Thus h_x satisfies

- $$h_x(\nu) = O\left(e^{-\pi|\mathrm{Im}\,\nu|}(1+|\mathrm{Im}\,\nu|)^{K+2|\mathrm{Re}\,\nu|-\ell}\, x^{-|\mathrm{Re}\,\nu|}\, \zeta^{\ell-1}\right)$$

  for $|\mathrm{Im}\,\nu| \geq 1$ and $\forall\, \ell \geq 1$

If $\nu_j \in (S \cap (0,\rho) \cap \mathbb{Z})$ (a finite set)

- $$h_x(\nu_j) = \frac{M\Psi(-\nu_j)}{g(\nu_j)}\, x^{-\nu_j} + O\left(x^{\zeta_0}(1+|\log x|)^{k_{\zeta_0}-1} + x^{\nu}(1+|\log x|)^{k_0}\right)$$

The function $h_x(\nu)$ goes in the sum formula. We look at the $\int$ in the Kloosterman term.

- $$\hat{h}_x(ma) = \frac{1}{2\pi i}\int_{\mathrm{Re}\,\nu=\sigma_2} M(\Psi_x-\beta_x)(-\nu)\, a^{\nu+\rho}\left(1+O\left(\frac{a^{\alpha}}{1+|\mathrm{Im}\,\nu|}\right)\right)d\nu$$

$$+ \frac{1}{2\pi i}\int_{\mathrm{Re}\,\nu=\sigma_2} M(\Psi_x-\beta_x)(\nu)\,\frac{g(\nu)}{g(-\nu)}\, a^{\nu+\rho}\, O(1)\, d\nu$$

In the second $\int$ use $\sigma_2=\sigma$, get (by the estimates above):

$$\ll \int_{-\infty}^{+\infty}(1+|y|)^{-4\sigma}dy\ (x^{\sigma}+x^{\langle\zeta_0\rangle})\, a^{\rho+\sigma} \quad \text{as } x\downarrow 0.$$