

PERIODS OF AUTOMORPHIC FORMS OVER A COMPACT UNITARY GROUP

OMER OFFEN (JOINT WITH EREZ LAPID)

Let me start by stating our main result in "adelic" language and then discuss its interpretation in more classical terms.

We consider the following setting. Let F be a totally real field and E a purely imaginary quadratic extension of F . In other words, E is a CM-field and F its maximal totally real subfield. We make the following technical assumption:

E/F is unramified at ALL finite places.

Denote by $x \mapsto \bar{x}$ the Galois action of E/F and by \mathbb{A} the ring of adèles of F . Set $G' = GL_n$ regarded as an algebraic group defined over F and let G be the restriction of scalars of G' from E to F . The associated groups of F -rational points are then

$$G'(F) = GL_n(F) \quad ; \quad G(F) = GL_n(E).$$

Let K (resp. K') denote the standard maximal compact of $G(\mathbb{A})$ (resp. $G'(\mathbb{A})$). We denote by

$$\text{bc} : \mathcal{A}(G'(\mathbb{A})) \rightarrow \mathcal{A}(G(\mathbb{A}))$$

the transfer of automorphic forms from $G'(\mathbb{A})$ to $G(\mathbb{A})$ given by quadratic base change. Let π' be a cuspidal automorphic representation of $G'(\mathbb{A})$ which is everywhere unramified. We assume further that $\pi' \not\cong \pi' \otimes \eta$, where $\eta = \eta_{E/F}$ is the quadratic idèle class character attached to E/F by class field theory. In this case $\pi = \text{bc}(\pi') = \text{bc}(\pi' \otimes \eta)$ is a cuspidal automorphic representation of $G(\mathbb{A})$. Clearly it is also everywhere unramified. Let ϕ_0 be the K -invariant, L^2 -normalized cusp form in π . Let $\alpha = {}^t\bar{\alpha} \in G(F)$ be a hermitian matrix which is either positive or negative definite at all real embeddings, let

$$H_\alpha = \{g \in G : g\alpha {}^t\bar{g} = \alpha\}$$

be the anisotropic unitary group associated with α and set $\theta = (\theta_v) \in G(\mathbb{A})$ with

$$\begin{cases} \theta_v {}^t\bar{\theta}_v = \pm\alpha_v & \infty \mid v \\ \theta_v = e & v < \infty. \end{cases}$$

Theorem 1 ([LO]).

$$\left| \int_{H_\alpha(F) \backslash H_\alpha(\mathbb{A})} \phi_0(h\theta) \right|^2 = c_E |P_\alpha(\pi)|^2 \frac{L(1, \pi' \times \tilde{\pi}' \otimes \eta)}{\operatorname{Res}_{s=1} L(s, \pi' \times \tilde{\pi}')}.$$

The constant c_E is explicit on the nose. The term $P_\alpha(\pi)$ is a finite product of local terms that is also explicit, but dependent on π . In particular for the unitary group determined by the identity matrix we have

$$P_e(\pi) = 1.$$

The quotient is of the completed Rankin-Selberg L -functions. We obtain a similar formula dropping our technical assumption on E/F in the cost of the local terms $P_\alpha(\pi)$ remaining unknown at finitely many places. For the case of $n = 2$, those local terms are known and we may in this case, in particular, provide a formula for a general CM -field over \mathbb{Q} .

The unitary period of a cusp form has another interpretation as a finite weighted sum of point evaluations. Recall that ϕ_0 is an automorphic form on the locally symmetric space $G(F) \backslash G(\mathbb{A})/K$. For simplicity, assume that $F = \mathbb{Q}$ and that $\alpha = e$ and set $H = H_e$. The locally symmetric space is then identified with an arithmetic quotient of several copies, depending on the class number of E , of the symmetric space $GL_n(\mathbb{C})/U_n$. Under the above interpretation, we can write

$$\int_{H(F) \backslash H(\mathbb{A})} \phi_0(h) dh = \operatorname{vol}(H(\mathbb{A}) \cap K) \sum \frac{1}{|K \cap x_i^{-1} H(F) x_i|} \phi_0(x_i)$$

the sum being over the genus of the hermitian form defined by e , i.e.

$$H(\mathbb{A}) = \bigsqcup_i H(F) x_i (K \cap H(\mathbb{A})).$$

Thus we express a finite sum of point evaluations of a cusp form in terms of special values of L -functions.

On the upper half plane there is a well known and extremely important formula of Waldspurger. Let ϕ be a cusp form on $\Gamma \backslash \mathbb{H}$ – a quotient of the upper half plane by a congruent subgroup and let d be a negative integer. Then

$$\left| \sum_{z \in \Lambda_d} \phi(z) \right|^2 \sim L\left(\frac{1}{2}, \operatorname{bc}_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{d})}(\pi_\phi)\right)$$

where, Λ_d is the set of Heegner points of discriminant d and π_ϕ is the automorphic representation emanating from ϕ .

Our formula is of a similar nature, except that it involves the special value at $s = 1$ of a quotient of L -functions. The size of these L -values

for standard automorphic L -functions is well understood. From Langlands functoriality it would follow that Rankin-Selberg L -functions are standard. We may therefore obtain information on the size of the periods on the left hand side of our formula.

As an application of the formula, let me point out a connection to a recent conjecture of Sarnak about the L^∞ -norm of automorphic forms.

In the case of an arithmetic co-compact quotient of the upper half plane Iwaniec and Sarnak predict in [IS95] that

$$\|\phi\|_\infty < \lambda^\epsilon$$

for all $\epsilon > 0$, where ϕ is an L^2 -normalized eigenfunction with Laplace eigenvalue λ . Here the convexity bound is with $\epsilon = \frac{1}{4}$. This was improved by Sarnak-Iwaniec to $\epsilon = \frac{5}{24}$.

In higher rank the situation is different. Let ϕ be a cusp form on $\Gamma \backslash GL_n(\mathbb{C})/U_n$ where Γ is arithmetic. Assume that ϕ is L^2 -normalized and an eigenfunction of the ring of invariant differential operators as well as the Hecke operators. One expects bounds on the L^∞ -norm, of the form

$$\|\phi\|_\infty < \lambda_\phi^\delta.$$

Here λ_ϕ is given by

$$\lambda_\phi = \prod_{i < j} (1 + |\lambda_i - \lambda_j|)$$

where $(\lambda_1, \dots, \lambda_n)$ are the eigen-values of a minimal set of generators of the ring of invariant differential operators. In other words $(\lambda_1, \dots, \lambda_n)$ is the infinitesimal character in Harish-Chandra's parameterization of the corresponding representation of $GL_n(\mathbb{C})$.

From purely differential geometric considerations on the symmetric space, Sarnak showed in this case that one can take $\delta = 1$ ([Sar04]). In a recent work, in a more general setting Sarnak and Venkatesh show that one can take $\delta < 1$.

On the other hand, there are precise conjectures regarding the size of the quotient of L -functions that appears in the formula. From estimates obtained by Molteni ([Mol]), granted that Rankin-Selberg L -functions are standard, the finite part of the L -functions should be bounded above and below by a term of size λ_ϕ^ϵ for every $\epsilon > 0$. The archimedean factors can be estimated using Stirling's formula. They are roughly of size λ_ϕ . So the formula gives

$$\|\phi\|_\infty \gg \lambda_\phi^{\frac{1}{2} + \epsilon}$$

for cusp forms in the image of base change. In particular we see that one cannot expect $\delta < \frac{1}{2}$. In fact this was already known. Let me give

a heuristic argument for the case $n = 2$. For any cusp form ϕ which is not a base change, by a result of Harder-Langlands-Rapoport its unitary period vanishes ([HLR86]). By a local Weyl Low type formula (which is known at least for compact quotients) if $\{y_i\}_{i=1}^m$ is any finite set of points then

$$\sum_{\mu_\phi < R^2} \left| \sum_i \phi(y_i) \right|^2 \sim cR^3.$$

Here the sum of point evaluations (absolute value squared) is averaged over an orthonormal basis of eigenfunctions with Laplace eigenvalue $\mu_\phi < R^2$. There are roughly R^3 forms in the sum. If we now apply this to the points $\{x_i\}$ of the genus of the hermitian form, then only those forms which are base change will contribute. There are roughly R^2 of them. We therefore obtain that on average the weighted sum satisfies

$$\left| \sum' \phi(x_i) \right| \sim R^{\frac{1}{2}}.$$

This is an argument of Rudnick-Sarnak([RS94]). However, even in this case, our formula gives a sharper result, since it says that every single cusp form which is base change has a large L^∞ -norm.

I remark that Molteni's bounds for the L -values do apply in the $n = 2$ case. Thanks to the Gelbart-Jacquet lifting ([GJ78]) we know in this case that the Rankin-Selberg L -functions are automorphic.

This example shows the connection between large L^∞ -norms and functoriality. In general Sarnak conjectures that cusp forms with large L^∞ -norms are all lifts from smaller subgroups. This is somewhat similar to the situation with the Ramanujan conjecture. But as we have seen, in this case, exceptional automorphic forms already exist for GL_2 .

In the rest of the talk I will discuss the proof. Let me first list the main ingredients that we use to compute the periods.

- (1) The fundamental-Lemma of Jacquet – [Jac04], [Jac05].
- (2) The fine spectral expansion of the Relative Trace Formula (RTF) – [Lap]
- (3) Local identities of Bessel distributions for principal series – [Off]
- (4) Explicit formulas for spherical functions on Hermitian matrices (p -adic case) – [Hir88a], [Hir89], [Hir88b], [Hir90], [Hir99].

The last ingredient for general n is only available if E/F is an unramified quadratic extension of p -adic fields. This is the reason for our technical assumption on E/F . For $n = 2$ Hironaka computed the spherical functions for a general quadratic extension, and we can then provide a formula for any CM-field.

The point of departure of our computation is a global identity of distributions that comes from the Relative trace formula of Jacquet. It is a consequence of the first two ingredients (1) and (2) above. The RTF is a tool developed by Jacquet to study periods of automorphic forms. I wish to make a short retreat and roughly explain, in a few words, the general setting for periods in this context. For now let G be a connected reductive algebraic group defined over F and let H be the group of fixed points of an involution on G . For a cusp form ϕ on $G(F)\backslash G(\mathbb{A})$ we consider the period

$$P^H(\phi) = \int_{H(F)\backslash H(\mathbb{A})} \phi(h)dh.$$

Definition 1. A cuspidal representation π of $G(\mathbb{A})$ is distinguished by H if there exists a cusp form ϕ in the space of π , so that $P^H(\phi) \neq 0$.

In many known examples, as in the example we have seen, these periods are related to special values of L -functions. In general to the symmetric space $H\backslash G$ there is attached another group G' and a transfer of automorphic forms from $G'(\mathbb{A})$ to $G(\mathbb{A})$. It is expected that being distinguished characterizes the image of the transfer of automorphic forms. This project was recently completed by Jacquet in the case of unitary periods.

Going back to the more specific notation where $G' = GL_n$ over F and $G = \text{Res}_{E/F}(G')$, the transfer of automorphic forms is quadratic base change.

Theorem 2 ([Jac05]). *A cuspidal representation π of $G(\mathbb{A})$ is a base change from $G'(\mathbb{A})$ if and only if it is distinguished by some unitary group.*

The result of Jacquet follows from the RTF. Jacquet had to consider all unitary groups. Since we compute a specific unitary period, we use a slightly simpler version of the RTF that I now wish to describe. Recall that $H = H_e$ is the unitary group determined by the identity matrix. We add some more notation. Let ψ be an additive character of $F\backslash\mathbb{A}$. Let U (resp. U') be the group of upper triangular unipotent matrices in G (resp. G') and set

$$\psi_U(u) = \psi(\text{Tr}_{E/F}(u_{1,2}) + \cdots + \text{Tr}_{E/F}(u_{n-1,n}))$$

and

$$\psi_{U'}(u) = \psi(u_{1,2} + \cdots + u_{n-1,n}).$$

We compare between distributions on G and on G' that are defined as follows. For $f = \otimes_v f_v \in C_c^\infty(G(\mathbb{A}))$ we consider

$$RTF(f) = \int_{U(F)\backslash U(\mathbb{A})} \int_{H(F)\backslash H(\mathbb{A})} K_f(h, u) dh \psi_U(u) du$$

and for $f' = \otimes_v f'_v \in C_c^\infty(G'(\mathbb{A}))$ we consider the Kuznetsov trace formula

$$KTF(f') = \int \int_{(U'(F)\backslash U'(\mathbb{A}))^2} K_{f'}(u_1, u_2) \psi_{U'}(u_1 u_2) du_1 du_2$$

where K_f and $K_{f'}$ are the standard kernel functions associated with the test functions f and f' respectively. As in the case of the Arthur-Selberg trace formula, both distributions have an expansion as a sum of distributions parameterized by geometric data and an expansion in terms of spectral data. There is a certain geometric correspondence between the relevant double cosets in

$$U'\backslash G'/U' \quad " \leftrightarrow " \quad H\backslash G/U$$

and the geometric expansions are in terms of orbital integrals, the orbits being the double cosets. We get an identity

$$RTF(f) = KTF(f')$$

whenever

$$f \leftrightarrow f'$$

have matching orbital integrals. The matching reduces to a local condition. Locally, we say that f and f' match if

$$\int_{U'} \int_{U'} f'(u_1 w a u_2) \psi_{U'}(u_1 u_2) du_1 du_2 = \begin{cases} \gamma(a) \int_U \int_H f(h\xi u) \psi_U(u) dh du & \text{if } a = {}^t \bar{\xi} \xi, \\ 0 & \text{if } a \notin \{{}^t \bar{g} g : g \in G\} \end{cases}$$

where the transfer factor $\gamma(a)$ is defined in terms of η . A standard strategy in the business of trace formula identities is to try to compare between the corresponding distributions coming from the geometric side and to apply this comparison in order to obtain identities in the spectral side. This, in general, turned out to be more difficult than people first thought. The main obstacle is the so called fundamental lemma. At the case at hand, the fundamental lemma is now available thanks to a recent work of Jacquet. Recall that locally base change defines a homomorphism from the bi- K -invariant Hecke algebra of G to the bi- K' -invariant Hecke algebra of G'

$$\text{bc} : \mathcal{H}_G \rightarrow \mathcal{H}_{G'}.$$

Jacquet showed that

$$f \leftrightarrow \text{bc}(f), f \in \mathcal{H}_G$$

in particular, he showed that the identity elements of the Hecke algebras match

$$\mathbf{1}_K \leftrightarrow \mathbf{1}_{K'}.$$

In addition Jacquet proved smooth matching ([Jac03]), i.e. that every $f' \in C_c^\infty(G')$ matches a function $f \in C_c^\infty(G)$ and vice versa. He then obtains the identity of distributions for enough test functions.

In order to study periods of cusp forms, one has to isolate the discrete part of the identity, i.e. the contribution of the discrete spectrum. This can be done thanks to the result of Lapid (2). The discrete contribution to the KTF consists of the so called Bessel distributions. They involve the Fourier (Whittaker) coefficients of automorphic forms. For the RTF the discrete part consists of relative analogues that involve a combination of Fourier coefficients and unitary periods. We refer to those as relative Bessel distributions. From (1) and (2) we then obtain that if $\pi = \text{bc}(\pi')$ then for $f \leftrightarrow f'$ we have

$$\tilde{B}_\pi(f) = B_{\pi'}(f').$$

The Bessel distribution is defined by

$$B_{\pi'}(f') = \sum_{\text{ob}(\pi')} \mathcal{W}^\psi(\pi'(f')\phi') \overline{\mathcal{W}^\psi(\phi)}$$

where the Fourier coefficient is defined by

$$\mathcal{W}^\psi(\phi') = \int_{U'(F) \backslash U'(\mathbb{A})} \phi'(u) \overline{\psi_U(u)} du.$$

The relative Bessel distribution is

$$\tilde{B}_\pi(f) = \sum_{\text{ob}(\pi)} P^H(\pi(f)\phi) \overline{\mathcal{W}^\psi(\phi)}.$$

For our computation we now pick specific matching functions. Let S be a large enough finite set of places of F containing the real places the even places and the places where ψ ramifies. We fix a bi- K -invariant function f and a matching f' of the following form

$$f = \prod_{v \in S} f_v \prod_{v \notin S} \mathbf{1}_{K_v} \leftrightarrow f' = \prod_{v \in S} f'_v \prod_{v \notin S} \mathbf{1}_{K'_v}$$

where $f_v \in \mathcal{H}_{G_v}$ for all v . That such matching functions exist is a consequence of the local results of Jacquet. We stress, however, that

the functions f'_v , $v \in S$ are not necessarily Hecke functions. Since the function f is bi- K -invariant, it follows from our definition that

$$\tilde{B}_\pi(f) = \hat{f}_S(\pi_S) P^H(\phi_0) \overline{\mathcal{W}}(\phi_0)$$

where $\hat{f}_S(\pi_S)$ is the spherical Fourier transform of f . The Fourier coefficient can also be computed up to finitely many terms. In what follows $(*)$ will always stand for explicitly known constants. We have

$$|\mathcal{W}(\phi_0)|^2 = (*) \frac{1}{\text{Res}_{s=1} L^S(s, \pi \times \tilde{\pi})} \prod_{v \in S} |W_v(e)|^2.$$

Here the local terms involve the value at the identity of the L^2 -normalized spherical Whittaker function W_v . The invariant inner product we use on the local Whittaker models

$$(W_1, W_2) = \int_{U_{n-1} \backslash G_{n-1}} W_1(\text{diag}(g, 1)) \overline{W_2}(\text{diag}(g, 1)) dg$$

is due to Bernstein at the finite places and due to Baruch at the infinite places. The formula for the Fourier coefficient follows from a formula for the inner product of cusp forms, based on the unfolding of Rankin-Selberg integrals and the local computations of Jacquet-Shalika ([JS81]). On the other side, based on the same technique, the Bessel distribution is explicitly factorizable. We have

$$B_{\pi'}(f') = (*) \frac{1}{\text{Res}_{s=1} L^S(s, \pi' \times \tilde{\pi}')} \prod_{v \in S} B_{\pi'_v}(f'_v)$$

where the local bessel distributions are defined with respect to the above inner product on the Whittaker model. It is left to compute the local Bessel distribution $B_{\pi'_v}(f'_v)$. Since the function f'_v is not necessarily bi- K'_v -invariant, directly this may be difficult. But we do know that the matching function f_v is a Hecke function. This is where (3) comes into play. Since our representations are assumed everywhere unramified, it is enough to consider principal series representations. I obtain an identity

$$\tilde{B}_{\pi'_v}(f_v) = \gamma(\pi'_v) B_{\pi'_v}(f'_v)$$

where γ is a certain gamma factor and $\tilde{B}_{\pi'_v}(f_v)$ is a local analogue of the relative Bessel distribution. As in the global case, the local relative Bessel distribution involves a combination of a Whittaker functional and a certain unitary period. We interpret the unitary period as the spherical functions of Hironaka and we can therefore apply her explicit formulas (4). We obtain that

$$|B_{\pi'_v}(f'_v)|^2 = (*) \frac{L(1, \pi_v \times \tilde{\pi}_v)}{L(1, \pi'_v \times \tilde{\pi}'_v)^2} \left| \hat{f}_v(\pi_v) W_v(e) \right|^2.$$

Combining all this we obtain

$$|P^H(\phi_0)|^2 = (*) \frac{\text{Res}_{s=1} L(s, \pi \times \tilde{\pi})}{\text{Res}_{s=1} L(s, \pi' \times \tilde{\pi}')^2}.$$

Our formula follows since

$$L(s, \pi \times \tilde{\pi}) = L(s, \pi' \times \tilde{\pi}') L(s, \pi' \times \tilde{\pi}' \otimes \eta).$$

REFERENCES

- [GJ78] Stephen Gelbart and Hervé Jacquet. A relation between automorphic representations of $GL(2)$ and $GL(3)$. *Ann. Sci. École Norm. Sup. (4)*, 11(4):471–542, 1978.
- [Hir88a] Yumiko Hironaka. Spherical functions of Hermitian and symmetric forms. I. *Japan. J. Math. (N.S.)*, 14(1):203–223, 1988.
- [Hir88b] Yumiko Hironaka. Spherical functions of Hermitian and symmetric forms. III. *Tohoku Math. J. (2)*, 40(4):651–671, 1988.
- [Hir89] Yumiko Hironaka. Spherical functions of Hermitian and symmetric forms. II. *Japan. J. Math. (N.S.)*, 15(1):15–51, 1989.
- [Hir90] Yumiko Hironaka. Spherical functions of Hermitian and symmetric forms over 2-adic fields. *Comment. Math. Univ. St. Paul.*, 39(2):157–193, 1990.
- [Hir99] Yumiko Hironaka. Spherical functions and local densities on Hermitian forms. *J. Math. Soc. Japan*, 51(3):553–581, 1999.
- [HLR86] G. Harder, R. P. Langlands, and M. Rapoport. Algebraische Zyklen auf Hilbert-Blumenthal-Flächen. *J. Reine Angew. Math.*, 366:53–120, 1986.
- [IS95] H. Iwaniec and P. Sarnak. L^∞ norms of eigenfunctions of arithmetic surfaces. *Ann. of Math. (2)*, 141(2):301–320, 1995.
- [Jac03] Hervé Jacquet. Smooth transfer of Kloosterman integrals. *Duke Math. J.*, 120(1):121–152, 2003.
- [Jac04] Hervé Jacquet. Kloosterman identities over a quadratic extension. *Ann. of Math. (2)*, 160(2):755–779, 2004.
- [Jac05] Hervé Jacquet. Kloosterman identities over a quadratic extension II. *Ann. Sci. École Norm. Sup.*, 38(4):609–669, 2005.
- [JS81] H. Jacquet and J. A. Shalika. On Euler products and the classification of automorphic representations. I. *Amer. J. Math.*, 103(3):499–558, 1981.
- [Lap] Erez Lapid. On the fine spectral expansion of Jacquet’s relative trace formula. *J. Inst. Math. Jussieu*. To Appear
- [LO] Erez Lapid and Omer Offen. Compact unitary periods. *Submitted*.
- [Mol] Giuseppe Molteni. Upper and lower bounds at $s = 1$ for certain Dirichlet series with Euler product. *Duke Math. J.*, 111(1):133–158, 2002.
- [Off] Omer Offen. Stable relative Bessel distributions on $gl(n)$ over a quadratic extension. *Submitted*.
- [RS94] Zeév Rudnick and Peter Sarnak. The behaviour of eigenstates of arithmetic hyperbolic manifolds. *Comm. Math. Phys.*, 161(1):195–213, 1994.
- [Sar04] P. Sarnak. A letter to Cathleen Morawetz. 2004.