Thus, the $\int$ in the Kloosterman term is:

$$\tilde{k}(ma) = \underbrace{\int_{Re\,\nu=0} k(\nu)\, g(\nu)\, a^{\nu+\rho}\, d\nu}_{①} + \underbrace{\int_{Re\,\nu=0} k(\nu)\, g(\nu)\, a^{\nu+\rho+\alpha}\, O\left(\frac{1}{1+|\nu|}\right) d\nu}_{②}$$

Now ① is a Mellin transform and ② should hopefully be of smaller order.

If Ψ_x is a smooth approx of a charact. fn of $[x, 2x] \subset (0, +\infty) \simeq A$ then ① suggests

$$k(\nu) = \frac{M\Psi_x(-\nu)}{g(\nu)} \qquad \left(M\Psi_x(\nu) = \int_0^\infty y^{-\nu}\, \Psi_x(y)\, \frac{dy}{y} \right)$$

but, need $k(\nu)$ even, holo on a strip and with the right decay. Note that $g(\nu)$ has zeroes in $[-\rho, 0)$.

Choice of a test function

Start with $\Psi \in C_c^\infty(0, +\infty)$, $0 \le \Psi \le 1$

$$\sum_{m \in \mathbb{Z}} \Psi(y 2^m) = 1$$

$Y > 0$ controls the slope of Ψ