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Theorem'. Let $\psi_1 \in C^\infty(0,\infty)$, $\psi_1(y) = \begin{cases} 0 & y \le \frac{3}{4} \\ \psi(y) & \frac{3}{4} \le y \le 1 \\ 1 & y \ge 1 \end{cases}$

Then

$$-\sum_{a_\gamma m_\gamma \mathcal{J}} a_\gamma S(\gamma)\, \chi\!\left(\frac{a_\gamma^\alpha}{x}\right) = \sum_{\nu_j \in \mathcal{E}} \sigma_{\nu_j} \frac{M\psi_1(-\nu_j)}{g(\nu_j)} x^{-\nu_j} + O_\varepsilon\!\left(x^{1-\rho+\varepsilon} + |\log x|^2\right)$$

Notes. (1) The sum in l.h.s is an approximation of

$$\sum_{a_\gamma^\alpha \ge x} a_\gamma^\rho S(\gamma) \quad \text{or else of} \quad \boxed{\sum_{a_\gamma^{-\rho} \le X} \frac{S(\gamma)}{a_\gamma^{-\rho}}}$$

with $X = x^{-1}$.

- For $SL_2(\mathbb{R})$ $a_\gamma^{-\rho} = |c|$ if $\gamma = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$
- For $SL_2(\mathbb{C})$ $a_\gamma^{-\rho} = |c|^2$.

(2) The error term is (if $X = x^{-1}$)

- $O(|\log X|^2)$ for $G = SL(2,\mathbb{R})$
- $O(X^\varepsilon)$ $\forall \varepsilon$ $G = SL(2,\mathbb{C})$
- $O(X^{\rho-1+\varepsilon})$ other groups

Get a smooth version of the Linnik conjecture.