Lower bounds for moments of L-functions

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Plan:

§1 Moments of $\zeta(\frac{1}{2}+it)$

§2 A lower bound on moments in families

§3 Application: Fluctuations around quantum ergodicity for the modular domain

- Joint work with K. Soundararajan

§1 Moments of $\zeta(\frac{1}{2}+it)$

$$I_k(T) := \frac{1}{T}\int_0^T |\zeta(\frac{1}{2}+it)|^{2k} dt$$

old conjecture:

$$I_k(T) \sim C_k (\log T)^{k^2}, \qquad T \to \infty$$

-implies Lindelöf Hypothesis: $|\zeta(\frac{1}{2}+it)| \ll |t|^{\epsilon}$, $\forall \epsilon > 0$.

Known:

- Hardy & Littlewood (1918): second moment ($k = 1$)
- Ingham (1926): fourth moment ($k = 2$)

for $k > 2$, no good **UPPER** bounds are known.

Lower bound: Titchmarsh (1928), Ramachandra (1980):

$$I_k(T) \geq \text{const} \cdot (\log T)^{k^2}$$

old conjecture for C_k

$I_k(T) \sim C_k(\log T)^{k^2}$ where

$$C_k = a(k) f(k) \ ,$$

$$a(k) = \prod_p \left\{ (1 - \frac{1}{p})^{k^2} \sum_{m=0}^{\infty} (\frac{\Gamma(k+m)}{m!\Gamma(k)})^2 p^{-m} \right\}$$

$$\boxed{\text{f(k) = ??}}$$

Known: H & L: $f(1) = 1$, Ingham $f(2) = \frac{1}{12}$.

Conjectures for $f(k)$:

- Conrey& Ghosh (1992) $f(3) = \frac{42}{9!}$
- Conrey & Gonek (2001): $f(4) = \frac{24024}{16!}$.

Keating-Snaith conjecture (2000):

$f(k)$ can be extracted from Random Matrix Theory!

$$f(k) = \lim_{N\to\infty} \frac{1}{N^{k^2}} \int_{U(N)} |\,\mathrm{det}(I - e^{i\theta}A)|^{2k} d\mu(A) \ ,$$

$d\mu$ = Haar measure on unitary group $U(N)$

- implies $f(k) = \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$

§2 Ensembles of L-functions

One can replace $\zeta(s)$ by any L-function with similar results.

Rather than studying value distribution of a fixed L-function up the critical line, it is useful to fix $s = 1/2$ and vary the L-function over a "family" or "ensemble" $\mathcal{F}$.

We would then study say the moments $|L(\frac{1}{2}, f)|^{2k}$ as f varies over the ensemble $\mathcal{F}$ in the limit $\#\mathcal{F} \to \infty$.

Examples of ensembles of L-functions:

• Dirichlet L-functions modulo q (prime): Take $\chi : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}^*$ a (primitive) Dirichlet character mod q. Let $L(s,\chi) = \sum_n \chi(n) n^{-s}$, $\Re(s) > 1$.

Here we look at all (primitive) characters mod q, $q \to \infty$. The ensemble average is

$$\langle F(\chi) \rangle := \frac{1}{q-2} \sum_{\chi}^{*} F(\chi)$$

e..g the moments of the central values are:

$$\left\langle |L(\frac{1}{2},\chi)|^{2k} \right\rangle = \frac{1}{q-2} \sum_{\chi}^{*} |L(\frac{1}{2},\chi)|^{2k}$$

• Maass cusp forms: Take $\psi_j \in L^2(SL_2(\mathbb{Z})\backslash\mathbb{H})$, an eigenfunction of the Laplacian $-\Delta\psi_j = E_j\psi_j$, and eigenfunction of all Hecke operators:

$$T(n)\psi_j = \lambda_j(n)\psi_j$$

The (standard) L-function is

$$L(s,\psi_j) = \sum_n \lambda_j(n)n^{-s}$$

has analytic continuation & FE $s \mapsto 1-s$.

Ensemble average - e.g. average $\left\langle L(\frac{1}{2},\psi_j)^k \right\rangle_E$ over eigenvalues in a window $E < E_j < 2E$

Katz & Sarnak: model the behaviour of low-lying zeros of L-functions by random matrices drawn from suitable classical groups.

Keating & Snaith, Conrey & Farmer: use these classical groups to model moments of central values of L-functions.

Moments of central values:

In many cases there are conjectures for the moments of central values

$$\left\langle |L(\frac{1}{2}, f)|^{2k} \right\rangle_{\mathcal{F}}$$

(Conrey, Farmer, Keating, Rubinstein, Snaith).

e.g. for Dirichlet ensemble ("unitary")

$$\left\langle |L(\frac{1}{2}, \chi)|^{2k} \right\rangle \sim c_k (\log q)^{k^2}$$

For Maass cusp forms (orthogonal):

$$\left\langle L(\frac{1}{2}, \psi_j)^k \right\rangle_E \sim c_k (\log E)^{k(k-1)/2}$$

Known results - **at best** for small values of k (depending on family).

Upper bounds ??

Lower bound (ZR & Soundararajan):

If we know how to compute **first moment** then we can get **lower** bound of right order of magnitude (with wrong constant) for **all** moments!

Lower bound for Dirichlet ensemble

Thm (ZR & Sound.): $\left\langle |L(\frac{1}{2},\chi)|^{2k} \right\rangle \gg (\log q)^{k^2}$

Approximate Functional Equation: replace $L(\frac{1}{2},\chi)$ by sum of length $q^{1/2}$:

$$L(\frac{1}{2},\chi) \approx \sum_{n<\sqrt{q}} \frac{\chi(n)}{\sqrt{n}} + \epsilon(\chi) \sum_{n<\sqrt{q}} \frac{\overline{\chi(n)}}{\sqrt{n}}$$

where $\epsilon(\chi) =$ "sign" of Gauss sum: $|\epsilon(\chi)| = 1$.

Thus

$$|L(\frac{1}{2},\chi)|^{2k} = \sum_{n_i,m_j<\sqrt{q}} \frac{\chi(n_1)\dots\chi(n_k)\overline{\chi(m_1)\dots\chi(m_k)}}{\sqrt{n_1\dots m_k}}$$

$$+ \text{ similar terms}$$

Problem: The sum contains too many terms ($\approx q^k$) to compute average.

Practice problem:

Replace $L(1/2, \chi)$ by "short" approximation

$$A(\chi) := \sum_{n<x} \frac{\chi(n)}{\sqrt{n}}, \qquad x = q^{\epsilon}$$

$$\Rightarrow A(\chi)^k = \sum_{n<x^k} \frac{d_k(n,x)}{\sqrt{n}} \chi(n)$$

where $d_k(n,x) = \#\{n = m_1 \cdots m_k, \quad m_i < x\}$.

Therefore:

$$\left\langle |A(\chi)|^{2k} \right\rangle = \sum_{m<x^k} \sum_{n<x^k} \frac{d_k(n,x) d_k(m,x)}{\sqrt{mn}} \left\langle \chi(m)\overline{\chi(n)} \right\rangle$$

Orthogonality: $\left\langle \chi(m)\overline{\chi(n)} \right\rangle \sim \begin{cases} 1 & m \equiv n \mod q \\ 0 & \text{otherwise} \end{cases}$

Therefore

$$\left\langle |A(\chi)|^{2k} \right\rangle = \sum_{m,n<x^k, m\equiv n \mod q} \frac{d_k(n,x)d_k(m,x)}{\sqrt{mn}}$$

Reduction to sum over $\boxed{\text{diagonal}}$:

$m, n < x^k < q$ <u>and</u> $m \equiv n \mod q \Rightarrow \boxed{\text{m=n}}$!

Recall: $x = q^\epsilon$.

Conclusion:

$$\left\langle |A(\chi)|^{2k} \right\rangle = \sum_{n<x^k} \frac{d_k(n,x)^2}{n} \approx (\log q)^{k^2}$$

Now use Hölder's inequality ($p = 2k$, $q = \frac{2k}{2k-1}$):

$$\left\langle L(\frac{1}{2},\chi)A(\chi)^{2k-1}\right\rangle \le \left\langle |L(\frac{1}{2},\chi)|^{2k}\right\rangle^{\frac{1}{2k}} \cdot \left\langle |A(\chi)|^{2k}\right\rangle^{\frac{2k-1}{2k}}$$

$\Rightarrow$ lower bound

$$\left\langle |L(\frac{1}{2},\chi)|^{2k}\right\rangle \ge \frac{\left\langle L(\frac{1}{2},\chi)A(\chi)^{2k-1}\right\rangle^{2k}}{\left\langle |A(\chi)|^{2k}\right\rangle^{2k-1}}$$

We saw $\left\langle |A(\chi)|^{2k}\right\rangle \approx (\log q)^{k^2}$ (for $x = q^{\epsilon}$), so all we need is a fuzzy <u>first moment</u>

$$\left\langle L(\frac{1}{2},\chi)A(\chi)^{2k-1}\right\rangle \approx (\log q)^{k^2}$$

Here range of summation is relatively short (about $q^{1/2+\epsilon}$), so reduces to $\boxed{\text{diagonal}}$ contribution, in contrast to that of $\left\langle |L(\frac{1}{2},\chi)|^{2k}\right\rangle$ where sum is of length q^k.

Conclusion: Method yields lower bounds on moments of several families of L-functions, provided we know "$1+\epsilon$" moments.

Other examples:

• quadratic characters $L(1/2, \chi_d)$,

• standard L-function attached to holomorphic forms of large weight

and.....

§3 Fluctuations around quantum ergodicity for the modular domain

Let $M = SL_2(\mathbb{Z})\backslash\mathbb{H}$ be the modular domain, with constant negative curvature metric.

Consider eigenfunctions of Laplacian

$$-\Delta\psi_j = E_j\psi_j$$

Weyl's law (Selberg, 1950's):
Let ψ_j be an orthonormal basis of the discrete spectrum (there is also a continuous part). Then:

$$N(T) := \#\{\sqrt{E_j} < T\} \sim \frac{\text{area}(M)}{4\pi}T^2, \qquad T \to \infty$$

Hecke operators: There is a commuting set of self-adjoint operators T_n, $n = 1, 2, \ldots$ which commute with Δ.

We take ψ_j to be joint eigenfunctions of these (Hecke-Maass forms).

Let $a \in C_0^\infty(M)$ be an "observable", and consider "matrix coefficients"

$$\langle \mathrm{Op}(a)\psi_j, \psi_j\rangle := \int_M a(x)|\psi_j(x)|^2 dx$$

Mean value of matrix elts = classical average:

$$\frac{1}{N(T)} \sum_{\sqrt{E_j}\sim T} \langle \mathrm{Op}(a)\psi_j, \psi_j\rangle \sim \int_{S^*M} a$$

<u>**Quantum ergodicity**</u> (Schnirelman, Zelditch, Colin de Verdiere):

the variance of the matrix elements vanishes:

$$\begin{aligned} \mathrm{Var}(T) &:= \frac{1}{N(T)} \sum_{\sqrt{E_j}\sim T} \left| \langle \mathrm{Op}(a)\psi_j, \psi_j\rangle - \int_{S^*M} a \right|^2 \\ &\to 0, \qquad T \to \infty \end{aligned}$$

<u>Problem:</u> understand the <u>approach</u> to the classical average

Quantum fluctuation Conjectures (generic M)

Feingold and Peres (1986), Eckhart, Fishman, Keating, Agam, Main and Müller (1995):

Normalize $\int_{S^*M} a = 0$. If the geodesic flow $\Phi : S^*M \to S^*M$ is chaotic then

1) $\text{Var}(T) \sim V_{cl}(a)/T^{\dim(M)-1}$, where

$$\begin{aligned} V_{cl}(a) &:= \int_{-\infty}^{\infty} \langle a \circ \Phi^t, a \rangle_{S^*M} dt \\ &= \int_{-\infty}^{\infty} dt \int_{S^*M} a(\Phi^t(x,\xi))\overline{a(x,\xi)} dx d\xi \end{aligned}$$

is the average auto-correlation of classical observable a.

2) The normalized matrix coefficients

$$F_j(a) := E_j^{\frac{\dim(M)-1}{4}} \langle \text{Op}(a)\psi_j, \psi_j \rangle$$

have a Gaussian distribution with mean zero and variance $V_{cl}(a)$.

Numerical tests ("generic" cases):

- Bäcker 1998

- Barnett 2004 deformed Sinai billiard

Theoretical results (non-generic cases):

- cat maps: Kurlberg & Z.R. (2002), Kelmer (2005)

- modular domain: Luo & Sarnak, Z.R. & Soundararajan

Luo & Sarnak: For the modular domain and Hecke-Maass forms ψ_j, the variance of normalized coefficients

$$F_j(a) := E_j^{1/4}\left(\langle \mathrm{Op}(a)\psi_j, \psi_j\rangle - \int_{S^*M} a\right)$$

exists:

$$\left\langle |F_j(a)|^2 \right\rangle_T < \infty$$

(so predicted rate of decay is valid here).

- related (but not identical) to classical variance $V_{cl}(a)$ predicted by Feingold & Peres.

- If in addition a is a Hecke-Maass form then

$$\left\langle |F_j(a)|^2 \right\rangle_T \sim V_{cl}(a) \cdot L(\frac{1}{2}, a)$$

Fluctuations (Z.R. & Soundararajan)

study higher moments of the normalized matrix elements

$$F_j(a) := E_j^{1/4}\left(\langle \mathrm{Op}(a)\psi_j, \psi_j\rangle - \int_{S^*M} a\right)$$

• Generically expect Gaussian moments with variance $V_{cl}(a)$.

• Instead we find the higher moments blow up:

Thm: If a is a joint eigenfunction of Δ and all T_n then for even $k \geq 2$

$$\left\langle |F_j(a)|^{2k} \right\rangle_T \gg (\log T)^{k(k-1)/2}$$

Main tool is Watson's formula, connecting matrix element to the value of an L-function at the center of the critical strip $s = 1/2$:

If a, ψ_j are eigenfunctions of Δ and of all Hecke operators then:

$$|F_j(a)|^2 = \frac{L(\frac{1}{2}, a \otimes \mathrm{Sym}^2 \psi_j)}{L(1, \mathrm{Sym}^2 \psi_j)^2} \cdot c(a)$$

Thus moments of matrix elts $\leftrightarrow$ moments of central values of L-functions in a suitable ensemble

$$\left\langle |F_j(a)|^{2k} \right\rangle_T \leftrightarrow \left\langle \left(\frac{L(\frac{1}{2}, a \otimes \mathrm{Sym}^2 \psi_j)}{L(1, \mathrm{Sym}^2 \psi_j)^2} \right)^k \right\rangle_T$$

The L-function $L(s, a\times \mathrm{Sym}^2\,\psi_j)$: If

$$\Delta\psi_j = (\frac{1}{4}+t_j^2)\psi_j, \quad T_n\psi_j = \lambda_j(n)\psi_j$$

then

$$L(s, a\times \mathrm{Sym}^2\,\psi_j) = \sum_{n=1}^{\infty}\frac{A_j(n)}{n^s}, \quad \Re(s) \gg 1$$

$$A_j(n) := \sum_{n=d^3t_1s_1^2(t_2s_2^2)^2} \mu(d)\lambda_a(dt_1s_1^2)\lambda_j(t_1^2)\lambda_j(t_2^2)$$

- Euler product:

$$\sum\frac{A_j(n)}{n^s} = \prod_p (1+\cdots+p^{-6s})^{-1}$$

- analytic continuation & functional equation

$$L^*(s) := G(s)L(s) = L^*(1-s)$$

$$G(s) = \pi^{-3s}\Gamma(\frac{s+it_a}{2})\Gamma(\frac{s-it_a}{2})\Gamma(\frac{s+it_a+2it_j}{2})\times$$
$$\Gamma(\frac{s+it_a-2it_j}{2})\Gamma(\frac{s-it_a+2it_j}{2})\Gamma(\frac{s-it_a-2it_j}{2})$$

Summary

• Z.R. & Sound: general method for getting lower bound for moments of L-functions if we know how to compute <u>first+ϵ</u> moment.

• Application: for the modular domain, fluctuations of matrix elements about ergodic average are non-Gaussian