# ON THE UNIQUENESS OF FOURIER JACOBI MODELS FOR REPRESENTATIONS OF $U(2,1)$

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*Dedicated to Fridrikh Izrailevich Karpelevich*

Abstract. We show that every irreducible admissible representation of $U_{2,1}(k)$, where $k$ is a $p$-adic field or $k = \mathbf{R}$ has at most one Fourier Jacobi model.

## 1. INTRODUCTION

The Shimura integral for $G = U(2,1)$ studied in [5], [6], [7] is given by

$$\int_{G(F)\backslash G(\mathbf{A})} f(g)\overline{\Theta(g)}E(g,s)dg.$$

Here $F$ is a number field, $\mathbf{A}$ is the adels of $F$, $\Pi$ is an irreducible cuspidal automorphic representation of $G(\mathbf{A})$, $f$ is a cusp form in the space of $\Pi$ and $\Theta$ and $E$ are certain theta and Eisenstein series respectively. The integral is known ([6],[7]) to represent a certain L-function (in the complex variable $s$) attached to $\Pi$. The unfolding of this integral [5], [6], [7], gives rise to a Fourier Jacobi coefficient which in turn gives rise to local Fourier Jacobi models for the local representations $\Pi_\nu$ which are the components of $\Pi$. The uniqueness of this model which was proved by Piatetski-Shapiro [10] in the $p$-adic case allows the factorization of this integral into an Euler product. This proof has never appeared in print and we reproduce it here. We also prove the uniqueness of the archimedean models. We remark that many cases of uniqueness of models were proved in the $p$-adic case while the archimedean proofs are still lagging behind. We hope that our archimedean result here will give more motivation to bridge the gap.

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## 2. The main result

Let $k$ be a local field of characteristic zero. If $k$ is archimedean then we assume that $k = \mathbf{R}$. Let $K$ be a quadratic extension of $k$ and let $x \mapsto \bar{x}$ be the nontrivial Galois involution of $K$ over $k$. There exist $\epsilon \in k$ such that

$$K = k[\sqrt{\epsilon}] \tag{2.1}$$

and we have that every element $x$ of $K$ can be written uniquely in the form $x = \lambda_1 + \lambda_2\sqrt{\epsilon}$ with $\lambda_1, \lambda_2 \in k$. Then $\bar{x} = \lambda_1 - \lambda_2\sqrt{\epsilon}$. Let $S^1 = \{x \in K : x\bar{x} = 1\}$. Let

$$w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and let $G = G(k) = U_{2,1}(k)$ be defined by

$$G = \{A \in GL_3(K) \ : \ \bar{A}^t w A = w\}.$$

Let $N$ be the subgroup of upper unipotent matrices in $G$. Then

$$N = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & -\bar{x} \\ 0 & 0 & 1 \end{pmatrix} \ : \ x, z \in K, z + \bar{z} = -x\bar{x} \right\}.$$

The center of $N$ is

$$Z = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ : \ z \in K, z + \bar{z} = 0 \right\}.$$

Let

$$A = \left\{ d(a,b) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \bar{a}^{-1} \end{pmatrix} \ : \ a, b \in K, b\bar{b} = 1 \right\},$$

and

$$S = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \ : \ b \in K, b\bar{b} = 1 \right\}.$$

Let $B = AN$ be a Borel subgroup of $G$ and let $J = SN$ be a Fourier Jacobi subgroup of $G$. Let $\psi$ be a non-trivial character of $Z$ and let $\theta_\psi$ be the (smooth part of the) oscillator representation of $N$ with central character $\psi$. We shall use the Schrödinger model (see ([9], 3.1) or [4]) for $\theta_\psi$. The space of $\theta_\psi$ can be identified with $S(k)$ which is the space of Bruhat-Schwartz functions on $k$. If $k = \mathbf{R}$ then we put on $S(k)$ the usual Frechét topology.

The (smooth part of the) representation which is contragredient to the oscillator representation with central character $\psi$ can be identified with $\theta_{\psi^{-1}}$ via the $L^2$ inner product on $S(k)$. Let $\chi$ be a character of $S^1$ which we view as a character of $S \cong S^1$. Then $\chi \otimes \theta_\psi$ is a smooth irreducible representation of $J$.

Let $(\pi, V)$ be a smooth irreducible representation of $G$. If $k = \mathbf{R}$ then we assume that $V = H_\infty$ is the smooth part of an irreducible representation of $G$ on a Hilbert space $H$. Our main result in this paper is the following.

**Theorem 2.1.**

$$dim(Hom_J(V, S(k))) \leq 1.$$

**Remark 2.2.** Here $S(k)$ is the oscillator representation of $J$ which we denoted by $\chi \otimes \theta_\psi$. If $k = \mathbf{R}$ then Hom denotes the space of continuous linear maps between the Frechét spaces $H_\infty$ and $S(\mathbf{R})$.

If the dimension of the above Hom space is one then $\pi$ can be embedded in the space

$$Ind_J^G \sigma \otimes \theta_\psi.$$

We call this unique embedding, a Fourier Jacobi model for $\pi$ corresponding to the Fourier Jacobi data $(\chi, \psi)$.

We shall denote by $\mathrm{Hom}_J(\pi, \sigma \otimes \theta_\psi)$ the Hom space in Theorem 2.1. In order to prove Theorem 2.1 we notice that there is a natural injection from $\mathrm{Hom}_J(\pi, \sigma \otimes \theta_\psi)$ to $\mathrm{Hom}_{J^\triangle}(\pi \otimes (\chi \otimes \theta_\psi)^\vee, 1)$ where $\pi \otimes (\chi \otimes \theta_\psi)^\vee$ is a representation of $G \times J$, $J^\triangle$ is the diagonal embedding of $J$ into $G \times J$ and $(\chi \otimes \theta_\psi)^\vee$ is the (smooth part of the) representation which is $J$ contragredient to $\chi \otimes \theta_\psi$. By the remarks above on the oscillator representation we have that

$$(\chi \otimes \theta_\psi)^\vee = \chi^{-1} \otimes \theta_{\psi^{-1}}$$

hence

$$\dim(\mathrm{Hom}_J(\pi, \sigma \otimes \theta_\psi)) \leq \dim(\mathrm{Hom}_{J^\triangle}(\pi \otimes (\chi \otimes \theta_\psi), 1)).$$

Thus, Theorem 2.1 will follow from

**Theorem 2.3.**

$$dim(Hom_{J^\triangle}(\pi \otimes (\chi \otimes \theta_\psi), 1)) \leq 1.$$

*for every character $\chi$ of $T$ and every nontrivial character $\psi$ of $Z$.*

To prove Theorem 2.3 we let $\tau$ be an anti involution on $G \times J$ defined by

$$\tau(g) = (\bar{g}^{-1}, \bar{j}^{-1}).$$

We shall prove that any distribution on $G \times J$ which is bi-invariant under $J^\triangle$ and is $\psi$ equivariant under translations of the second variable by $Z$ is fixed by $\tau$. (If $k = \mathbf{R}$ we also assume that the distribution is an eigendistribution under the action of the casimir differential operator. See Theorem 3.1 for the precise statement). This will imply Theorem 2.3 as in ([1], p.184) or ([8], p.67). (See also [12], p.183-185 for a treatment of the archimedean case).

The remainder of the paper is devoted to proving Theorem 3.2 which is equivalent to Theorem 3.1 which we used above.

## 3. Preliminaries

3.1. **Group actions.** Let $X$ be a real analytic manifold or a $p$-adic space. We denote by $C_c^\infty(X)$ the space of compactly supported and smooth (resp. locally constant) functions on $X$. If a group $G$ acts on $X$ then $G$ acts on $C_c^\infty(X)$ by

$$g(\phi(x)) = \phi(g^{-1}x), \quad g \in G, x \in X, \phi \in C_c^\infty(X).$$

In particular, if $X$ is a subset of $G$ and if $x \in X$ and $g \in G$ then we denote:

$$L_g(x) = gx$$
$$R_g(x) = xg^{-1}$$
$$g(x) = gxg^{-1}$$

$G$ acts on distributions by duality.

If $k = \mathbf{R}$ and $G = U_{2,1}(\mathbf{R})$ then we let $\mathfrak{g}$ be the Lie Algebra of $G$ given by

$$\mathfrak{g} = \{A \in M_2(\mathbf{C}) \ : \ \bar{A}^t w + wA = 0\}.$$

$\mathfrak{g}$ acts on $C_c^\infty(G)$ by left invariant (resp. right invariant) differential operators as follows. Let $\phi \in C_c^\infty(G)$, $x \in G$, $A \in \mathfrak{g}$. We denote:

$$(L_A\phi)(x) = \frac{d}{dt}\phi(e^{tA}x)|_{t=0}$$
$$(R_A\phi)(x) = \frac{d}{dt}\phi(xe^{tA})|_{t=0}$$

These actions extend to the universal enveloping algebra of $G$. Let $\Box$ be the casimir element in the universal enveloping algebra. Then $L_\Box$ is defined as above.

3.2. **Equivalent statements of the main result.**

**Theorem 3.1.** *Let $T$ be a distribution on $G \times J$. Assume that*
*(a) $R_{(j,j)}T = T$, $L_{(j,j)}T = T$, $\quad j \in J$.*
*(b) $R_{(e,z)}T = \psi(z)T$, $\ z \in Z$.*
*(c) If $k = \mathbf{R}$ then $L_{\Box\otimes 1}T = \beta T$ for some scalar $\beta \in \mathbf{C}$.*
*Then $T^\tau = T$.*

Our main tools for studying these invariant distributions are Harish-Chandra's submersive maps, Frobenius reciprocity and in the $p$-adic case, Bernstein's localization principle. A rough and short statement of these principles together with references to the precise statements can be found in ([1], Lemma 2.3, Lemma 2.2 and Lemma 2.1).

Applying Frobenius reciprocity (see ([1], Theorem 2.5 and Theorem 2.6) for a similar situation) to the space of invariant distributions satisfying the conditions of Theorem 3.1 we get that Theorem 3.1 is equivalent to

**Theorem 3.2.** *Let $T$ be a distribution on $G$. Assume that*

*(a) $j(T) = T$, $j \in J$.*

*(b) $L_z T = \psi(z)T$, $z \in Z$.*

*(c) If $k = \mathbf{R}$ then $L_\square T = \beta T$ for some scalar $\beta \in \mathbf{C}$.*

*Then $T^\tau = T$.*

Notice that the action of $j$ on $T$ denoted by $j(T)$ above is the action induced by conjugation. To prove Theorem 3.2 we will assume that $T$ is a distribution on $G$ satisfying (a), (b), (c) above and that $T$ is skew invariant under $\tau$, that is, $T^\tau = -T$, and we will show that $T = 0$.

Before proceeding to the proof we state two Lemmas which apply only in the $p$-adic case and follow from Bernstein's localization principle [2]. For analog statements and proofs see ([1], Lemma 2.7 and Lemma 2.8). We shall assume for the following Lemmas that $k$ is $p$-adic. A $p$-adic set will be an $l$-space in the sense of [3].

**Lemma 3.3.** *Let $X$ be a $J$ stable $p$-adic subset of $G$ which is stable under $\tau$. If every $J$ orbit in $X$ is stable under $\tau$ then every $J$ invariant distribution $T$ on $X$ which is $\tau$ skew invariant vanishes identically.*

**Lemma 3.4.** *Let $X$ be a $J$ stable $p$-adic subset of $G$ which is stable under $Z$. (Notice that $J$ acts by conjugation and $Z$ by left translations). If every $J$ orbit in $X$ is stable under $Z$ then every $J$ invariant distribution on $X$ which is $\psi$ equivariant under $Z$ (see (a),(b) of Theorem 3.2) vanishes identically.*

## 4. DISTRIBUTIONS ON THE OPEN BRUHAT CELL

Our strategy in the proof of Theorem 3.2 is to restrict our skew-invariant distribution $T$ to the open Bruhat cell and show that it vanishes there. Since the proof in the $p$-adic case and in the archimedean case are quite similar we will treat them at the same time.

Let $X = BwB$ be the open Bruhat cell.

**Proposition 4.1.** *Let $T$ be a distribution on $X$ and assume that $T$ satisfies (a) and (b) of Theorem 3.2 and that $T^\tau = -T$. Then $T = 0$.*

*Proof.* We define a map from $N \times B$ to $X = BwN$ by

$$(n, b) \mapsto nbwn^{-1}$$

It is easy to check that this map is submersive hence by ([1], Lemma 2.3) it induces an onto mapping (which in this case is an isomorphism) from $C_c^\infty(N \times B)$ to $C_c^\infty(X)$. In particular, if $\alpha \in C_c^\infty(N)$ and $\beta \in C_c^\infty(B)$ then $\alpha \otimes \beta \in C_c^\infty(N \times B)$ is mapped to $f_{\alpha\otimes\beta} \in C_c^\infty(X)$ which is given by

$$f_{\alpha\otimes\beta}(bwn) = \alpha(n)\beta(nb)$$

Since $T$ is invariant under conjugation by $N$ we get that there exist a distribution $\sigma_T$ on $B$ such that

$$T(f_{\alpha\otimes\beta}) = \left(\int_N \alpha(n)dn\right)\sigma_T(\beta)$$

for every $\alpha$ and $\beta$ as above. We will show that $\sigma_T = 0$. Since $B$ is isomorphic to $N \times A$ via multiplication it follows that we can identify $\sigma_T$ with a distribution which we again call $\sigma_T$ on $N \times A$. Since $T$ is invariant under conjugation by $S$ it follows that $\sigma_T$ is invariant under conjugation by $S$ in the $N$ coordinate. Since $T$ is skew-invariant under $\tau$ it follows that $\sigma_T$ is skew invariant under $\tilde{\tau}$ where $\tilde{\tau}$ is given by

$$\tilde{\tau}(n,a) = (\bar{n}^{-1}, a)$$

We identify $N$ with $K \times k$ in the following way: For $x \in K$ and $z \in k$ we let $n(x,z) \in N$ be defined by

$$n(x,z) = \begin{pmatrix} 1 & x & -x\bar{x}/2 + z\sqrt{\epsilon} \\ 0 & 1 & -\bar{x} \\ 0 & 0 & 1 \end{pmatrix}.$$

Here $\epsilon$ is defined in (2.1). Thus we can identify $\sigma_T$ with a distribution $Q$ on $K \times k \times A$. The invariance of $\sigma_T$ under $S$ implies the invariance of $Q$ under the action of $S^1$ on $K \times k \times A$ given by

$$\lambda(x,z,a) = (\lambda x, z, a), \quad \lambda \in S^1, x \in K, z \in k, a \in A.$$

The skew invariance of $\sigma_T$ under $\tilde{\tau}$ implies that $Q$ is skew invariant under

$$(x,z,a) \mapsto (-\bar{x}, z, a)$$

Hence our Proposition will follow from the following Lemma. □

**Lemma 4.2.** *Let $Y$ be a manifold (resp. p-adic set). Let $Q$ be a distribution on $K \times Y$ and assume that $Q$ is invariant under the action of $S^1$ on $K \times Y$ given by*

$$\lambda(x,y) = (\lambda x, y), \quad \lambda \in S^1, x \in K, y \in Y.$$

*Then $Q$ is invariant under the involution*

$$(x,y) \mapsto (-\bar{x}, y), \quad x \in K, y \in Y.$$

*Proof.* We separate into the $p$-adic case and the archimedean case. First assume that $k$ is nonarchimedean. The involution maps $(x,y)$ to $(-\bar{x},y)$. If $x = 0$ then the involution fixes $(x,y)$. If $x \neq 0$ then

$$(-\bar{x}, y) = \frac{-\bar{x}}{x}(x,y)$$

hence in both cases the involution stabilizes the $S^1$ orbit of $(x,y)$. By Bernstein's localization principle it follows that $Q$ is invariant under the involution.

Now assume $k = \mathbf{R}$, $K = \mathbf{C}$, $\epsilon = -1$ and $\sqrt{\epsilon} = i$. Let $\mathbf{R}^* = \mathbf{R} - \{0\}$, $\mathbf{C}^* = \mathbf{C} - \{0\}$. We will also assume that $Q$ is skew invariant under the above involution. We define a map from $S^1 \times \mathbf{R}^* \times Y$ to $\mathbf{C}^* \times Y$ by

$$(\lambda, x, y) \mapsto (\lambda x i, y), \quad \lambda \in S^1, x \in \mathbf{R}^*, y \in Y.$$

It is easy to check that this map is submersive onto $\mathbf{C}^* \times Y$. It induces a map from $C_c^\infty(S^1 \times \mathbf{R}^* \times Y)$ to $C_c^\infty(\mathbf{C}^* \times Y)$. In particular, if $\alpha \in C_c^\infty(S^1)$ and $\beta \in C_c^\infty(\mathbf{R}^*)$ and $\gamma \in C_c^\infty(Y)$ and if $\alpha \otimes \beta \otimes \gamma \in C_c^\infty(S^1 \times \mathbf{R}^* \times Y)$ is mapped to $f_{\alpha\otimes\beta\otimes\gamma} \in C_c^\infty(\mathbf{C}^* \times Y)$ then there exist a distribution $\sigma_Q$ on $\mathbf{R}^* \times Y$ such that

$$Q(f_{\alpha\otimes\beta\otimes\gamma}) = \left( \int_{S^1} \alpha(\lambda) d\lambda \right) \sigma_Q(\beta \otimes \gamma).$$

It is easy to check that the induced involution is trivial on $\mathbf{R}^* \times Y$ and that $\sigma_Q$ is skew-invariant under the induced involution hence $\sigma_Q = 0$ and $Q$ is supported on $0 \times Y$.

We arite $\mathbf{C} = \{x + iy \ : \ x, y \in \mathbf{R}\}$. If a distribution $Q$ on $\mathbf{C} \times Y$ is supported on $0 \times Y$ then by a well known theorem of L. Schwartz, [11], there exist distributions $Q_{i,j}$, $i \geq 0, j \geq 0$ on $Y$ for which at most a finite number are nonzero such that

$$Q = \sum_{i,j \geq 0} \frac{\partial^i}{\partial x^i} \frac{\partial^j}{\partial y^j} Q_{i,j}.$$

Let $Z = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Since $Q$ is invariant under translations by $S^1$ in the first component it follows that there exist distributions $R_n$, $0 \leq n \leq N$ on $Y$ such that

$$Q = \sum_{n=0}^{N} Z^n R_n.$$

Since the involution sends $\frac{\partial}{\partial x}$ to $-\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ to $\frac{\partial}{\partial y}$ it follows that the involution fixes the differential operator $Z$ hence it fixes $Q$. Since $Q$ is skew invariant we get that $Q = 0$. $\square$

## 5. Distributions supported on the closed Bruhat cell

Our strategy in the proof of Theorem 3.2 is to restrict the sqew-invariant distribution $T$ to the open Bruhat cell and show that it vanishes there. After that we would like to show that distributions with the invariance conditions on $T$ with support in the closed Bruhat cell vanish identically. Here the proof in the $p$-adic case is very different from the proof in the archimedean case. In the $p$-adic case, a distribution supported on the closed cell comes from a distribution on the closed cell and we can continue treating such distributions using the same techniques as in the open cell. In the archimedean case, it is no longer true that a distribution supported on the closed cell comes from a distribution on the closed cell since the transversal derivatives come into play. However, in the archimedean case we have an extra hypothesis, that is, that $T$ is an eigendistribution for the casimir differential operator. Using this hypothesis and ideas as in [12] we will show that such distributions vanish.

5.1. **The $p$-adic case.**

**Proposition 5.1.** *Assume that $k$ is nonarchimedean and let $T$ be a distribution on $B$ satisfying the conditions of Theorem 3.2, that is, $T$ satisfies the invariance conditions (a) and (b) of Theorem 3.2 and $T^\tau = -T$. Then $T = 0$.*

*Proof.* Let $A' = \{d(a,b) \in G \ : \ a\bar{a} \neq 1\}$ and let $B' = A'N$. Then it is easy to check that $B'$ satisfies the conditions of Lemma 3.4, that is, every $J$ orbit in $B'$ (under conjugation) is stable under left translations by $Z$. Hence by Lemma 3.4, the restriction of $T$ to $B'$ vanishes and $T$ is supported on $B - B'$. Hence $T$ comes from a distribution $Q$ on $B - B'$ satisfying the same invariance conditions as $T$. It is easy to check that each $J$ orbit in $B - B'$ (under conjugation) is stable under $\tau$ hence by Lemma 3.3, $Q = 0$ and $T = 0$. $\square$

5.2. **The archimedean case.** Here we assume $k = \mathbf{R}$. We first give a realization of $\square$. We introduce some elements of $\mathfrak{g} = \mathfrak{u}_{2,1}(\mathbf{R})$. Let

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$D_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}, \quad D_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\square = \sqrt{2}Z^tZ + X^tX - Y^tY + \frac{1}{2}D_1^2 + (1 + \frac{1}{\sqrt{2}})D_1 - \frac{1}{2}D_2^2 - \frac{1}{\sqrt{2}}D_3^2$$

We let $D = \frac{1}{2}D_1^2 + (1 + \frac{1}{\sqrt{2}})D_1 - \frac{1}{2}D_2^2 - \frac{1}{\sqrt{2}}D_3^2$. Then

$$\square = \sqrt{2}Z^tZ + X^tX - Y^tY + D$$

Let $\mathfrak{b} = \mathrm{Lie}(B)$ and let $U(\mathfrak{b})$ be the universal enveloping algebra of $\mathfrak{b}$. We think of $U(\mathfrak{b})$ as an algebra of differential operators on $B$.

**Proposition 5.2.** *Let $k = \mathbf{R}$ and assume that a distribution $T$ satisfies (a),(b),(c) in Theorem 3.1 and that $T$ is supported on $B$. Then $T = 0$.*

*Proof.* The crucial observations for this proof are the following. We first notice that by (b) of Theorem 3.1,

$$L_Z T = cT$$

for some nonzero $c \in \mathbf{C}$ depending on $\psi$. Also, using (a) of Theorem 3.1 we get that

$$L_X T = R_X T, \quad L_Y T = R_Y T.$$

It will turn out to be essential to replace $L_X$ with $R_X$ and $L_Y$ with $R_Y$ as above. The reason is that $R_X$ commutes with all the differential operators $L_A$ for $A \in \mathfrak{g}$ while $L_X$ does not commute for example with $L_{X^t}$.

We can now write the equation $L_\Box T = \beta T$ in the form

$$\sqrt{2} c Z^t T = -L_{X^t} R_X T + L_{Y^t} R_Y T + (\beta - D)T. \tag{5.1}$$

Let $b \in B$. Since $T$ is supported on $B$, it follows from the theory of distributions of L. Schwartz [11] (see Lemma 2.4 in [12] for the relevant formulation) that there exist an open set $U$ around $b$ such that

$$T = \sum L^i_{Z^t} L^j_{Y^t} L^k_{X^t} T_{i,j,k} \tag{5.2}$$

on $U$. Here $T_{i,j,k}$ are distributions on $B$. Also, $T_{i,j,k}$ are determined uniquely and at most a finite number of them are nonzero. We shall think of the $T_{i,j,k}$ as the coefficients of the expression in (5.2) or the coefficients of $T$ at $b$. We notice that the distribution that appears in equation (5.1) is also supported on $B$ hence can be written around a neighborhood of $b$ as in (5.2) in a unique way. Our goal is to show that if $T$ is nonzero then the left hand side and the right hand side of (5.1) yield different coefficients contrary to the uniqueness of (5.2). In particular we will show that if $T \neq 0$ then a certain coefficient of $Q = \sqrt{2} c Z^t T$ is nonzero on the left hand side of (5.1) while it is zero on the right hand side of (5.1). Write

$$Q = \sum L^i_{Z^t} L^j_{Y^t} L^k_{X^t} Q_{i,j,k}$$

around $b$ as in (5.2). Then it is clear that

$$Q_{i,j,k} = \sqrt{2} c T_{i-1,j,k} \tag{5.3}$$

where we set $T_{i,j,k} = 0$ if $i < 0$. We now study the right hand side of (5.1)

We first notice that if $A, B \in \mathfrak{g}$ then $L_A$ commutes with $R_B$. Hence we have

(5.4)

$$Q = L_{X^T} \sum L^i_{Z^t} L^j_{Y^t} L^k_{X^t} (R_X T_{i,j,k}) + L_{Y^T} \sum L^i_{Z^t} L^j_{Y^t} L^k_{X^t} (R_Y T_{i,j,k}) + (\beta - D)T$$

We notice that $R_X T_{i,j,k}$ and $R_Y T_{i,j,k}$ are some new distributions on $B$. Assume $T \neq 0$. Let $(i_0, j_0, k_0)$ be such that

(1) $T_{i_0,j_0,k_0} \neq 0$,
(2) $i_0$ is maximal among all $(i,j,k)$ for which $T_{i,j,k} \neq 0$,
(3) $i_0 + j_0 + k_0$ is maximal among $(i_0, j, k)$ satisfying (2).

Then by (5.3) the $(i_0+1, j_0, k_0)$ coefficient of $Q$ is nonzero.

On the other hand it follows from (5.4) that each nonzero coefficient $Q_{i,j,k}$ satisfies either $i < i_0+1$ or $i+j+k < i_0+j_0+k_0+1$. To see this we must compute the contributions of each summand in (5.4). First we notice that the distribution $(\lambda - D)T$ does not contribute coefficients with $i > i_0$. This follows from the fact that

$$DL_{Z^t}^i L_{Y^t}^j L_{X^t}^k = L_{Z^t}^i L_{Y^t}^j L_{X^t}^k D'$$

for some differential operator $D' \in U(\mathfrak{b})$.

To compute the contributions of the first summand we must write

$$L_{X^T} L_{Z^t}^i L_{Y^t}^j L_{X^t}^k = \sum_{l,m,n} c_{l,m,n} L_{Z^t}^l L_{Y^t}^m L_{X^t}^n.$$

For example

$$L_{X^T} L_{Z^t} L_{Y^t} L_{X^t} = L_{Z^t} L_{Y^t} L_{X^t}^2 + 2iL_{Z^t}^2 L_{X^t}$$

Here we can see that a nonzero coefficient $(1,1,1)$ will produce nonzero coefficients $(1,1,2)$ and $(2,0,1)$ and we see that either the first coefficient did not increase or the sum did not increase. The reason for that is that $X^t$ and $Z^t$ commute while $[X^t, Y^t] = -2iZ^t$ so when we commute $L_{X^t}$ and $L_{Y^t}$ we produce summands that either do not increase the power of $L_{Z^t}$ or summands that do not increase the degree of the differential operator.

Similar arguments apply when we look at the contributions of the second summand which will involve commuting expressions of the form

$$L_{Y^T} L_{Z^t}^i L_{Y^t}^j L_{X^t}^k.$$

In fact this case is simpler since

$$L_{Y^T} L_{Z^t}^i L_{Y^t}^j L_{X^t}^k = L_{Z^t}^i L_{Y^t}^{j+1} L_{X^t}^k$$

and it is clear that the power of $L_{Z^t}$ did not grow.

Hence we get a contradiction and $T = 0$. □

5.3. **Proof of Theorem 3.1.** We are now ready to prove Theorem 3.1. Let $T$ be a distribution satisfying the (a), (b), (c) of Theorem 3.1 and such that $T^\tau = -T$. We restrict $T$ to the open Bruhat cell $BwB$. By Proposition 4.1, $T = 0$ on $BwB$. Hence $T$ is supported on $B$. By Proposition 5.1 and Proposition 5.2, $T = 0$.

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