Now the l.h.s is what we were looking for + the 1st term in the r.h.s occurs in the sum formula. It equals

$$-\int_{\mathcal{L}} h_x(r)\, d\sigma(r) - \int_{Re\, r=0} h_x(r)\, d\delta(r)$$

spectral delta

It remains to estimate these.

For the delta term split

$$h_x(iy) << \begin{cases} 1+|\log x| & |y|\le \varepsilon_0 \\ 1 & \varepsilon_0 < |y| \le 1 \\ e^{-\pi|y|}(1+|y|)^{K-1} & 1 \le y \le Y \\ e^{\pi|y|}(1+|y|)^{K-\ell} Y^{\ell-1} & |y| \ge Y \end{cases}$$

Get

$$-A\int_{-\infty}^{+\infty} h_x(iy)\, y \sinh(\pi y)\, dy = O(Y^{K+1} + |\log x|)$$

To estimate the spectral term choose

$$-h_t(r) = \frac{e^{tr^2} p(r)}{\cos \pi r} \qquad p(r) = \prod_0^{\ell-1} \left(\left(\tfrac{1}{2}+j\right)^2 - r^2\right)$$

ℓ suff large

even, holo on strip, with fast decay.