda_p + p(p+1))F$$

$$T_{p^2}F = \Big((p+1)p^{3/2}\lambda_p + (p-1)(p+1)\Big)\,F$$

Classical to Adelic :

- Set $G = GSpin(1,4)$
- Strong approximation (O'Meara)

$$G(\mathbb{A}) \cong G(\mathbb{Q})G^+(\mathbb{R})K_0 \quad \text{where } K_0 := \prod_{p<\infty} G(\mathbb{Z}_p)$$

- F defines a cuspidal automorphic form Φ_F on $G(\mathbb{A})$

$$F \longrightarrow \Phi_F(g_{\mathbb{Q}} g_\infty k_0) := F(g_\infty(i_3))$$

- Consider the space of functions obtained from Φ_F by right translation and denote by π_F an irreducible component. Write $\pi_F \cong \otimes'_p \pi_p$ where π_p is an irreducible unramified representation of G_p for odd p.
- $G_p \cong GSp_4(\mathbb{Q}_p)$ and $G(\mathbb{Z}_p) \cong GSp_4(\mathbb{Z}_p)$
- From Cartier we know that an irreducible unramified representation of G_p is the unique spherical constituent of the representation obtained by induction from an unramified character of the Borel subgroup B

- Borel subgroup $B = NA$ with unipotent radical N and torus $A := \{a = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_1^{-1}a_0 & \\ & & & a_2^{-1}a_0 \end{pmatrix} : a_0, a_1, a_2 \in \mathbb{Q}_p^\times\}$

- Given unramified characters χ_0, χ_1, χ_2 on $\mathbb{Q}_p^\times$ define the character χ on A by $\chi(a) := \chi_0(a_0)\chi_1(a_1)\chi_2(a_2)$. Extend χ to a character of $B = NA$ by setting it to be trivial on N. Define $I(\chi) := Ind_B^{G_p}(\chi)$.

<u>Theorem</u> <u>4</u> **:** π_p is the unique spherical constituent of $I(\chi)$ where, upto the action of the Weyl group, the character χ is given by

$$\chi_1(p) = p^{1/2}\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}, \ \chi_2(p) = p^{1/2}\frac{\lambda_p - \sqrt{\lambda_p^2 - 4}}{2},$$

$$\chi_0(p) = p^{-1/2}$$

<u>CAP representation</u> :

Let G_1 and G_2 be two groups such that $G_{1,\nu} \cong G_{2,\nu}$ for almost all places ν and P_2 be a parabolic subgroup of G_2. Then we will call an irreducible **<u>cuspidal</u>** automorphic representation π_1 of G_1 a CAP representation associated to P_2 if there is an irreducible cuspidal automorphic representation σ of M_2, the Levi component of P_2, such that $\pi_{1,\nu} \cong \pi_{2,\nu}$ for almost all places ν, where π_2 is an irreducible component of $Ind_{P_2}^{G_2}(\sigma)$.

- Set $G_1 = GSpin(1,4)$ and $G_2 = GSp(4)$. Let

  $$P_2 := \{ \begin{pmatrix} g & B \\ & \mu^t g^{-1} \end{pmatrix} : g \in GL_2, B \text{ symmetric matrix } \}$$

  be the Siegel parabolic subgroup.

- $f \in S^+_{1/2}(4) \stackrel{\text{Shim. corr.}}{\longrightarrow} h \in S_0(1) \stackrel{\text{Strg. Approx.}}{\longrightarrow}$
  $\phi_f \in Aut(GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})) \longrightarrow \begin{matrix}\sigma \text{ Aut.} \\ \text{rep. of } GL_2(\mathbb{A})\end{matrix}$

<u>Theorem 5</u>: With F as in Theorem 1, the automorphic representation π_F of $GSpin(1,4)(\mathbb{A})$ is CAP to a subrepresentation of $Ind_{P_2}^{G_2}(\sigma \otimes |det/\mu|^{-1/2})$ of $GSp_4(\mathbb{A})$.

Questions : Does there exist

1. Cuspidal automorphic representation π' of $GSp_4(\mathbb{A})$ which is CAP to $Ind_{P_2}^{G_2}(\sigma \otimes |det/\mu|^{-1/2})$ above ?

2. Cuspidal representation σ' of the Levi subgroup of the only parabolic subgroup P of $GSpin(1,4)$ such that π_F is CAP to $Ind_P^{G_1}(\sigma')$?

Answer : NO

Theorem (P-S) : The representation $Ind_{P_2}^{G_2}(\sigma \otimes |det/\mu|^{-1/2})$ has a CAP on GSp_4 associated to it if and only if atleast one of the following conditions is satisfied :

1. $L(1/2, \sigma) = 0$ and there exists a quadratic character χ such that $L(1/2, \sigma \otimes \chi) \neq 0$.
2. $\Delta(\sigma) := |\{\tau \text{ aut. rep. of } \widetilde{SL}_2 : Wd(\tau) = \sigma\}| > 1$.

With σ as in our case neither of the above two conditions are satisfied (Katok-Sarnak and Waldspurger).

Comments and Future Steps

- Generalize quaternion algebra and input data.
- Characterize the image of our lift as a Maaß space where the Fourier coefficients $A(\beta)$ satisfy suitable linear relations among themselves
- Limitation of Maaß Converse Theorem : Analogue of Hecke Converse theorem for triangle subgroups
- Adelic converse theorem for Spin group in terms of L-functions