The Condition of Moderate Growth

"*Without the condition of moderate growth you can't do anything.*"

Harish-Chandra

G a real reductive group, $K \subset G$ a maximal compact subgroup. (σ, V) a finite dimensional representation of G with compact kernel.

$\langle \ldots, \ldots \rangle$ a K-invariant Hilbert space structure on V. $\|\ldots\|$the corresponding operator nome on $End(V \bigoplus V)$.Write $\|g\|$for $\left\| \begin{bmatrix} \sigma(g) & 0 \\ 0 & \sigma(g^{-1})^* \end{bmatrix} \right\|$. The function $g \mapsto \|g\|$ will be called a *norm* on G.

1. $\|1\| = 1$, $\|k_1 g k_2\| = \|g\|$, $k_1, k_2 \in K$, $g \in G$.

2. $\|xy\| \leq \|x\| \|y\|$.

3. $\left\|x^{-1}\right\| = \|x\|$.

4. If $r > 0$ the set $B_r = \{g \in G | \, \|g\| \leq r\}$ is compact.

5. $\left\|xy^{-1}\right\| \geq \|x\| / \|y\|$.

6. If (π, W) is a representation of G on a Banach space then $\|\pi(g)\| \leq C \|g\|^k$ for some $C > 0, k \geq 0$.

We will say that a smooth function $f : G \to \mathbb{C}$ is of *moderate growth* if there exits k such that

$$|R_x f(g)| \leq C_x \|g\|^k$$

for all $x \in U(Lie(G))$ and all $g \in G$ and $R_x f$ is the usual action of $U(Lie(G))$ on $C^\infty(G)$ as left invariant differenctial operators.

Key Example:

(π, W) a Banach representation of G. W^∞ the space of C^∞ vectors with the standard Fréchet space structure. If $\lambda \in (\ W^\infty)'$ (continuous functionals) and $w \in W^\infty$ then the matrix coefficient

$$g \mapsto c_{\lambda,v}(g) = \lambda(\pi(g)v)$$

is of moderate growth. Indeed there exists k and a continuous seminorm ξ_λ on W^∞ such that

$$|c_{\lambda,v}(g)| \leq \|g\|^k \, \xi_\lambda(v) \ (*)$$

for all, $g \in G$, $v \in W$.

A smooth Fréchet representation of G is a representation $(\mu, \mathcal{V})$ of G on a Fréchet space $\mathcal{V}$ such that the map

$$g \to \mu(g)\phi$$

is smooth for all $\phi \in \mathcal{V}$. We will say that it is of moderate growth if the functions $c_{\lambda,v}$ for $\lambda \in \mathcal{V}', v \in \mathcal{V}$ satisfy the condition (*) above with ξ_λ a continuous seminorm on $\mathcal{V}$.

We will say that $\mathcal{V}$ is admissible if dim $Hom_K(F, \mathcal{V}) < \infty$ for all finite dimensional representations of K.

We will use the notation $\mathcal{V}_K$ for the space of all K-finite vectors in $\mathcal{V}$.

We will say that $\mathcal{V}$ is finitely generated and admissible if $\mathcal{V}_K$ is finitely generated as a $U(Lie(G))$-module.

Theorem of Casselman and Wallach:

$\mathfrak{g} = Lie(G) \bigotimes \mathbb{C}$.

$\mathcal{HC}(\mathfrak{g}, K)$ the category of finitely generated admissible $(\mathfrak{g}, K)$-modules.

$\mathcal{FHC}(G)$ the category of all finitely generated smooth, admissible, Fréchet representations of G with moderate growth and morphisms that are topologically split.

There exists an equivalence of categories $W \rightsquigarrow T(W)$ between $\mathcal{HC}(\mathfrak{g}, K)$ and $\mathcal{FHC}(G)$ such that if (π, W) is a Banach representation of G such that $(W^{\infty})_K$ is finitely generated and admissible then $T((W^{\infty})_K)$ is isomorphic with W^{∞}. In fact, the functor T is naturally the inverse functor to $W \rightsquigarrow W_K$.

Let $\mathcal{A}(G)$ denote the space of all functions on G of moderate growth. We set $\mathcal{A}_k(G)$ equal the space of all $f \in \mathcal{A}(G)$ such that

$$|R_x f(g)| \leq C_x \left\| g \right\|^k$$

for all $x \in U(Lie(G))$ and all $g \in G$.

On $\mathcal{A}_k(G)$ we define the seminorms

$$\xi_{k,x}(f) = \sup_{g\in G} \|g\|^{-k} |R_x f(g)|$$

This defines a Fréchet topology on $\mathcal{A}_k(G)$. We give $\mathcal{A}(G)$ the direct limit topology.

Let W be a closed G-invariant subspace (under the right action G) of $\mathcal{A}(G)$ with the following property:

If $Z(\mathfrak{g})$ is the center of $U(\mathfrak{g})$ then if $\phi \in W$ then $\dim Z(\mathfrak{g})\phi < \infty$ and W_K is finitely generated (i.e. an object in $\mathcal{HC}(\mathfrak{g}, K)$).

Theorem. *There exists $k \geq 0$ such that $W \subset \mathcal{A}_k(G)$ and with the induced topology W is isomorphic with $T(W_K)$ as a smooth Fréchet module.*

We also note

Theorem. *If $M \subset \mathcal{A}(G)$ is a $\mathfrak{g}$ and K-invariant subspace that is in $\mathcal{HC}(\mathfrak{g}, K)$ then the closure of M in $\mathcal{A}(G)$ is isomorphic with $T(M)$ as a smooth Fréchet module.*

Let $H \subset G$ be a closed sugroup.

Let $\mathcal{A}(H\backslash G) = \{f \in \mathcal{A}(G) | f(hg) = f(g)$ for all $h \in H, g \in G\}$.

Then $\mathcal{A}(H\backslash G)$ is closed in $\mathcal{A}(G)$.

Let W be a subspace of $\mathcal{A}(H\backslash G)$ that is $\mathfrak{g}$ and K-invariant and in $\mathcal{HC}(\mathfrak{g}, K)$.

Then there is an isomorphism, B, of $T(W)$ onto the closure, $\overline{W}$,of Was smooth Fréchet modules.

Let $\delta(f) = f(1)$ for any function on G. Then δ defines a continuous functional on $\mathcal{A}(G)$ and $f \in \mathcal{A}(G)$ is tautologically $c_{\delta,f}$.

Thus if we set $\lambda = B^*\delta \in (T(W)')^H$ then if $f \in W$ and if $B(\phi) = f$ then $f = c_{\lambda,\phi}$. This implies that there is a canonical isomorphism

$$Hom_{\mathfrak{g},K}(W, \mathcal{A}(H\backslash G)) \cong (T(W)')^H.$$

A variant of Frobenius reciprocity

A variant of this is:

Let P be a minimal parabolic subgroup of G, $P = MN$ a Levi decomposition and (σ, H_σ) an irreducible representation of M extended to N trivially.

Let $I^\infty_{P,\sigma}$ (resp. $I_{P,\sigma}$) be the smooth (resp. K-finite) induced representation of σ from P to G.

On $I^\infty_{P,\sigma}$ we put the usual Fréchet topology as smooth functions from K to H_σ.

Casselman's subrepresentation theorem implies that there exists σ and $B : I_{P,\sigma} \to W$ a surjective $(\mathfrak{g}, K)$. homomorphism.

We note that $T(I_{P,\sigma}) \cong I^\infty_{P,\sigma}$ Thus the CW-theorem implies that B extends to a homomorphism of $I^\infty_{P,\sigma}$ onto $\overline{W}$ as smooth Fréchet representations.

Let $\delta(f) = f(1)$ for any function on G. Then δ defines a continuous functional on $\mathcal{A}(G)$ and $f \in \mathcal{A}(G)$ is tautologically $c_{\delta,f}$. Thus if we set $\lambda = B^*\delta \in (I^\infty_{P,\sigma})'$ and let $B(\phi) = f$ then $f = c_{\lambda,\phi}$.

Let $\chi : U(\mathfrak{g})^K \to \mathbb{C}$ be an algebra homomorphism and let$\mathcal{A}(K\backslash G)_\chi$ be the space of $f \in \mathcal{A}(K\backslash G)$ such that $L_x f = \chi(x)f$ for $x \in U(\mathfrak{g})^K$ (here L is the left regular representation). Let ν be the Harish-Chandra parameter of χ in the Langlands chamber. Then the result we just described specializes to the assertion that if $p_\nu(x,y)$ is the Poisson kernel with $x \in K \cap P\backslash K$ and $y \in K\backslash G$ then if $f \in \mathcal{A}(K\backslash G)_\chi$ there exists a unique $T \in C^\infty(K \cap P\backslash K)'$ such that $f(g) = T(p_\nu(\cdot, Kg))$ for all $g \in G$.

This is a theorem proved almost simultaneously by Oshima-Sekiguchi and me (1980, announced by me in the 1980 Park City meeting).

We now look at the situation when G is the group of real points in a reductive algebraic group, $\mathbf{G}$, defined over $\mathbb{Q}$. Let Γ be subgroup of $\mathbf{G}_{\mathbb{Q}}$.

Let P be the real points of a parabolic subgroup, $\mathbf{P}$, of $\mathbf{G}$ defined over $\mathbb{Q}$. We will say that P is cuspical.

Let $\mathbf{N}$ be the nilradical of $\mathbf{P}$ and N the real points of $\mathbf{N}$.

Let M be the group of real points of $\mathbf{P}/\mathbf{N}$. Finally set $\Gamma_M = \Gamma \cap P/\Gamma \cap N$.

If $f \in \mathcal{A}(\Gamma\backslash G)$ then we define

$$\Lambda_P(f)(m) = \int_{\Gamma\backslash N} f(\overline{n}m)d\overline{n}$$

with $d\overline{n}$ a choice of invariant measure on $\Gamma\backslash N$.

$$\Lambda_P : \mathcal{A}(\Gamma\backslash G) \to \mathcal{A}(\Gamma_M\backslash M)$$

is continous in the induced topologies.

Furthermore, if W is a closed invariant subspace that is an object in $\mathcal{FHC}(G)$ then $\Lambda_P(W) \in \mathcal{FHC}(M)$.

There is a partial inverse to this map. Let ${}^oP = \cap Ker|\chi|$ the intersection is over all $\chi : P \to \mathbb{C}^\times$ a continuous homomorphism Let W be a closed subrepresentation of $\mathcal{A}(\Gamma_M \backslash M)$ that is in $\mathcal{FHC}(G)$. Let $\nu : P/{}^oP \to \mathbb{C}^\times$ be a continuous homomorphism. We form

$$I^{\infty}_{P,W,\nu}$$

the corresponding induced representation. We note that $G = PK$ so we note that as a Fréchet space it is isomorphic with $I^{\infty}_{P\cap K,W}$ with W thought of as just a $P \cap K$ representation on a Fréchet space. If $f \in I^{\infty}_{P\cap K,W}$ then define $f_\nu(pk) = \nu(p)\det(Ad(p)_{|Lie(N)})^{\frac{1}{2}} L_p f(k)$.

Consider $\delta : W \to \mathbb{C}$ as above. Set for $f \in I^{\infty}_{P\cap K,W}$

$$E(P,W,\nu)(f)(g) = \sum_{\gamma \in \Gamma\cap P\backslash \Gamma\cap G} \delta(L_g f_\nu(\gamma)).$$

Unfortuately the series rarely converges.

However, if we start with V a closed invariant subspace of $\mathcal{A}(\Gamma\backslash G)$ that is an object in $\mathcal{FHC}(G)$ then if $f \in V_K$ is such that $\Lambda_P(f) = 0$ for all cuspical P as above with $P \neq G$ then Langlands showed using the condition of moderate growth that f is rapidly decreasing in a neighborhood of each cusp.

This implies that if we start with V a closed, invariant and in $\mathcal{FHC}(G)$ and if P is minimal among the cuspidal parabolic subgroups such that $\Lambda_P(V) \neq 0$ then there is an open tube of ν such that the series defining $E(P, \Lambda_P(V), \nu)$ coverges uniformly and absolutely defining a $(\mathfrak{g}, K)$-homomorphism

$$E_\nu : I_{P\cap K,,\Lambda_P(V)_{K_M}} \to \mathcal{A}(\Gamma\backslash G)_K.$$

Langlands proved that this family of $(\mathfrak{g}, K)$-homomorphisms has a meromorphic continuation to all ν.

The CW theorem implies that at a point of holomorphy E_ν extends to a continuous homomorphism

$$E_\nu : I^\infty_{P\cap K,\Lambda_P(V)} \to \mathcal{A}(\Gamma\backslash G).$$

We can replace $\Lambda_P(V)$ with W a closed subrepresentation of $\mathcal{A}(\Gamma_M\backslash M)$ that is in $\mathcal{FHC}(G)$. That has the property that $\Lambda_Q(W) = 0$ for all Q proper and cuspidal. The same meromorphic continuation therem is true.

Open question: The meromorphic continuation can be done in the context of

$$E_\nu : I^\infty_{P\cap K,W} \to \mathcal{A}(\Gamma\backslash G).$$