In the first $\int$ split parts of Ψ_x, β_x

$$\int_{\mathrm{Re}\,\nu=\sigma} M\beta_x(-\nu)\,d\nu \cdot a^{\rho+\sigma} \underset{\sigma_2=\sigma,\ l=2}{\ll} x^{\langle \xi_0 \rangle}\int_{-\infty}^{+\infty}(1+|y|)^{-2}\,dy\; a^{\rho+\sigma}$$

For the integral involving Ψ_x get $\left(\substack{\text{use}\\ l=1}\right)$

$$a^{\rho}\Psi_x(a^{\alpha}) + O\left(\int_{-\infty}^{+\infty} x^{-\sigma_2}(1+|y|)^{-2}\,dy\; a^{\rho+\sigma_2+1}\right).$$

Later will need $\rho+\sigma_2+1 > 2\rho$, so take $\sigma_2=\sigma-1$

($\sigma_2 \in [-\varepsilon, \rho]$ f' some $\varepsilon < \frac{1}{2}$)

\- Get $O(x^{1-\sigma}a^{\rho+\sigma})$

That is,

\- $\tilde{h}_x(ma) = a^{\rho}\Psi_x(a^{\alpha}) + a^{\rho+\sigma}O\left(x^{1-\sigma} + x^{\langle \xi_0 \rangle}\right)$ as $x \downarrow 0$.

Thus finally get

Proposition

$$\sum_{\xi_\gamma \in \mathfrak{I}} a_\gamma^{\rho}\, S(m_\gamma a_\gamma)\,\Psi_x(a_\gamma^{\alpha}) = \sum_\gamma S(m_\gamma a_\gamma)\,\tilde{h}_x(m_\gamma a_\gamma) + O\left(x^{\langle \xi_0 \rangle} + x^{1-\sigma}\right)$$