

Mean Values of Zeta-Functions via Representation Theory

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0. The aim of the present article is to reveal a structure shared by two basic zeta-functions in their fourth power moments. It might induce one to ponder over the possibility to go beyond.

To be precise, let Γ be a discrete subgroup of a Lie group G in the framework of GL_2 , and let L_ψ be the L -function associated with a Γ -automorphic function ψ on G . Then a major subject in analytic number theory is offered by the mean value

$$\mathcal{M}(L_\psi, g) = \int_{-\infty}^{\infty} |L_\psi(\tfrac{1}{2} + it)|^2 g(t) dt. \quad (0.1)$$

The principal issue is to establish an explicit spectral decomposition, and the following two cases have so far been discussed in greater detail:

- (1) the fourth power moment of the Riemann zeta-function ([9][24][25][30]),
- (2) the fourth power moment of the Dedekind zeta-function of the Gaussian number field ([8]),

which correspond, respectively, to the specifications

$$G = \mathrm{PSL}_2(\mathbb{R}), \quad \Gamma = \mathrm{PSL}_2(\mathbb{Z}), \quad (0.2)$$

$$G = \mathrm{PSL}_2(\mathbb{C}), \quad \Gamma = \mathrm{PSL}_2(\mathbb{Z}[\sqrt{-1}]). \quad (0.3)$$

We shall see that in these cases the spectra come from irreducible representations of G occurring in $L^2(\Gamma \backslash G)$, and that the resulting integral transform of g has a kernel composed of Bessel functions of representations of G . It should be noted that (2) can readily be extended to any imaginary quadratic number field of class number one. Other examples that share the same structure and have been more or less worked out are

- (3) the mean square of the Dedekind zeta-function of any quadratic number field ([29]),
- (4) the fourth power moment of the Dedekind zeta-function of any real quadratic number field with class number one ([6]),
- (5) the spectral fourth power moment of all Hecke L -functions, under either (0.2) or (0.3) ([34]),

As far as our present purpose is concerned, (1) is the most fundamental, and (2) comes next endorsing our conceptual view about $\mathcal{M}(L_\psi, g)$ in general. The situation with (3) and (4) is much similar to (1), though (3) requires Γ be replaced by a Hecke congruence subgroup, and with (4) we need to move to Hilbert modular groups. The case (5) might appear different from others, but can in fact be regarded as an extension of either (1) or (2), though the situation with (0.3) is still under investigation. Note that in (2)–(4) as well as in (5) under (0.3) the

twist and the average with respect to Grössencharakteren can also be taken into account. Concerning this, an interesting argument has been developed by P. Sarnak [37], which seems to indicate another way to view (2)–(5). Despite this, we shall mostly concentrate on (1) and (2).

To the list above one may wish to add, for instance, the mean square of individual Hecke L -functions associated with cusp forms, and further more any extension to the SL_3 environment ([10]). It appears, however, somewhat premature to discuss these subjects fully in the perspective described or suggested in the present work. We plan to return to them in the near future; nevertheless, see [18][26][31][35][36] for instance.

CONVENTION: The weight function g in (0.1) is assumed to be even, entire, real on \mathbb{R} , and of rapid decay in any fixed horizontal strip. The symbol ζ is reserved for the Riemann zeta-function, as usual. We shall often use F to denote various functions which are specified in local contexts. Other notations are introduced where they are needed for the first time, and will remain effective thereafter unless otherwise stated. Also we stress that we are concerned with the structural aspect, and the asymptotic study is being more or less left aside; by the same token we shall often skip the discussion of convergence, naturally under the premise that no confusion be brought in.

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1. First we shall describe an old observation ((2.3)–(2.5) below) made in [24] (see also [30, Section 4.2]), a proper exploitation of which was only recently actuated in our joint work [9] with Bruggeman on $\mathcal{M}(\zeta^2, g)$ the fourth power moment of the Riemann zeta-function.

Thus we begin with an idea of H. Weyl: In analytic number theory we deal mainly with sums over rational integers

$$\sum_{n \in I} F(n), \tag{1.1}$$

where I is an interval in \mathbb{R} , and F an arbitrary function; and to estimate this there is a general principle due to Weyl. A version of it is attributed to J.G. van der Corput and built upon the following triviality:

$$\sum_{n \in I} F(n) = \frac{1}{M} \sum_{n=-\infty}^{\infty} \sum_{m=1}^M F(n+m) \delta_I(n+m), \tag{1.2}$$

where $M \geq 1$ is arbitrary, and δ_I the characteristic function of I . If some effective inequalities are applied to the right side, then the result does not look trivial any more but can be a sharp tool (see [16, Chapter 2]). In fact it could yield subconvexity bounds for ζ which have played basic rôles in many problems in analytic number theory. For instance the bound

$$\zeta\left(\frac{1}{2} + it\right) \ll t^{1/6} \log t, \quad t \geq 2, \tag{1.3}$$

is relatively easy to prove with (1.2), which is significantly better than the convexity bound following from the functional equation (1.7) below.

One may regard (1.2) as a kind of lifting of a one dimensional sum to a two dimensional sum. The original problem has been transformed into the one of finding interaction among the non-diagonal entries. This observation leads us to another triviality: For general double sums we have the decomposition

$$\sum_{m,n} F(m,n) = \left\{ \sum_{m=n} + \sum_{m<n} + \sum_{m>n} \right\} F(m,n). \quad (1.4)$$

This is exactly the same as what F.V. Atkinson did in his important investigation [1] on the mean square of ζ . In [30, Section 4.1], an essentially equivalent assertion is formulated as

$$\begin{aligned} \mathcal{M}(\zeta, g) &= \int_{-\infty}^{\infty} \left[\operatorname{Re} \left\{ \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} + it \right) \right\} + 2\gamma_E - \log(2\pi) \right] g(t) dt + 2\pi \operatorname{Re} \left\{ g \left(\frac{1}{2}i \right) \right\} \\ &+ 4 \sum_{n=1}^{\infty} d(n) \int_0^{\infty} (r(r+1))^{-1/2} g_c(\log(1+1/r)) \cos(2\pi nr) dr, \end{aligned} \quad (1.5)$$

where γ_E is the Euler constant, $d(n)$ the number of divisors of n , and

$$g_c(x) = \int_{-\infty}^{\infty} g(t) \cos(xt) dt. \quad (1.6)$$

We should note that (1.5) implies (1.3); see Ivić's lecture notes [15] for the details of various consequences of Atkinson's result. We stress also that the last infinite sum, which is of a spectral nature, corresponds to the non-diagonal parts in (1.4), and that (1.5) could be regarded as a completion of a particular application of Weyl's idea to ζ .

This completion was realized in [30] via the functional equation

$$\zeta(1-s) = 2(2\pi)^{-s} \cos\left(\frac{1}{2}\pi s\right) \Gamma(s) \zeta(s). \quad (1.7)$$

On the other hand, in [1] had been applied the Poisson summation formula

$$\sum_{n=1}^{\infty} F(n) = \int_0^{\infty} F(r) dr + 2 \sum_{n=1}^{\infty} \int_0^{\infty} F(r) \cos(2\pi nr) dr, \quad (1.8)$$

where F is assumed to be smooth and of fast decay on $(0, \infty)$. However, this difference is superficial, for (1.7) and (1.8) are equivalent to each other. In fact, the kernel function $\cos(2\pi r)$ in (1.8), as well as in (1.5), is related to (1.7) via the Mellin transform:

$$\int_0^{\infty} \cos(2\pi r) r^{s-1} dr = (2\pi)^{-s} \cos\left(\frac{1}{2}\pi s\right) \Gamma(s), \quad 0 < \operatorname{Re} s < 1, \quad (1.9)$$

or, more precisely, via the convolution relation

$$\int_0^{\infty} F(r) \cos(2\pi nr) dr = \frac{1}{2\pi i} \int_{(\alpha)} F^*(1-s) \cos\left(\frac{1}{2}\pi s\right) \Gamma(s) (2\pi n)^{-s} ds, \quad (1.10)$$

where F^* is the Mellin transform of F , and (α) the vertical line $\operatorname{Re} s = \alpha$ with $\alpha > 0$. Summing (1.10) over n , and applying (1.7), we are led to (1.8). Reversing the reasoning, one may reach (1.7) with a combination of (1.8) and (1.9).

In terms of the representation theory of the Lie group \mathbb{R} , the identity (1.8) is understood as a transformation formula of F that gives a way to compute the values of projections of the Poincaré series

$$\sum_{n \in \mathbb{Z}} F(n+x), \quad x \in \mathbb{R}, \quad (1.11)$$

to irreducible subspaces of $L^2(\mathbb{Z} \backslash \mathbb{R})$. The appearance in (1.8) of the Fourier-cosine transformation indicates the nature of the harmonic analysis over the group \mathbb{R} in which \mathbb{Z} is a discrete subgroup.

A motivation of the present article is to see how far extends the above harmonic structure lying behind the important expansion (1.5).

2. In this context, one may ponder whether the dissection mode in (1.4) is optimal or not. It is certainly not in general. An option to tune it up is to refine the notion of being diagonal. A simple arithmetic way to do this is to replace (1.4) by

$$\sum_{m,n} F(m,n) = \left\{ \sum_{km=ln} + \sum_{km<ln} + \sum_{km>ln} \right\} F(m,n), \quad (2.1)$$

where k, l are arbitrary non-zero integers. To extract information from all of these dissections we multiply both sides by a weight $W(k, l)$ and sum over all k, l . We get

$$\left(\sum_{k,l} W(k,l) \right) \left(\sum_{m,n} F(m,n) \right) = \left\{ \sum_{km=ln} + \sum_{km<ln} + \sum_{km>ln} \right\} W(k,l) F(m,n). \quad (2.2)$$

As before, we may regard (2.2) to be a lifting of a double sum to a four dimensional sum; and we are led to the following trivial decomposition:

$$\sum_{k,l,m,n} F(k,l,m,n) = \left\{ \sum_{km=ln} + \sum_{km<ln} + \sum_{km>ln} \right\} F(k,l,m,n). \quad (2.3)$$

Then we take a new viewpoint: We regard quadruple sums as sums over 2×2 integral matrices M . The last identity thus becomes

$$\sum_M F(M) = \left\{ \sum_{|M|=0} + \sum_{|M|>0} + \sum_{|M|<0} \right\} F(M). \quad (2.4)$$

Invoking Hecke's representatives of integral matrices with a given determinant, we have further

$$\sum_{|M|>0} F(M) = \sum_{f=1}^{\infty} \sqrt{f} T_f P_{F_f}(1), \quad P_F(g) = \sum_{\gamma \in \Gamma} F(\gamma g), \quad g \in G, \quad (2.5)$$

with (0.2), where $F_f(g) = F(\sqrt{f} \cdot g)$ and

$$T_f F(g) = \frac{1}{\sqrt{f}} \sum_{d|f} \sum_{b=1}^d F \left(\left[\begin{array}{cc} \sqrt{f}/d & b \\ d/\sqrt{f} & \end{array} \right] g \right). \quad (2.6)$$

Hereby we find a relation lying between the dissection (1.4) and the theory of Γ -automorphic functions on G . If F is sufficiently smooth one may apply the harmonic analysis over $\Gamma \backslash G$ to the Poincaré series P_F . Thus, we are now to find the transformation formula of F that gives a way to compute the values of projections of P_F to irreducible subspaces of $L^2(\Gamma \backslash G)$. In other words, our interest is in fixing the kernel function corresponding to $\cos(2\pi r)$ of (1.8), via the theory of irreducible representations of G occurring in $L^2(\Gamma \backslash G)$. In the application to $\mathcal{M}(\zeta^2, g)$, as developed in [9], those four integral variables in (2.3) correspond to the four zeta-values in an obvious way. In the subsequent sections we shall show the salient points of the pertinent parts of [9] and briefly describe a solution to the last problem.

It should be stressed here that the restriction of the sum (2.5) to those M with $|M| = f$, i.e., $T_f P_{F_f}$, is perhaps more worth investigating. A typical example is the additive divisor sum

$$\sum_{n=1}^{\infty} \sigma_{\lambda}(n) \sigma_{\mu}(n+f) W(n/f), \quad (2.7)$$

where $\sigma_{\lambda}(n) = \sum_{d|n} d^{\lambda}$ with $\lambda \in \mathbb{C}$; and the smooth weight W has a support in the positive reals. A detailed discussion of the spectral decomposition of (2.7) is developed in [27], and the results there have recently found important applications in our joint work [19] (see also [20]) with M. Jutila, where is considered an instance of the case (5) above, and established a new uniform bound for Hecke L -functions associated with cusp forms under (0.2), which generalizes the classical subconvexity bound (1.3) to a vast family of L -functions.

3. In this and the next two sections, we shall deal with the Poincaré series P_F , with an analogy of (1.8) in mind. To begin with, we shall collect elements of the theory of Γ -automorphic representations of G , under the specification (0.2): Thus we write

$$\mathfrak{n}[x] = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}, \quad \mathfrak{a}[y] = \begin{bmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{bmatrix}, \quad \mathfrak{k}[\theta] = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}. \quad (3.1)$$

Let $N = \{\mathfrak{n}[x] : x \in \mathbb{R}\}$, $A = \{\mathfrak{a}[y] : y > 0\}$, and $K = \{\mathfrak{k}[\theta] : \theta \in \mathbb{R}/\pi\mathbb{Z}\}$, so that $G = NAK$ be the Iwasawa decomposition of G . We read it as $G \ni g = \mathfrak{n} \mathfrak{a} \mathfrak{k} = \mathfrak{n}[x] \mathfrak{a}[y] \mathfrak{k}[\theta]$. The Haar measures on the groups N, A, K, G are defined, respectively, by $dn = dx$, $da = dy/y$, $dk = d\theta/\pi$, $dg = dn da dk / y$, with Lebesgue measures $dx, dy, d\theta$. The space $L^2(\Gamma \backslash G)$ is composed of all left Γ -automorphic functions on G , vectors for short, which are square integrable over $\Gamma \backslash G$ against dg . Elements of G act unitarily on vectors from the right, and we have the orthogonal decomposition into invariant subspaces

$$L^2(\Gamma \backslash G) = \mathbb{C} \cdot 1 \oplus {}^c L^2(\Gamma \backslash G) \oplus {}^e L^2(\Gamma \backslash G), \quad (3.2)$$

where ${}^c L^2$ is the cuspidal subspace, and ${}^e L^2$ is spanned by integrals of Eisenstein series.

The cuspidal subspace splits into irreducible subspaces:

$${}^c L^2(\Gamma \backslash G) = \overline{\bigoplus V}; \quad \Omega|_V = (\nu_V^2 - \frac{1}{4}) \cdot 1, \quad (3.3)$$

where $\Omega = y^2 (\partial_x^2 + \partial_y^2) - y \partial_x \partial_\theta$ is the Casimir operator. Under (0.2), we can restrict our attention to two cases: either $\nu_V \in i(0, \infty)$ or ν_V is equal to half a positive odd integer. According to the right action of K , the space V is decomposed into K -irreducible subspaces

$$V = \overline{\bigoplus_{p=-\infty}^{\infty} V_p}, \quad \dim V_p \leq 1. \quad (3.4)$$

If it is not trivial, V_p is spanned by a Γ -automorphic function φ_p such that $\varphi_p(\mathrm{gk}[\theta]) = \exp(2ip\theta)\varphi(\mathrm{g})$; it is called a Γ -automorphic form of spectral parameter ν_V and weight $2p$.

Let us assume temporarily that V belongs to the unitary principal series, i.e., $\nu_V \in i(0, \infty)$ under (0.2). Then $\dim V_p = 1$ for all $p \in \mathbb{Z}$, and there exists a complete orthonormal system $\{\varphi_p \in V_p : p \in \mathbb{Z}\}$ of V such that

$$\begin{aligned} \varphi_p(\mathrm{g}) &= \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} |n|^{-\nu} \varrho_V(n) \mathcal{A}_n \phi_p(\mathrm{g}; \nu_V) \\ &= \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\varrho_V(n)}{\sqrt{|n|}} \mathcal{A}_{\mathrm{sgn}(n)} \phi_p(\mathfrak{a}[|n|]\mathrm{g}; \nu_V), \end{aligned} \quad (3.5)$$

where $\phi_p(\mathrm{g}; \nu) = y^{1/2+\nu} \exp(2ip\theta)$, and \mathcal{A}_u is the Jacquet operator:

$$\mathcal{A}_u \phi_p(\mathrm{g}; \nu) = \int_N \exp(-2\pi iuv) \phi_p(\mathrm{wn}[v]\mathrm{g}; \nu) dn, \quad \mathrm{w} = \mathfrak{k} \left[\frac{1}{2} \pi \right]. \quad (3.6)$$

It should be observed that the coefficients $\varrho_V(n)$ in (3.5) do not depend on the weight. We note that for $u \in \mathbb{R}^\times$

$$\begin{aligned} \mathcal{A}_u \phi_p(\mathrm{g}; \nu) &= y^{1/2-\nu} \exp(2\pi iux) \exp(2pi\theta) \int_{-\infty}^{\infty} \frac{\exp(2\pi yuv)}{(v^2+1)^{\frac{1}{2}+\nu}} \left(\frac{v+i}{v-i} \right)^p dv \\ &= (-1)^p \pi^{1/2+\nu} |u|^{\nu-1/2} \exp(2\pi iux) \exp(2pi\theta) \frac{W_{\mathrm{sgn}(u)p, \nu}(4\pi|u|y)}{\Gamma(\mathrm{sgn}(u)p + \frac{1}{2} + \nu)}, \end{aligned} \quad (3.7)$$

where $W_{\lambda, \mu}(y)$ is the Whittaker function (see [WW, Chapter XVI]). The first line is valid for $\mathrm{Re} \nu > 0$, while the second defines $\mathcal{A}_n \phi_p$ for all $\nu \in \mathbb{C}$. In particular, we have the expansion

$$\varphi_0(\mathrm{g}) = \frac{2\pi^{1/2+\nu_V}}{\Gamma(\frac{1}{2} + \nu_V)} \sqrt{y} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \varrho_V(n) K_{\nu_V}(2\pi|n|y) \exp(2\pi inx), \quad (3.8)$$

with K_ν being the K -Bessel function of order ν . This is a cusp-form on the hyperbolic upper half plane G/K .

Next, let us consider a V in the discrete series; that is, $\nu_V = \ell - \frac{1}{2}$, $1 \leq \ell \in \mathbb{Z}$. We have, in place of (3.4),

$$\text{either } V = \overline{\bigoplus_{p=\ell}^{\infty} V_p} \quad \text{or} \quad V = \overline{\bigoplus_{p=-\infty}^{-\ell} V_p}, \quad (3.9)$$

with $\dim V_p = 1$, corresponding to the holomorphic and the antiholomorphic discrete series. The involution $\mathrm{g} = \mathfrak{n}a\mathfrak{k} \mapsto \mathfrak{n}^{-1}a\mathfrak{k}^{-1}$ maps one to the other. In the holomorphic case, we have a complete orthonormal system $\{\varphi_p : p \geq \ell\}$ in V such that

$$\varphi_V(\mathrm{g}) = \pi^{1/2-\ell} \left(\frac{\Gamma(p+\ell)}{\Gamma(p-\ell+1)} \right)^{1/2} \sum_{n=1}^{\infty} n^{-\nu_V} \varrho_V(n) \mathcal{A}_n \phi_p(\mathrm{g}; \nu_V). \quad (3.10)$$

In particular, we have

$$\varphi_\ell(\mathfrak{g}) = (-1)^\ell \frac{2^{2\ell} \pi^{1/2+\ell}}{\sqrt{\Gamma(2\ell)}} \exp(2i\ell\theta) y^\ell \sum_{n=1}^{\infty} \varrho_V(n) n^{\ell-1/2} \exp(2\pi i n(x+iy)). \quad (3.11)$$

This infinite sum is a holomorphic cusp-form of weight 2ℓ on G/K .

With this, we may assume further that all V be Hecke invariant, so that there exists, for any integer $n \geq 1$, a real number $\tau_V(n)$ such that

$$T_n|_V = \tau_V(n) \cdot 1, \quad (3.12)$$

where T_n is as in (2.6). Thus, for any non-zero integer n ,

$$\varrho_V(n) = \varrho_V(\operatorname{sgn}(n)) \tau_V(|n|). \quad (3.13)$$

We may introduce the convention that $\varrho_V(-1) = 0$ and $\varrho_V(1) = 0$ for V in the holomorphic and antiholomorphic discrete series, respectively, and $\varrho_V(-1) = \epsilon_V \varrho_V(1)$ with $\epsilon_V = \pm 1$ for V in the unitary principal series. We associate, with each V , the Hecke series

$$H_V(s) = \sum_{n=1}^{\infty} \tau_V(n) n^{-s}, \quad \operatorname{Re} s > 1, \quad (3.14)$$

which continues to an entire function.

4. With this, we are going to decompose P_F via (3.2). We may restrict ourselves to the orthogonal projection ϖ_V with V in the unitary principal series. In fact, the discrete series is highly analogous, the projection to the subspace ${}^e L^2(\Gamma \backslash G)$ is facilitated by that Eisenstein series are explicitly defined, and the space of constant functions gives no specific problem.

We are of course concerned with how to compute $\varpi_V P_F$ in terms of F , or more generally with the harmonic analysis on V while the variable $\mathfrak{g} \in G$ is restricted to the big Bruhat cell. To this end we shall employ a reasoning which we term the Kirillov scheme. This is because we utilize an operator \mathcal{K} , defined by (4.3) below, that apparently originated in A.A. Kirillov [21]. We shall exploit two basic properties of \mathcal{K} , and they are embodied here in two lemmas, respectively:

Lemma 1. *Let $U = U_\nu$ with $\nu \in i\mathbb{R}$ be the Hilbert space*

$$\overline{\bigoplus_{p=-\infty}^{\infty} \mathbb{C}\phi_p}, \quad \phi_p(\mathfrak{g}) = \phi_p(\mathfrak{g}; \nu), \quad (4.1)$$

equipped with the norm

$$\|\phi\|_U = \sqrt{\sum_{p=-\infty}^{\infty} |c_p|^2}, \quad \phi = \sum_{p=-\infty}^{\infty} c_p \phi_p. \quad (4.2)$$

For $u \in \mathbb{R}^\times$ and smooth $\phi \in U$, that is, with c_p decaying faster than any negative power of $|p|$, we let

$$\mathcal{K}\phi(u) = |u|^{1/2-\nu} \mathcal{A}_u \phi(1) = \mathcal{A}_{\text{sgn}(u)} \phi(\mathfrak{a}[|u|]). \quad (4.3)$$

Then the operator \mathcal{K} maps U unitarily onto $L^2(\mathbb{R}^\times, (1/\pi)d^\times)$, where $d^\times u = du/|u|$.

Lemma 2. *Let us define the Bessel function of representations of $\text{PSL}_2(\mathbb{R})$ as to be*

$$j_\nu(u) = \pi \frac{\sqrt{|u|}}{\sin \pi \nu} \left(J_{-2\nu}^{\text{sgn}(u)}(4\pi\sqrt{|u|}) - J_{2\nu}^{\text{sgn}(u)}(4\pi\sqrt{|u|}) \right), \quad (4.4)$$

with $J_\nu^+ = J_\nu$ and $J_\nu^- = I_\nu$ in the ordinary notation for Bessel functions. Then, for any smooth $\phi \in U_\nu$, we have

$$\mathcal{K}R_w \phi(u) = \int_{\mathbb{R}^\times} j_\nu(uv) \mathcal{K}\phi(v) d^\times v, \quad u \in \mathbb{R}^\times. \quad (4.5)$$

Proof. It should be stressed that the definition (4.3) is taken from [9][35], and somewhat different from that employed in [32][33]. A proof of the unitarity of \mathcal{K} is given in [32] [33, Theorem 1]. It depends on the following integral formula via the second line of (3.7): For any $\alpha, \beta \in \mathbb{C}$ and $|\text{Re } \nu| < \frac{1}{2}$

$$\begin{aligned} \int_0^\infty W_{\lambda,\nu}(u) W_{\mu,\nu}(u) \frac{du}{u} &= \frac{\pi}{(\lambda - \mu) \sin(2\pi\nu)} \\ &\times \left[\frac{1}{\Gamma(\frac{1}{2} - \lambda + \nu) \Gamma(\frac{1}{2} - \mu - \nu)} - \frac{1}{\Gamma(\frac{1}{2} - \lambda - \nu) \Gamma(\frac{1}{2} - \mu + \nu)} \right], \end{aligned} \quad (4.6)$$

which is tabulated as [14, eq. 7.611(3)]. The verification of this made in [32] [33] employs the Whittaker differential equation ([41, p. 337]) which is related to the Casimir operator. The surjectivity of \mathcal{K} is proved in [9] via the first line of (3.7) and the completeness of the system $\{((v+i)/(v-i))^p : p \in \mathbb{Z}\}$ in the space $L^2(\mathbb{R}, dv/(\pi(v^2+1)))$. As to Lemma 2, the realization (4.5) of the action of w the Weyl element in terms of the space $L^2(\mathbb{R}^\times, (1/\pi)d^\times)$ seems to have been published for the first time by N.Ja. Vilenkin (see [38, Section 7 of Chapter VII] as well as [39, eq. (17) on p. 454]), though the concept of the Bessel function of representations had been coined by I.M. Gel'fand, M.I. Graev and I.I. Pyatetski-Shapiro [12]. Two independent proofs are known; they are conceptually different. One is due to M. Baruch and Z. Mao [2], which is along the line of [38] and fills a gap therein concerning a convergence issue. The other is due to ourselves [31][32][33], and seems more in line with the purpose of the present work. It is shown there that (4.5) is in fact equivalent to the Jacquet–Langlands local functional equation ([17, Theorem 5.15])

$$\begin{aligned} (-1)^p \Gamma_p(s) &= 2^{1-2s} \pi^{-2s} \Gamma(s+\nu) \Gamma(s-\nu) \\ &\times (\cos(\pi s) \Gamma_p(1-s) + \cos(\pi\nu) \Gamma_{-p}(1-s)) \end{aligned} \quad (4.7)$$

for the Mellin transform

$$\Gamma_p(s) = \int_0^\infty \mathcal{A}_u \phi_p(1) u^{s-\nu-1} du, \quad (4.8)$$

which continues meromorphically to \mathbb{C} . The Mellin inversion of (4.7) coupled with (4.9)–(4.10) below gives (4.5) for $\phi = \phi_p$; the extension to any smooth $\phi \in U$ is easy. Thus, if $|\operatorname{Re} \nu| - \frac{1}{2} < \operatorname{Re} s$, then

$$\int_{-\infty}^0 j_\nu(u) |u|^{s-1} du = \frac{1}{\pi} (2\pi)^{-2s} \cos(\pi\nu) \Gamma\left(s + \frac{1}{2} + \nu\right) \Gamma\left(s + \frac{1}{2} - \nu\right); \quad (4.9)$$

and if $|\operatorname{Re} \nu| - \frac{1}{2} < \operatorname{Re} s < -\frac{1}{4}$, then

$$\int_0^\infty j_\nu(u) u^{s-1} du = -\frac{1}{\pi} (2\pi)^{-2s} \sin(\pi s) \Gamma\left(s + \frac{1}{2} + \nu\right) \Gamma\left(s + \frac{1}{2} - \nu\right); \quad (4.10)$$

The former follows from [40, eq. (8) in Section 13.21], and the latter from [ibid, eq. (1) in Section 13.24]. This ends our brief discussion on the proof of the above lemmas.

The above extends not only to the discrete and the complementary series but also to the complex situation, i.e., to $\operatorname{PSL}_2(\mathbb{C})$, as is to be shown in Section 7.

5. Now we shall carry out the computation of $\varpi_V P_F$ via the Kirillov scheme. It should be stressed that absolute convergence required below can readily be confirmed, provided F is sufficiently smooth.

Thus, the projection to V_p is, by the unfolding argument,

$$\begin{aligned} \langle P_F, \varphi_p \rangle_{\Gamma \backslash G} &= \int_G F(\mathfrak{g}) \overline{\varphi_p(\mathfrak{g})} d\mathfrak{g} \\ &= \overline{\varrho_V(1)} \sum_{m=1}^{\infty} \frac{\tau_V(m)}{\sqrt{m}} (\Phi_p^+ + \epsilon_V \Phi_p^-) F_m(\nu_V), \end{aligned} \quad (5.1)$$

where (3.5), (3.13) are used; $F_m(\mathfrak{g}) = F(\mathfrak{a}[m]^{-1}\mathfrak{g})$ and

$$\Phi_p^\delta F(\nu) = \int_G F(\mathfrak{g}) \overline{\mathcal{A}_\delta \phi_p(\mathfrak{g})} d\mathfrak{g}. \quad (5.2)$$

Thus

$$\begin{aligned} \varpi_V P_F(\mathfrak{g}) &= \sum_{p=-\infty}^{\infty} \langle P_F, \varphi_p \rangle_{\Gamma \backslash G} \varphi_p(\mathfrak{g}) \\ &= |\varrho_V(1)|^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\tau_V(m) \tau_V(n)}{\sqrt{mn}} \\ &\quad \times \left(\mathcal{B}^{(+,+)} + \mathcal{B}^{(-,-)} + \epsilon_V \mathcal{B}^{(+,-)} + \epsilon_V \mathcal{B}^{(-,+)} \right) F_m(\mathfrak{a}[n]\mathfrak{g}; \nu_V), \end{aligned} \quad (5.3)$$

where

$$\begin{aligned} \mathcal{B}^{(\delta_1, \delta_2)} F(\mathfrak{g}; \nu) &= \sum_{p=-\infty}^{\infty} \Phi_p^{\delta_1} F(\nu) \mathcal{A}_{\delta_2} \phi_p(\mathfrak{g}; \nu) \\ &= \exp(2\pi i \delta_2 x) \sum_{p=-\infty}^{\infty} \Phi_p^{\delta_1} F(\nu) \mathcal{A}_{\delta_2} \phi_p(\mathfrak{a}[y]) \exp(2ip\theta). \end{aligned} \quad (5.4)$$

Since it can be asserted, with an appropriate change of F , that our interest is in the value $\varpi_V P_F(1)$ (see (2.5)), we may restrict ourselves to the subgroup A . Namely, it suffices to consider $\mathcal{B}^{(\delta_1, \delta_2)} F(\mathfrak{a}[y]; \nu)$, with a new F ; and this can be expressed in terms of the Kirillov operator:

$$\mathcal{B}^{(\delta_1, \delta_2)} F(\mathfrak{a}[y]; \nu) = \mathcal{K}\mathcal{L}^{\delta_1} F(\delta_2 y), \quad \mathcal{L}^\delta F = \sum_{p=-\infty}^{\infty} \Phi_p^\delta F(\nu) \phi_p, \quad (5.5)$$

where $\mathcal{L}^\delta F \in U$ is smooth. Hence, we may proceed in the sense of weak convergence, appealing to Lemma 1; and we observe

$$\Phi_p^\delta F(\nu) = \langle \mathcal{L}^\delta F, \phi_p \rangle_U = \frac{1}{\pi} \int_{\mathbb{R}^\times} \mathcal{K}\mathcal{L}^\delta F(u) \overline{\mathcal{K}\phi_p(u)} d^\times u. \quad (5.6)$$

This means that if we are able to transform (5.2) into

$$\Phi_p^\delta F(\nu) = \frac{1}{\pi} \int_{\mathbb{R}^\times} Y^\delta(u) \overline{\mathcal{K}\phi_p(u)} d^\times u, \quad (5.7)$$

then it should follow that

$$\mathcal{B}^{(\delta_1, \delta_2)} F(\mathfrak{a}[y]; \nu) = Y^{\delta_1}(\delta_2 y), \quad (5.8)$$

because of the surjectivity assertion in the lemma.

Since the integral in (5.2) is in fact over the big Bruhat cell, we perform the change of variables accordingly. We have instead

$$\Phi_p^\delta F(\nu) = \int_0^\infty \int_{N_{\mathbb{W}N}} F(\mathfrak{a}[u]g) \overline{R_g \mathcal{A}_\delta \phi_p(\mathfrak{a}[u])} d\dot{g} \frac{du}{u}. \quad (5.9)$$

Here R_g is the right translation with $g = n[x_1] \mathfrak{w}n[x_2]$, and $d\dot{g} = dx_1 dx_2 / \pi$. We observe that

$$\begin{aligned} R_g \mathcal{A}_\delta \phi_p(\mathfrak{a}[u]) &= \exp(2\pi i \delta x_1 u) \mathcal{A}_\delta R_{\mathfrak{w}} R_{n[x_2]} \phi_p(\mathfrak{a}[u]) \\ &= \exp(2\pi i \delta x_1 u) \mathcal{K} R_{\mathfrak{w}} R_{n[x_2]} \phi_p(\delta u). \end{aligned} \quad (5.10)$$

By Lemma 2 this is replaced by

$$R_g \mathcal{A}_\delta \phi_p(\mathfrak{a}[u]) = \exp(2\pi i \delta x_1 u) \int_{\mathbb{R}^\times} \exp(2\pi i x_2 v) j_\nu(\delta u v) \mathcal{K}\phi_p(v) d^\times v, \quad (5.11)$$

and (5.9) by

$$\begin{aligned} \Phi_p^\delta F(\nu) &= \frac{1}{\pi} \int_0^\infty \int_{\mathbb{R}^2} F(\mathfrak{a}[u]n[x_1] \mathfrak{w}n[x_2]) \exp(-2\pi i \delta x_1 u) \\ &\quad \times \int_{\mathbb{R}^\times} \exp(-2\pi i x_2 v) j_\nu(\delta u v) \overline{\mathcal{K}\phi_p(v)} d^\times v dx_1 dx_2 \frac{du}{u}. \end{aligned} \quad (5.12)$$

Hence we find via (5.8) that

$$\begin{aligned} \mathcal{B}^{(\delta_1, \delta_2)} F(\mathfrak{a}[y]; \nu) &= \int_0^\infty j_\nu(\delta_1 \delta_2 y u) \\ &\quad \times \left\{ \int_{\mathbb{R}^2} F(\mathfrak{a}[u]n[x_1] \mathfrak{w}n[x_2]) \exp(-2\pi i \delta_1 u x_1 - 2\pi i \delta_2 y x_2) dx_1 dx_2 \right\} \frac{du}{u}, \end{aligned} \quad (5.13)$$

which ends the application of the Kirillov scheme.

We may compare (4.7) with (1.7). Then (4.5) may also be compared with (1.8). That is, the formula (5.3) coupled with (5.13) which is a local assertion derived from (4.5) corresponds to (1.8). As remarked above already, analogues of Lemmas 1 and 2 are shown in [9][32] for the discrete series representations and the complementary series, although the latter is irrelevant under (0.2). Hence, returning to (3.2), we obtain a genuine extension of (1.8). The final result is, however, too complicated to be stated as an independent assertion. We should instead be content with the local expression (5.13) and with the fact that we have found that the combination of Lemmas 1 and 2 is the key implement.

Hence, what corresponds to the cosine-transform in (1.8) is (5.3) with (5.13). One may desire to compute the double sum (5.3) and the last double integral into closed forms. In the applications to $\mathcal{M}(\zeta^2, g)$ and to the sum (2.7), which will be briefly dealt with below, we are in a fortuitous situation that the double sum is transformed into a product of two values of H_V . As to the double integral, it is a Fourier transform over the Euclidean plane, and thus might be expressed in terms of a Bessel transform. With $\mathcal{M}(\zeta^2, g)$ as well as (2.7), the situation turns out in fact to be as such, and we shall see that (5.13) is expressed as an integral transform whose kernel is a convolution of two instances of the Bessel function of representations (see (6.2) below).

Thus the matter seems to depend much on the specific nature of the seed F . Nevertheless, with any smooth F , one might appeal to Mellin transform of several variables, and the above could be pushed into a more closed form.

6. We are now at the stage to render the spectral decomposition of $\mathcal{M}(\zeta^2, g)$ in terms of notions from representation theory: Thus, let us put

$$\Theta(\nu; g) = \frac{1}{4 \cos(\pi\nu)} \int_0^\infty \left(\frac{u}{u+1} \right)^{1/2} g_c(\log(1+1/u)) \Xi(u; \nu) d^\times u, \quad (6.1)$$

$$\Xi(u; \nu) = \int_{\mathbb{R}^\times} j_0(-v) j_\nu \left(\frac{v}{u} \right) \frac{d^\times v}{\sqrt{|v|}}. \quad (6.2)$$

Then we have

$$\mathcal{M}(\zeta^2, g) = \left\{ \mathcal{M}^{(r)} + \mathcal{M}^{(c)} + \mathcal{M}^{(e)} \right\} (\zeta^2, g), \quad (6.3)$$

where

$$\mathcal{M}^{(c)}(\zeta^2, g) = \sum_V \alpha_V H_V \left(\frac{1}{2} \right)^3 \Theta(\nu_V; g), \quad (6.4)$$

$$\mathcal{M}^{(e)}(\zeta^2, g) = \int_{(0)} \frac{|\zeta(\frac{1}{2} + \nu)|^6}{|\zeta(1 + 2\nu)|^2} \Theta(\nu; g) \frac{d\nu}{2\pi i}, \quad (6.5)$$

with $\alpha_V = |\varrho_V(1)|^2 + |\varrho_V(-1)|^2$. The V runs over a maximal orthogonal system of Hecke-invariant cuspidal Γ -automorphic representations of G . Apart the term $2\pi \operatorname{Re} \{ (\log(2\pi) - \gamma_E)g(\frac{1}{2}i) - \frac{1}{2}ig'(\frac{1}{2}i) \}$, the $\mathcal{M}^{(r)}(\zeta^2, g)$ is an integral transform of g whose kernel is given explicitly in terms of logarithmic derivatives of the Gamma function.

The proof of (6.3) as developed in [9] starts with the integration of $\zeta(z_1 + it)\zeta(z_2 + it)\zeta(z_3 - it)\zeta(z_4 - it)$ against $g(t)dt$ over \mathbb{R} , where (z_1, z_2, z_3, z_4) is to remain in the region

of absolute convergence. The device (2.3)–(2.5) is to be applied. However, the naive choice of the seed does not work well, because we require it be smooth and of rapid decay on G . Such a choice of the seed is not difficult but somewhat subtle. Thus, we are forced to employ instead a sequence of suitable F 's and a limiting procedure with respect to F . Nevertheless, those F chosen in [9] are nice in that they are A -equivariant, i.e., $F(a[y]g) = y^\omega F(g)$ with an $\omega \in \mathbb{C}$ being independent of F . Hence the summation over f in (2.5) is the same as the multiplication by a value of H_V at each V ; and also the sum over m in (5.1) can be written in terms of a product of a value of H_V and $(\Phi_p^+ + \epsilon_V \Phi_p^-)F(\nu_V)$. In particular, the omission of m in (5.4) is possible without changing the specification of F , which amounts to a considerable simplification in the subsequent discussion leading to (5.13). Besides, this makes (5.13) easier to handle; taking the limit in F we come already close to the expression (6.2). Moreover, the sum over n in (5.3) yields now another factor in a value of H_V . Then it remains to perform analytic continuation and specialization with respect to (z_1, z_2, z_3, z_4) . Thus the factor j_0 in (6.2) stands for the integral kernel of the Bessel transform that emerges from the inner double integral of (5.13) at the end of the whole procedure (see [9, (7.34)]). This explains how (6.2) originates and reveals especially the mechanism behind (6.4).

The same argument can be applied to the additive divisor sum (2.7). We are led to a spectral decomposition analogous to (6.3). Specifically, the cuspidal part is found to be

$$\frac{1}{4} f^{(\lambda+\mu+1)/2} \sum_V \alpha_V \tau_V(f) H_V \left(\frac{1}{2}(1 - \lambda - \mu) \right) H_V \left(\frac{1}{2}(1 + \lambda - \mu) \right) \times (\Psi_+ + \epsilon_V \Psi_-)(\nu_V; \lambda, \mu; W), \quad (6.6)$$

where

$$\Psi_\delta(\nu; \lambda, \mu; W) = \int_0^\infty W(u) \Lambda_\delta(u; \nu; \lambda, \mu) u^{(\lambda+\mu)/2+1} d^\times u, \quad (6.7)$$

with

$$\Lambda_\delta(u; \nu; \lambda, \mu) = \int_0^\infty j_{\lambda/2}(-\delta v) j_\nu(\delta v/u) \frac{d^\times v}{v^{(\mu+1)/2}}. \quad (6.8)$$

It is safe to keep both $|\operatorname{Re} \lambda|$, $|\operatorname{Re} \mu|$ sufficiently small so that (6.6) holds with the expression (6.8). However, as can be seen from (4.9)–(4.10), Λ_δ can be expressed in terms of the Mellin inversion of a product of four Gamma factors, and then (6.7) allows us to continue (6.6) analytically to quite a wide domain of (λ, μ) .

It has been explained in the above how the factor j_0 in (6.2) turns up; the same can be applied to $j_{\lambda/2}$ in (6.8). However, it is not done in any framework of metric theory. This is sharply different from the situation with another factor j_ν shared in both equations. Thus it remains still to find a genuine characterization of the factors j_0 , $j_{\lambda/2}$. In passing, we note that instead of (6.2) we may write

$$\Xi(u; \nu) = 2\operatorname{Re} \left\{ u^{-1/2-\nu} \left(1 - \frac{1}{\sin(\pi\nu)} \right) \frac{\Gamma^2(\frac{1}{2} + \nu)}{\Gamma(1 + 2\nu)} {}_2F_1 \left(\frac{1}{2} + \nu, \frac{1}{2} + \nu; 1 + 2\nu; -1/u \right) \right\}, \quad (6.9)$$

with the Gaussian hypergeometric function ${}_2F_1$ (see [30, (4.7.2)]). This reminds us of the free-space resolvent kernel of the hyperbolic Laplacian (see [ibid, (1.1.49)]), a fact that appears mysterious to us.

It might be expedient to make here a digression on a historical background: A prototype of the spectral decomposition of $\mathcal{M}(\zeta^2, g)$ was obtained by the present author in [24][25], which was afterwards improved to (6.3) in [30, Theorem 4.1]. However, the assertion there did not reach the expression (6.2); it was stated with (6.9). A reason for this is in that there we used the Kloosterman-Spectral sum formula of N.V. Kuznetsov [30, Theorems 2.3 and 2.5], which is pretty handy but hides the mechanism working behind the integral transform appearing on the spectral side. Note that in the above we dispensed with Kuznetsov's sum formula. The argument of [9], whose most salient part is depicted in the previous section, is admittedly more involved than that in [30], but this is much due to the fact that we started from the very fundamental assertion (3.2), whereas the discussion in [30] lacks the perspective offered by representation theory. Thus, the appearance in Kuznetsov's sum formula and consequently in [30, Theorem 4.1] of the contribution of holomorphic cusp forms was just an accidental byproduct of a technical marvel and remained mysterious there. Our discussion of $\mathcal{M}(\zeta^2, g)$ in terms of a Poincaré series on the group G allows us to see all contributions of cusp forms in a fairly equal term, since our method is based on (3.2), where all irreducible representations have equal rights.

It was Bruggeman [3][4] who tried for the first time to understand, via (3.2), all the terms on the spectral side in Kuznetsov's sum formula. However, the real comprehension of the structure supporting the sum formula appears to have been done by J.W. Cogdell and I. Piatetski-Shapiro in [11]. In particular, the Kirillov scheme together with the rôle of the Bessel function of representations was developed there, and Kuznetsov's sum formula was newly proved, though their discussion appears sketchy to us. The authors of [9] were inspired by the work [11].

7. The aim of this and the next sections is to show that the above discussion extends to the situation (0.3). In particular, we are going to show the complex analogues of Lemmas 1 and 2. Note that some symbols used under (0.2) are now assigned to corresponding notions under (0.3); this convention should not cause any confusion.

Thus, let $G = \mathrm{PSL}_2(\mathbb{C})$, and put

$$n[z] = \begin{bmatrix} 1 & z \\ & 1 \end{bmatrix}, \quad h[u] = \begin{bmatrix} u & \\ & 1/u \end{bmatrix}, \quad k = \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}, \quad (7.1)$$

where $z, u, \alpha, \beta \in \mathbb{C}$ with $u \neq 0$, $|\alpha|^2 + |\beta|^2 = 1$; and also $N = \{n[z] : z \in \mathbb{C}\}$, $A = \{a[r] : r > 0\}$, $K = \mathrm{PSU}(2) = \{k[\alpha, \beta] : \alpha, \beta \in \mathbb{C}\}$ with $a[r] = h[\sqrt{r}]$. In terms of the Euler angles φ, θ, ψ , we have

$$k[\alpha, \beta] = h[e^{i\varphi/2}]v[i\theta]h[e^{i\psi/2}], \quad v[\theta] = k[\cosh(\theta/2), \sinh(\theta/2)]. \quad (7.2)$$

The Iwasawa decomposition $G = NAK$ is read as $G \ni g = n[z]a[r]k[\alpha, \beta]$. The Haar measures on respective groups are given by $dn = dz$, $da = dr/r$, $dk = \sin\theta d\varphi d\theta d\psi/(8\pi^2)$, and $dg = dndadk/r^2$. With this, the Hilbert space $L^2(\Gamma \backslash G)$ is formed, to which G acts from the right; and we have an exact analogue of (3.2). See [5][8] for more details of what follows.

The cuspidal subspace decomposes into irreducible subspaces

$${}^cL^2(\Gamma \backslash G) = \overline{\bigoplus V}. \quad (7.3)$$

To classify representations V , we need two Casimir operators Ω_{\pm} , $\Omega_{-} = \overline{\Omega_{+}}$, where

$$\begin{aligned} \Omega_{+} = & \frac{1}{2}r^2\partial_z\partial_{\bar{z}} + \frac{1}{2}re^{i\varphi}\cot\theta\partial_z\partial_{\varphi} - \frac{1}{2}ire^{i\varphi}\partial_z\partial_{\theta} - \frac{re^{i\varphi}}{2\sin\theta}\partial_z\partial_{\psi} \\ & + \frac{1}{8}r^2\partial_r^2 - \frac{1}{4}ir\partial_r\partial_{\varphi} - \frac{1}{8}\partial_{\varphi}^2 - \frac{1}{8}r\partial_r + \frac{1}{4}i\partial_{\varphi}. \end{aligned} \quad (7.4)$$

They become constant multiplications in each V :

$$\Omega_{\pm}|V = \chi_V^{\pm} \cdot 1 = \frac{1}{8}((p_V \mp \nu_V)^2 - 1) \cdot 1, \quad p_V \in \mathbb{Z}, \nu_V \in i[0, \infty). \quad (7.5)$$

The pair (p_V, ν_V) is called the spectral parameter of V , but in the sequel we shall write simply $(p_V, \nu_V) = (p, \nu)$.

According to the action of K , the space V decomposes into K -irreducible subspaces

$$V = \overline{\bigoplus_{|p| \leq l, |q| \leq l} V_{l,q}}, \quad \dim V_{l,q} = 1. \quad (7.6)$$

To describe this precisely, let Ω_K be the Casimir element of the universal enveloping algebra of K defined by

$$\Omega_K = \frac{1}{2\sin^2\theta} (\partial_{\varphi}^2 + \sin^2\theta\partial_{\theta}^2 + \partial_{\psi}^2 - 2\cos\theta\partial_{\varphi}\partial_{\psi} + \sin\theta\cos\theta\partial_{\theta}). \quad (7.7)$$

Then

$$V_{l,q} = \{F \in V : \Omega_K F = -\frac{1}{2}l(l+1), \partial_{\psi} F = -iqF\}. \quad (7.8)$$

Any non-zero element of $V_{l,q}$ is called a Γ -automorphic form of spectral parameter (p, ν) and K -type (l, q) .

Next, we define functions $\Phi_{p,q}^l$ on K by

$$(\alpha X - \bar{\beta})^{l-q}(\beta X + \bar{\alpha})^{l+q} = \sum_{p=-l}^l \Phi_{p,q}^l(\mathfrak{k}[\alpha, \beta])X^{l-p}. \quad (7.9)$$

The system $\{\Phi_{p,q}^l : |p|, |q| \leq l, 1 \leq l\}$ is a complete orthogonal basis of $L^2(K)$ with norms

$$\|\Phi_{p,q}^l\|_K = \frac{1}{\sqrt{l + \frac{1}{2}}} \binom{2l}{l-p}^{1/2} \binom{2l}{l-q}^{-1/2}. \quad (7.10)$$

Here are some of its properties which we shall need later: Under the convention that $\Phi_{p,q}^l \equiv 0$ if the condition $|p|, |q| \leq l$ is violated, we have

$$\Phi_{p,q}^l(\mathfrak{k}[\alpha, \beta]) = e^{-ip\varphi - iq\psi} \Phi_{p,q}^l(\mathfrak{v}[i\theta]), \quad (7.11)$$

$$\begin{aligned} 2\partial_{\theta}\Phi_{p,q}^l(\mathfrak{v}[i\theta]) &= i(l+p+1)\Phi_{p+1,q}^l(\mathfrak{v}[i\theta]) + i(l-p+1)\Phi_{p-1,q}^l(\mathfrak{v}[i\theta]) \\ &= i(l-q)\Phi_{p,q+1}^l(\mathfrak{v}[i\theta]) + i(l+q)\Phi_{p,q-1}^l(\mathfrak{v}[i\theta]), \end{aligned} \quad (7.12)$$

$$2\frac{p-q\cos\theta}{\sin\theta}\Phi_{p,q}^l(\mathfrak{v}[i\theta]) = i(l-q)\Phi_{p,q+1}^l - i(l+q)\Phi_{p,q-1}^l(\mathfrak{v}[i\theta]), \quad (7.13)$$

$$2\frac{q-p\cos\theta}{\sin\theta}\Phi_{p,q}^l(\mathfrak{v}[i\theta]) = i(l+p+1)\Phi_{p+1,q}^l - i(l-p+1)\Phi_{p-1,q}^l(\mathfrak{v}[i\theta]). \quad (7.14)$$

For a verification of (7.12)–(7.14) see [5, Lemma 5].

With this, we put $\phi_{l,q}(\mathfrak{g}; \nu) = r^{1+\nu} \Phi_{p,q}^l(\mathfrak{k}) / \|\Phi_{p,q}^l\|_K$; and its Jacquet transform is defined by

$$\mathcal{A}_u \phi_{l,q}(\mathfrak{g}; \nu) = \int_N \exp(-2\pi i \operatorname{Re}(uv)) \phi_{l,q}(\operatorname{wn}[v]\mathfrak{g}; \nu) dn. \quad (7.15)$$

Let $\varphi_{l,q}$ be a generating vector of $V_{l,q}$, so that $\{\varphi_{l,q} : |p| \leq l, |q| \leq l\}$ is a complete orthonormal system in V . Then we have

$$\varphi_{l,q}(\mathfrak{g}) = \sum_{\omega \neq 0} |\omega|^{-\nu} \varrho_V(\omega) \mathcal{A}_\omega \phi_{l,q}(\mathfrak{g}; \nu), \quad \omega \in \mathbb{Z}[i], \quad (7.16)$$

which is precisely an analogue of the first line of (3.5).

Further, we define the Hecke operator for each non-zero $f \in \mathbb{Z}[i]$ by

$$T_f F(\mathfrak{g}) = \frac{1}{4|f|} \sum_{d|f} \sum_{b \bmod d} F \left(\begin{bmatrix} \sqrt{f}/d & b \\ & d/\sqrt{f} \end{bmatrix} \mathfrak{g} \right); \quad (7.17)$$

and we assume that all V are Hecke invariant so that there exists a real number $\tau_V(f)$ such that $T_f|_V = \tau_V(f) \cdot 1$. In particular we have, for all non-zero $n \in \mathbb{Z}[i]$,

$$\varrho_V(n) = \varrho_V(1)(n/|n|)^p \tau_V(n), \quad (7.18)$$

We have

$$\tau_V(-n) = \tau_V(n), \quad \tau_V(in) = \epsilon_V \tau_V(n), \quad \epsilon_V = \pm 1 \quad (7.19)$$

The Hecke L -function of the space V is defined by

$$H_V(s) = \frac{1}{4} \sum_{n \neq 0} \tau_V(n) |n|^{-2s}, \quad (7.20)$$

which continues to an entire function; note that $H_V \equiv 0$ whenever $\epsilon_V = -1$.

Now, the complex analogues of Lemmas 1 and 2 are as follows:

Lemma 3. *Let $U = U_{p,\nu}$ be the Hilbert space*

$$\overline{\bigoplus_{|p| \leq l, |q| \leq l} \mathbb{C} \phi_{l,q}}, \quad \phi_{l,q}(\mathfrak{g}) = \phi_{l,q}(\mathfrak{g}; \nu) \quad (7.21)$$

equipped with the norm

$$\|\phi\|_U = \sqrt{\sum_{|p| \leq l, |q| \leq l} |c_{l,q}|^2}, \quad \phi = \sum_{|p| \leq l, |q| \leq l} c_{l,q} \phi_{l,q}. \quad (7.22)$$

For $u \in \mathbb{C}^\times$ and smooth $\phi \in U$, we let

$$\mathcal{K}\phi(u) = |u|^{1-\nu} (u/|u|)^p \mathcal{A}_u \phi(1). \quad (7.23)$$

Then the operator \mathcal{K} maps U unitarily onto $L^2(\mathbb{C}^\times, (2/\pi)d^\times)$, where $d^\times u = du/|u|^2$.

Lemma 4. *Let us define the Bessel function of representations of $\mathrm{PSL}_2(\mathbb{C})$ as to be*

$$j_{p,\nu}(u) = 2\pi^2 \frac{|u|^2}{\sin(\pi\nu)} (J_{p-\nu}(2\pi u)J_{-p-\nu}(2\pi\bar{u}) - J_{-p+\nu}(2\pi u)J_{p+\nu}(2\pi\bar{u})). \quad (7.24)$$

Then we have, for any smooth $\phi \in U_{p,\nu}$,

$$\mathcal{K}R_w\phi(u^2) = \int_{\mathbb{C}^\times} j_{p,\nu}(uv)\mathcal{K}\phi(v^2)d^\times v, \quad u \in \mathbb{C}^\times. \quad (7.25)$$

Here $J_{p-\nu}(u)J_{-p-\nu}(\bar{u})$ is understood to be equal to $(u/|u|)^p|u|^{-2\nu}J_{p-\nu}^*(u)J_{-p-\nu}^*(\bar{u})$ where $J_\mu^*(u)$ is the entire function that coincides with $u^{-\mu}J_\mu(u)$ when $u > 0$.

With this, the analogues of (6.1)–(6.5) for the Dedekind zeta-function ζ_k of the Gaussian number field $k = \mathbb{Q}(i)$ can be rendered as follows: Let us put

$$\Theta(p, \nu; g) = \frac{\nu}{16 \sin(\pi\nu)} \int_{\mathbb{C}} \frac{|u|}{|u+1|} g_c(2 \log |1 + 1/u|) \Xi(u; p, \nu) d^\times u, \quad (7.26)$$

$$\Xi(u; p, \nu) = \int_{\mathbb{C}^\times} j_{0,0}(\sqrt{-v}) j_{p,\nu}(\sqrt{v/u}) \frac{d^\times v}{|v|}. \quad (7.27)$$

Then we have

$$\mathcal{M}(\zeta_k^2, g) = \left\{ \mathcal{M}^{(r)} + \mathcal{M}^{(c)} + \mathcal{M}^{(e)} \right\} (\zeta_k^2, g), \quad (7.28)$$

with V running over a maximal orthogonal system of Hecke-invariant cuspidal Γ -automorphic representations of G . Here $\mathcal{M}^{(r)}(\zeta_k^2, g)$ is analogous to $\mathcal{M}^{(r)}(\zeta^2, g)$, and

$$\mathcal{M}^{(c)}(\zeta_k^2, g) = \sum_V |\varrho_V(1)|^2 H_V \left(\frac{1}{2}\right)^3 \Theta(p_V, \nu_V; g), \quad (7.29)$$

$$\mathcal{M}^{(e)}(\zeta_k, g) = \sum_{p=-\infty}^{\infty} \int_{(0)} \frac{|\zeta_k(\frac{1}{2}(1+\nu), p)|^6}{|\zeta_k(1+\nu, 2p)|^2} \Theta(4p, \nu; g) \frac{d\nu}{2\pi i}, \quad (7.30)$$

where $\zeta_k(s, p)$ is defined by

$$\zeta_k(s, p) = \frac{1}{4} \sum_{n \neq 0} (n/|n|)^{4p} |n|^{-2s}, \quad \mathrm{Re} s > 1, \quad (7.31)$$

which continues meromorphically to \mathbb{C} .

The spectral decomposition (7.28) was proved in [8]. The argument was a faithful extension of the older proof of (6.3); that is, it depended on the sum formula of Kloosterman sums under the situation (0.3) that was established also in [8]. Thus, in much the same mechanism as Lemmas 1 and 2 did with Kuznetsov's sum formula, the last two lemmas should allow us to dispense with the sum formula of Kloosterman sums, in deriving (7.28). This claim is still to be checked fully; but it is certain such a new proof is available and conceptually simpler than that in [8].

Comparing (7.26)–(7.30) with (6.1)–(6.5), the outward similarity is striking. However, our good luck ends there. That is, the asymptotic nature of (6.3) is much superior than that of its counterpart (7.28). In fact, (7.28) does not seem suitable to be utilized as a means to derive quantitative assertions on the fourth power moment of ζ_k . Concerning this difficulty, Sarnak kindly suggested us to try to take further averaging:

$$\sum_{q=-\infty}^{\infty} \mathcal{M}(\zeta_k^2(\cdot, q), g)h(q), \quad (7.32)$$

with a smooth weight h . The argument of [8] should extend to this sum. The same can be said about an obvious analogue of (2.7). The latter is expected to yield an extension of the main result of [19] to L -functions associated with the group $\mathrm{PSL}_2(\mathbb{Z}[i])$. In fact this has been in our current investigation. However, we have not worked out all the details yet.

8. Now, we are about to prove Lemmas 3 and 4. This section is a reworking of our joint work [7] with Bruggeman; a few corrections are made. It should be stressed that the surjectivity assertion in Lemma 3 is a new addition, and that the definition (7.23) is somewhat different from that employed in [32].

We begin with the unitarity of \mathcal{K} : Naturally it is sufficient to show the orthogonality relation:

$$\frac{2}{\pi} \int_{\mathbb{C}^\times} \mathcal{K}\phi_{l,q}(u) \overline{\mathcal{K}\phi_{l',q'}(u)} d^\times u = \delta_{l,l'} \delta_{q,q'}, \quad (8.1)$$

with Kronecker deltas. By definition,

$$\mathcal{K}\phi_{l,q}(u) = (u/|u|)^{-q} \mathcal{A}_1 \phi_{l,q}(\mathfrak{a}[|u|]) / \|\Phi_{p,q}^l\|_K. \quad (8.2)$$

Note

$$\begin{aligned} \mathcal{A}_1 \phi_{l,q}(g) &= \exp(2\pi i \mathrm{Re}(z)) \sum_{|m| \leq l} v_m^l(r) \Phi_{m,q}^l(\mathfrak{k}) \\ &= \exp(2\pi i \mathrm{Re}(z)) \sum_{|m| \leq l} e^{-im\varphi - iq\psi} v_m^l(r) \Phi_{m,q}^l(v[i\theta]), \end{aligned} \quad (8.3)$$

where (7.11) has been used, and

$$\begin{aligned} v_m^l(r) &= \mathcal{A}_1 \phi_{l,m}(\mathfrak{a}[r]) \\ &= r^{1-\nu} \int_{\mathbb{C}} \frac{\exp(-2\pi i r \mathrm{Re} v)}{(1+|v|^2)^{1+\nu}} \Phi_{p,m}^l \left(\mathfrak{k} \left[\frac{\bar{v}}{\sqrt{1+|v|^2}}, \frac{-1}{\sqrt{1+|v|^2}} \right] \right) d^+v \\ &= 2\pi i^{-p-m} r^{1-\nu} \int_0^\infty \frac{J_{p+m}(2\pi r v)}{(1+v^2)^{1+\nu}} \Phi_{p,m}^l \left(\mathfrak{k} \left[\frac{v}{\sqrt{1+v^2}}, \frac{-1}{\sqrt{1+v^2}} \right] \right) v dv, \end{aligned} \quad (8.4)$$

where $d^+v = (d\mathrm{Re} v)(d\mathrm{Im} v)$. The left side of (8.1) is equal to

$$4 \frac{\delta_{q,q'}}{\|\Phi_{p,q}^l\|_K^2} \int_0^\infty v_q^l(r) \overline{v_q^{l'}(r)} \frac{dr}{r}. \quad (8.5)$$

Then we observe that the functions $v_q^l(r)$ satisfy the differential equations

$$\begin{aligned} D_q^+ v_q^l(r) &= -4\pi i(l-q)r^{-1}v_{q+1}^l(r), \\ D_q^- v_q^l(r) &= 4\pi i(l+q)r^{-1}v_{q-1}^l(r), \end{aligned} \quad (8.6)$$

where $D_q^- = \overline{D_{-q}^+}$ and

$$D_q^+ = \left(\frac{d}{dr}\right)^2 - (2q+1)r^{-1}\frac{d}{dr} + r^{-2}(q^2 + 2q - 4\pi^2 r^2 - 8\chi_V^+). \quad (8.7)$$

To show this, we apply Ω_+ to (8.3). We have $\Omega_+ \mathcal{A}_1 \phi_{l,q}(g) = \mathcal{A}_1 \Omega_+ \phi_{l,q}(g)$. Thus, by (7.4),

$$\begin{aligned} \chi_V^+ \mathcal{A}_1 \phi_{l,q}(g) &= \exp(2\pi i \operatorname{Re} z) \sum_{|m| \leq l} \left\{ -\frac{1}{2}\pi^2 r^2 + \frac{1}{2}\pi m r e^{i\varphi} \cot \theta + \frac{1}{2}\pi r e^{i\varphi} \partial_\theta - \pi q \frac{r e^{i\varphi}}{2 \sin \theta} \right. \\ &\quad \left. + \frac{1}{8}r^2 \partial_r^2 - \frac{1}{4}mr \partial_r + \frac{1}{8}m^2 - \frac{1}{8}r \partial_r + \frac{1}{4}m \right\} e^{-im\varphi - iq\psi} v_m^l(r) \Phi_{m,q}^l(v[i\theta]). \end{aligned} \quad (8.8)$$

On the other hand, invoking the first line of (7.12) and (7.14), we have

$$\left(m \cot \theta + \partial_\theta - \frac{q}{\sin \theta}\right) \Phi_{m,q}^l(v[i\theta]) = i(l-m+1) \Phi_{m-1,q}^l(v[i\theta]). \quad (8.9)$$

In the last two identities we set $g = a[r]$, and note that $\Phi_{m,q}^l(1) = \delta_{m,q}$. Then we get the first identity of (8.6). In much the same way we get the second as well.

Returning to the integral in (8.5), we see that it is equal to

$$\begin{aligned} & -\frac{1}{4\pi i(l-q+1)} \int_0^\infty D_{q-1}^+ v_q^l(r) \cdot \overline{v_q^l(r)} dr \\ &= -\frac{1}{4\pi i(l-q+1)} \int_0^\infty v_{q-1}(r) \cdot \overline{D_q^- v_q^l(r)} dr \\ &= \frac{l+q}{l-q+1} \int_0^\infty v_{q-1}^l(r) \overline{v_{q-1}^l(r)} \frac{dr}{r}. \end{aligned} \quad (8.10)$$

This procedure is valid only if $v_q^l(r)$ tends to 0 sufficiently fast as r tends to either 0 or ∞ , which is in fact implied by the second line of (8.4). Hence

$$\begin{aligned} \int_0^\infty v_q^l(r) \overline{v_q^l(r)} \frac{dr}{r} &= \frac{l-q}{l+q+1} \int_0^\infty v_{q+1}^l(r) \overline{v_{q+1}^l(r)} \frac{dr}{r} \\ &= \delta_{l,\nu} \left(\frac{2l}{l-q}\right)^{-1} \int_0^\infty |v_l^l(r)|^2 \frac{dr}{r}. \end{aligned} \quad (8.11)$$

On the other hand we have, by the third line of (8.4) and by a formula of N.J. Sonine ([40, eq. (2) on p. 434]),

$$\begin{aligned} v_l^l(r) &= 2(-1)^{l-p} i^{-l-p} \pi r^{1-\nu} \binom{2l}{l-p} \int_0^\infty \frac{J_{l+p}(2\pi r v)}{(1+v^2)^{l+1+\nu}} v^{l+p+1} dv \\ &= 2(-1)^{l-p} i^{-l-p} \pi r^{1-\nu} \binom{2l}{l-p} \frac{(\pi r)^{l+\nu}}{\Gamma(l+\nu+1)} K_{p-\nu}(2\pi r), \end{aligned} \quad (8.12)$$

which gives, via either [14, eq. 4 of Section 6.576] or [30, (2.6.11)],

$$\int_0^\infty |v_l^l(r)|^2 \frac{dr}{r} = \frac{1}{4(l + \frac{1}{2})} \binom{2l}{l-p}, \quad (8.13)$$

and via (7.10), (8.5) and (8.11) we end the proof of (8.1).

We turn to the surjectivity assertion. Thus, let $F(u)$, $u \in \mathbb{C}^\times$, be smooth and compactly supported, and such that

$$\int_{\mathbb{C}^\times} F(u) \mathcal{K} \phi_{l,q}(u) d^\times u = 0, \quad \text{for all } (q, l) \text{ with } |p| \leq l, |q| \leq l. \quad (8.14)$$

We are to show $F \equiv 0$. In view of (8.2) we may assume that F is radial, i.e., the Fourier expansion of F in $u/|u|$ has only one term, say, the q -th. With an obvious change in F , we consider instead

$$\int_0^\infty F(r) v_q^l(r) dr = 0, \quad \text{for all } (q, l) \text{ with } |p| \leq l, |q| \leq l. \quad (8.15)$$

We then invoke that in [8, Lemma 5.1] more than (8.12) is proved; thus there are non-zero $\eta_{p,q}^l(j; \nu)$ such that

$$v_q^l(r) = \sum_{j=0}^{l-\max\{|p|, |q|\}} \eta_{p,q}^l(j; \nu) r^{\max\{|p|, |q|\}+1+j} K_{\nu+\max\{|p|, |q|\}-|p+q|+j}(2\pi r). \quad (8.16)$$

Namely, we are given, for all integers $l \geq \max\{|p|, |q|\}$,

$$\int_0^\infty F(r) r^{l+1} K_{\nu+l-|p+q|}(2\pi r) dr = 0. \quad (8.17)$$

We replace the Bessel factor by a well-known integral representation (see e.g., [30, (1.1.17)]), and find, after some rearrangement, that (8.17) is the same as

$$\int_0^\infty \exp(-\pi\xi) \xi^{\nu+l-|p+q|-1} \left\{ \int_0^\infty F(r) r^{1-\nu+|p+q|} \exp(-\pi r^2/\xi) dr \right\} d\xi = 0. \quad (8.18)$$

Because of the completeness of polynomials over $[0, \infty)$, the member inside the braces should vanish for any $\xi > 0$; that is, for $\operatorname{Re} \xi > 0$ by analytic continuation. Hence, the choice $\xi = 1/(1+it)$, $t \in \mathbb{R}$, yields that the Fourier transform of a multiple of $F(\sqrt{r})$ by a non-zero factor vanishes constantly. This ends the proof of Lemma 3.

We now move to the proof of Lemma 4. It should be noted that with ordinary bounds for Bessel functions one may verify absolute convergence and analytic continuation needed to carry out the reasoning below.

Thus, we consider the integral

$$\Gamma_{l,q}(s) = \int_0^\infty v_q^l(r) r^{2(s-1)} dr. \quad (8.19)$$

The third line of (8.4) gives

$$\Gamma_{l,q}(s) = \pi^{1+\nu-2s} (-1)^{\min(0,p+q)} i^{-p-q} \frac{\Gamma(s + \frac{1}{2}(|p+q| - \nu))}{\Gamma(1-s + \frac{1}{2}(|p+q| + \nu))} \mathbf{L}_{l,q}(s), \quad (8.20)$$

with

$$\mathbf{L}_{l,q}(s) = \int_0^\infty \frac{v^{1+\nu-2s}}{(1+v^2)^{1+\nu}} \Phi_{p,q}^l \left(\mathbf{k} \left[\frac{v}{\sqrt{1+v^2}}, \frac{-1}{\sqrt{1+v^2}} \right] \right) dv, \quad (8.21)$$

since for $m \in \mathbb{Z}$ and $-\frac{1}{2}|m| < \operatorname{Re} s < \frac{1}{4}$

$$\int_0^\infty J_m(r) (r/2)^{2s-1} dr = (-1)^m \frac{\Gamma(s + \frac{1}{2}|m|)}{\Gamma(1-s + \frac{1}{2}|m|)}. \quad (8.22)$$

We have the functional equation

$$\mathbf{L}_{l,q}(s) = (-1)^{l-p} \mathbf{L}_{l,-q}(1-s), \quad (8.23)$$

which is a result of the change of variable $v \rightarrow v^{-1}$ in (8.21). The necessary absolute convergence, and the meromorphic continuation to \mathbb{C} of $\mathbf{L}_{l,q}(s)$ can be confirmed readily. Hence we have the local functional equation

$$\begin{aligned} (-1)^{l-q} \Gamma_{l,-q}(s) &= \pi^{2-4s} (-1)^{\max(|p|,|q|)} \Gamma_{l,q}(1-s) \\ &\times \frac{\Gamma(s + \frac{1}{2}(|p+q| + \nu)) \Gamma(s + \frac{1}{2}(|p-q| - \nu))}{\Gamma(1-s + \frac{1}{2}(|p+q| + \nu)) \Gamma(1-s + \frac{1}{2}(|p-q| - \nu))} \end{aligned} \quad (8.24)$$

(see [17, Theorem 6.4]).

Then we observe, by convolving (8.22), that

$$\begin{aligned} &\int_0^\infty \lambda^{2\nu-1} J_{|p+q|}(r\lambda) J_{|p-q|}(r/\lambda) d\lambda \\ &\longleftrightarrow 2^{s-3} \frac{\Gamma(\frac{1}{4}s + \frac{1}{2}(|p+q| + \nu)) \Gamma(\frac{1}{4}s + \frac{1}{2}(|p-q| - \nu))}{\Gamma(1 - \frac{1}{4}s + \frac{1}{2}(|p+q| + \nu)) \Gamma(1 - \frac{1}{4}s + \frac{1}{2}(|p-q| - \nu))} \end{aligned} \quad (8.25)$$

is a Mellin pair, provided $2|\operatorname{Re} \nu| < \operatorname{Re} s < 1 - 2|\operatorname{Re} \nu|$. Thus, denoting the left side by $K_{\nu,p}(r, q)$, we get, by (8.19), (8.24) and the Mellin–Parseval formula,

$$(-1)^{l-q} \lambda^{-2} v_{-q}^l(\lambda^2) = 8\pi^2 (-1)^{\max(|p|,|q|)} \int_0^\infty K_{\nu,p}(2\pi\lambda r, q) v_q^l(r^2) r dr. \quad (8.26)$$

In this we set $\lambda = |u|$ with $u \in \mathbb{C}^\times$, and multiply both sides by the factor $(u/|u|)^{2q} / \|\Phi_{p,q}^l\|_K$. On noting that $\Phi_{p,q}(\mathbf{k}[-\beta, \alpha]) = (-1)^{l-q} \Phi_{p,-q}(\mathbf{k}[\alpha, \beta])$ or $R_w \phi_{l,q} = (-1)^{l-q} \phi_{l,-q}$, we see by the definition (8.2) that (8.26) is identical to

$$\begin{aligned} &|u|^{-2} \mathcal{K} R_w \phi_{l,q}(u^2) \\ &= 4\pi (-1)^{\max(|p|,|q|)} \int_{\mathbb{C}^\times} K_{\nu,p}(2\pi|uv|, q) (uv/|uv|)^{2q} \mathcal{K} \phi_{l,q}(v^2) |v|^2 d^\times v. \\ &= 4\pi \int_{\mathbb{C}^\times} \left\{ \sum_{m=-\infty}^\infty (-1)^{\max(|p|,|m|)} K_{\nu,p}(2\pi|uv|, m) \left(\frac{uv}{|uv|} \right)^{2m} \right\} \mathcal{K} \phi_{l,q}(v^2) |v|^2 d^\times v. \end{aligned} \quad (8.27)$$

We then invoke that Graf's addition theorem ([40, eq. (1) on p. 359]) gives, for any $Z, z > 0$,

$$\begin{aligned} & \sum_{m=-\infty}^{\infty} (-1)^{\max(|p|, |m|)} J_{|m+p|}(Z) J_{|m-p|}(z) e^{2mi\theta} \\ &= (-1)^p J_{2p}(|Ze^{i\theta} + ze^{-i\theta}|) \left(\frac{Ze^{i\theta} + ze^{-i\theta}}{|Ze^{i\theta} + ze^{-i\theta}|} \right)^{2p}. \end{aligned} \quad (8.28)$$

We apply this to the member inside the braces of (8.27), and find that the proof of (7.25) with $\phi = \phi_{l,q}$ has been reduced to that of

$$\begin{aligned} & j_{p,\nu}(u)/(4\pi|u|^2) \\ &= (-1)^p \int_0^\infty \lambda^{2\nu-1} J_{2p}(2\pi|u| |\lambda e^{i\vartheta} + (\lambda e^{i\vartheta})^{-1}|) \left(\frac{\lambda e^{i\vartheta} + (\lambda e^{i\vartheta})^{-1}}{|\lambda e^{i\vartheta} + (\lambda e^{i\vartheta})^{-1}|} \right)^{2p} d\lambda, \end{aligned} \quad (8.29)$$

with $u = |u|e^{i\vartheta}$, which is, however, the same as [8, Theorem 12.1]. We end the proof; the extension to any smooth ϕ is immediate.

It does not seem that the identity (8.29) had been tabulated before [8], a fact somewhat bizarre against its classical appearance. This integral representation of the Bessel function of representations of $\mathrm{PSL}_2(\mathbb{C})$ is quite important, for it allows us to deal with test functions which not necessarily decay exponentially. This merit of (8.29) is indeed exploited fully in the proof of the spectral decomposition (7.28). The proof in [8] of (8.29) is conceptually involved, depending for instance on the Goodman–Wallach operator ([13]). The procedure above indicates the existence of a simpler approach, and in fact an alternative proof has been obtained in [7][32]. Any extension of (8.29) is highly desirable.

Lemmas 3 and 4 allow us to carry over the method of [11] to the complex situation, so that the proof of the Kloosterman–Spectral sum formula established in [8, Theorem 13.1] can now be proved in a simpler manner, although we have not worked out the details yet. Finally, we should mention that our argument seems to extend to Lie groups of real rank one; thus, the assertions due to R. Miatello and N.R. Wallach [22] are hoped to be included in our future discussion.

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