CENTRAL VALUE OF AUTOMORPHIC $L-$FUNCTIONS

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Abstract. We prove a generalization to the totally real field case of the Waldspurger's formula relating the Fourier coefficient of a half integral weight form and the central value of the L-function of an integral weight form. Our proof is based on a new interpretation of Waldspurger's formula as a combination of two ingredients–an equality between global distributions, and a dichotomy result for theta correspondence. As applications we generalize the Kohnen-Zagier formula for holomorphic forms and prove the equivalence of the Ramanujan conjecture for half integral weight forms and a case of the Lindelöf hypothesis for integral weight forms. We also study the Kohnen space in the adelic setting.

1. Introduction

In this paper, we prove a Waldspurger type formula in the totally real field case, and study some of its applications. A Waldspurger type formula is an identity relating the Fourier coefficients of a half integral weight form and the central twisted $L-$values of an integral weight form. Waldspurger first proved such a formula for holomorphic modular forms (over $\mathbb{Q}$). Explicit formulas for certain cases of Waldspurger type formula were obtained by many authors. An incomplete list includes the papers of Gross [Gr], Katok-Sarnak [Ka-S], Kohnen-Zagier [KZ],[K1], Khuri-Makdisi [Kh], Kojima [Ko1], [Ko2], Niwa [N], Shimura [Sh1],[Sh2].

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Both the Fourier coefficients of half integral weight forms and the central L-values are of much arithmetic interest. An identity between them has many pleasant applications, see for examples [C-PS-S], [Iw1], [KZ], [O-Sk], [Lu-Ra].

In this paper we work in the setting of automorphic forms over totally real fields. Thus our formula is more general and valid for the cases of Maass forms and Hilbert modular forms.

Another key point in this paper is a new interpretation of the Waldspurger type formula. We see it as a combination of two results, the basic Waldspurger's formula and Waldspurger's dichotomy result on theta correspondence. We interpret the basic Waldspurger's formula as an identity of global distributions, thus fitting it into a much more general family of identities. The use of the dichotomy result explains some of the subtle conditions in the Waldspurger type formula.

The main results in this paper are:

1. A Waldspurger type formula that is valid for automorphic forms over a totally real field.

2. Some applications of this formula:

a) Equivalence of the Ramanujan conjecture for half integral weight forms over totally real field with a case of the Lindelöf Hypothesis for integral weight forms.

b) A generalization of the Kohnen-Zagier formula by removing the restrictions on the fundamental discriminants appearing in that formula.

c) An explicit computation of the central values of the twisted $L-$functions associated to the elliptic curve $X_0(11)$ using our generalization of the Kohnen-Zagier formula.

We remark that our formula can be extended to all number fields once the local result in [B-M1] and [B-M2] is extended to the case of the complex field.

Below we discuss in more detail some of the results.

1.1. **An explicit version of the Waldspurger type formula.** The Shimura correspondence associates a cusp form $f(z)$ of integral weight $2k$ to a half integral weight cusp form $g(z)$ of weight $k+1/2$. For $f(z)=\sum_{n=1}^{\infty} a(n)e^{2\pi inz}$, D a fundamental discriminant, let

$$L(f,D,s)=\sum_{n=1}^{\infty}(\frac{D}{n})a(n)n^{-s}.$$

Waldspurger was the first to describe a relation between the twisted central $L-$values $L(f,D,k)$ and the Fourier coefficients of $g(z)$ [W2]. There are many later versions of the Waldspurger type formula. The result of Kohnen and Zagier is probably the easiest to describe. It states: [K1]

> Let $f(z)$ be a new form of weight $2k$, square free and odd level N, and of trivial character. There is a unique (up to a scalar multiple) weight $k+1/2$ form $g(z)=\sum_{n=1}^{\infty} c(n)e^{2\pi inz}$ corresponding to f and lying in the *Kohnen Space* ([K2]), such that when D is a fundamental discriminant with $(-1)^k D>0$ and $(\frac{D}{l})=w_l$ for all prime divisors l of N,

$$\frac{|c(|D|)|^2}{<g,g>}=\frac{(k-1)!}{\pi^k}|D|^{k-1/2}2^{\nu(N)}\frac{L(f,D,k)}{<f,f>}. \tag{1.1}$$

In the above statement, we have adopted the notations in [K1]; $\nu(N)$ is the number of prime divisors of N; w_l are the eigenvalues of the Atkin-Lehner involutions acting on $f(z)$.

Let us note some subtle conditions in the Kohnen-Zagier formula. First there is the restriction on the fundamental discriminant D. When $(\frac{D}{l})=-w_l$ for some l, we would have $c(|D|)=0$ while $L(f,D,k)$ does not necessarily vanish. When $l|D$ for some l, the constant in the formula needs to be modified (see [Gr]). Secondly to get the one to one correspondence between $f(z)$ and $g(z)$, there is the restriction on the level of $g(z)$ and the notion of Kohnen space.

For certain applications, it is crucial to remove the restriction on the fundamental discriminant D. For example, if $f(z)$ is a new form corresponding to an elliptic curve over $\mathbb{Q}$, then the weight is $k = 2$. The above formula works only in the case of $D < 0$. As an application of our main results, we derive a generalization of the Kohnen-Zagier formula without any restriction on D. Our result is:

Theorem 1.1. *Restatement of Theorem 10.1:*

Associated to $f(z)$ is a finite set of weight $k+1/2$ forms in the Kohnen space $\{g_S(z)|S \subset S_N\}$, here S_N is the set of all prime divisors of N. We divide the set of fundamental discriminants into a finite disjoint union of sets. Each of these sets will correspond to a subset S of S_N and to a form g_S in the following way: if the fundamental discriminants D is associated to S, then it satisfies $(\frac{D}{l}) = -w_l$ if and only if $l \in S$. For such a D we have: if $(-1)^{s+k} \neq sgn(D)$, then $L(f, D, k) = 0$; if $(-1)^{s+k} = sgn(D)$, then

$$\frac{|c_S(|D|)|^2}{< g_S, g_S >} = \frac{L(f, D, k)}{< f, f >}|D|^{k-1/2}\frac{(k-1)!}{\pi^k}2^{\nu(N)-t}\prod_{p\in S}\frac{p}{p+1}. \tag{1.2}$$

Here s is the size of S, t is the number of primes dividing both D and N, and $c_S(|D|)$ is the $|D|$-th Fourier coefficient of the form g_S.

Notice that the above theorem covers all fundamental discriminants D. In the process of deriving the theorem, we give an interpretation of the restrictions on D and an interpretation of the Kohnen space in the language of automorphic representations.

Example: Assume $N = p$ is an odd prime. Then $S_N = \{p\}$ is a singleton and we have two subsets of S_N: $S_1 = \emptyset$ and $S_2 = \{p\}$. If f is a new form of weight $2k$ and level $N = p$ then we attach to f two weight $k + 1/2$ forms g_{S_1} and g_{S_2}. Here g_{S_1} is the form that appears in (1.1) and g_{S_2} is a form of level $4p^2$.

Let D be a fundamental discriminant. Assume that $(\frac{D}{p}) = w_p$ or $p|D$. If $(-1)^k D < 0$ then $L(f, D, k) = 0$. If $(-1)^k D > 0$ then formula (1.2) gives (1.1) (and its generalization in [Gr] for the case $p|D$).

Now assume $(\frac{D}{p}) = -w_p$. If $(-1)^{k+1}D < 0$ then $L(f,D,k) = 0$. If $(-1)^{k+1}D > 0$ then

$$\frac{|c_{S_2}(|D|)|^2}{< g_{S_2}, g_{S_2} >} = \frac{L(f,D,k)}{< f,f >}|D|^{k-1/2}\frac{(k-1)!}{\pi^k}\frac{2p}{p+1}. \tag{1.3}$$

In § 10.1, an example of the set $\{g_S(z)\}$ is given when $p = 11$ and $k = 1$. The above theorem gives us an effective way of computing the family of $L-$values $L(f,D,k)$ for the unique weight 2 level 11 cusp form, namely through constructing the forms $g_S(z)$ and using the equation (1.2).

1.2. **The main result: basic Waldspurger's formula.** Our main result in this paper is a general relation between $L-$values and Fourier coefficients which we call the basic Waldspurger's formula. One can derive from it the explicit versions of Waldspurger type formula, including that of Theorem 1.1. Another important application of the formula is the equivalence of the Ramanujan conjecture for half integral weight forms and a case of Lindelöf hypothesis. This is an equivalence between the bound on Fourier coefficients of half integral weight forms and the bound on central $L-$values of integral weight forms. We establish this equivalence in the generality of totally real number fields. For this application we will need the full generality of our formula. This equivalence was recently used in [C-PS-S], in combination with their new bound on $L-$values, to give a solution for a classical problem about representing integers by ternary quadratic forms over totally real fields and to solve Hilbert's 11th problem.

In the basic formula, we work with the more general notion of automorphic representations over a totally real number field F. The relationship between the modular forms and the automorphic representations is as follows. A half-integral weight modular form is a vector in the space of an automorphic representation $\tilde{\pi}$ of $\widetilde{SL}_2$, the double cover of SL_2. An integral weight modular form with trivial character is a vector in the space of an automorphic representation π of PGL_2. The representation theoretic version of the Shimura correspondence is a theta correspondence relating π of PGL_2 to some $\tilde{\pi}$ of $\widetilde{SL}_2$ [W1].

Our first task is to define the constants associated to $\tilde{\pi}$ and π that are the analogues of the Fourier coefficients of the modular forms. Such constants are defined in § 2. For $D \neq 0 \in F^*$, we define the $D-$th "Fourier coefficients" of π and $\tilde{\pi}$ to be $d_\pi(S, \psi^D)$ and $d_{\tilde{\pi}}(S, \psi^D)$, (see § 2 for the definition). Here S is a finite set of bad local places and ψ^D is an additive character.

The basic Waldspurger's formula is the following simple statement.

Theorem 1.2. *Restatement of Theorem 4.1:*

Given a π and D, there is a representation $\tilde{\pi} = \tilde{\pi}(\pi, D)$ (under the theta correspondence) of $\widetilde{SL}_2$; we have

$$|d_\pi(S, \psi^D)|^2 L^S(\pi, 1/2) = |d_{\tilde{\pi}}(S, \psi^D)|^2. \tag{1.4}$$

Here $L^S(\pi, 1/2)$ is the central (partial) $L-$value. Some may prefer the following more explicit version (see (4.4)):

$$\frac{|\tilde{W}^D_{\tilde{\varphi}}(e)|^2}{||\tilde{\varphi}||^2} = \frac{|W_\varphi(e)|^2 L(\pi, 1/2)}{||\varphi||^2} \prod_{v \in S} E_v(\varphi_v, \tilde{\varphi}_v, \psi_v, D). \tag{1.5}$$

Here φ is any cusp form in the space of π and $\tilde{\varphi}$ is a cusp form in the space of $\tilde{\pi}$ with $\tilde{W}^D_{\tilde{\varphi}}(e) \neq 0$. $W_\varphi(e)$ and $\tilde{W}^D_{\tilde{\varphi}}(e)$ are the values of Whittaker functionals on φ and $\tilde{\varphi}$; they are directly related to Fourier coefficients of modular forms (see § 9). $E_v(\varphi_v, \tilde{\varphi}_v, \psi_v, D)$ are some local constants. One can compare this equation with the Kohnen-Zagier formula (1.1); the factors $\frac{(k-1)!}{\pi^k}|D|^{k-1/2}$ and 2 in (1.1) correspond to the local constants $E_v(\varphi_v, \tilde{\varphi}_v, \psi_v, D)$ when $v = \infty$ and $v = l|N$ respectively.

The basic formula can be interpreted as an equality between two global (adelic) distributions. Roughly, we define two global distributions I_π and $J_{\tilde{\pi}}$, and obtain the following factorization into products of distributions over local fields:

$$I_\pi = c_1 \prod_{v\in S} I_{\pi,v}, \quad J_{\tilde{\pi}} = c_2 \prod_{v\in S} J_{\tilde{\pi},v}.$$

Here $c_1 = |d_\pi(S,\psi^D)|^2 L^S(\pi,1/2)$ and $c_2 = |d_{\tilde{\pi}}(S,\psi^D)|^2$. For the more precise formulas, see Propositions 6.2 and 6.4. The basic Waldspurger's formula is then just the identity $c_1 = c_2$. It follows immediately from the comparison between the global distributions I_π and $J_{\tilde{\pi}}$ ([J1]), and the comparison between the local distributions $I_{\pi,v}$ and $J_{\tilde{\pi},v}$ ([B-M1],[B-M2]).

The basic formula thus fits into a more general family of formulas that should result from the comparison of the global distributions. We note there are many other comparisons of global distributions established or conjectured through works on the theory of the relative trace formula, see [Gu], [J2], [M], [M-R] etc.

1.3. **Use of Waldspurger's dichotomy result and the restriction on fundamental discriminants.** To derive a formula on twisted $L-$values from the basic formula (1.4), we simply apply it with π replaced by $\pi\otimes\chi_D$ where χ_D is a quadratic character associated to $D\in F^*$. This leads to the consideration of $\tilde{\pi}(\pi\otimes\chi_D, D)$, which is not necessarily the same for all D.

The dichotomy result of Waldspurger on theta correspondence ([W3]) gives a partition of the set F^* into a finite collection of subsets, such that the representation $\tilde{\pi}(\pi\otimes\chi_D, D)$ remains the same (or becomes zero dimensional) for D in a given subset. The equation (1.4) then gives a formula for the twisted $L-$value $L(\pi\otimes\chi_D,1/2)$ in terms of the $D-$th "Fourier coefficients" $d_{\tilde{\pi}}(S,\psi^D)$ of a fixed $\tilde{\pi}$, as long as D lies in this particular subset.

We now consider the conditions on D in the Kohnen-Zagier formula (1.1). The condition is to ensure that D lies in a given subset of $\mathbb{Q}^*$, so that the half-integral weight form appearing in (1.1) remains the same. With this understanding, we see that for D in other subsets of $\mathbb{Q}^*$, there should also be another version of the Kohnen-Zagier formula, involving a different half integral weight form. A detailed study of local representations leads to the generalization stated above.

1.4. **Ramanujan conjecture for half integral weight forms and Lindelőf hypothesis.** The usual Ramanujan conjecture for half integral weight form states a bound on Fourier coefficients in the case of holomorphic cusp forms over $\mathbb{Q}$. We will state in section (4.3) an automorphic form version of the Ramanujan conjecture for half integral weight forms. It states: (for notations see section (4.3))

Conjecture 1.3. *(Ramanujan conjecture). Let $\tilde{\pi}$ be an irreducible subrepresentation of $\tilde{A}_{00}$. Let $\tilde{\varphi}$ be a cusp form in the space of $\tilde{\pi}$. For D a square free integer in F^*, as $|D|_{S_\infty} \mapsto \infty$, for all $\alpha > 0$*

$$|\tilde{W}^D_{\tilde{\varphi}}(e)| \prod_{v \in S_\infty} e(\tilde{\varphi}_v, \psi^D_v) <<_{\tilde{\pi},\tilde{\varphi},\alpha} |D|^{\alpha-1/2}_{S\infty} \tag{1.6}$$

where the implied constant depends only on $\tilde{\pi}, \tilde{\varphi}$ and α.

This statement has the advantage of being applicable to non-holomorphic forms and Hilbert modular forms in the space of $\tilde{A}_{00}$. Note that $\tilde{W}^D_{\tilde{\varphi}}(e)$ is basically the $D-$th Fourier coefficient of $\tilde{\varphi}$. To see the equivalence with the usual Ramanujan conjecture for half integral weight cusp forms over $\mathbb{Q}$, we only need to compute the local constant $e(\tilde{\varphi}_\infty, \psi^D_\infty)$. This is done in sections (8.4) and (9.3).

The Lindelőf hypothesis is a conjecture on the bound of central value of $L-$functions. We state only a special case.

Conjecture 1.4. *(Lindelőf hypothesis) Let π be an irreducible cuspidal automorphic representation of PGL_2 with trivial central character, then for D square free integer, as $|D|_{S_\infty} \mapsto \infty$, for all $\beta > 0$*

$$|L^{S_\infty}(\pi \otimes \chi_D, 1/2)| <<_{\pi,\beta} |D|^{\beta}_{S_\infty} \tag{1.7}$$

where the implied constant depends only on π and β.

We will show the following Theorem.

Theorem 1.5. *Restatement of Theorem 4.6*

The inequality (1.6) holds for some $\alpha > 0$ *if and only if the inequality (1.7) holds for* $\beta = 2\alpha > 0$*. The above two conjectures are equivalent.*

1.5. **Structure of the paper.** The paper is organized as follows: In § 2, we define the constants $d_\pi(S, \psi)$ and $d_{\tilde{\pi}}(S, \psi)$. We recall Waldspurger's result on theta correspondence in § 3. In § 4 we state the main theorems. The relative trace formula of [J1] is reviewed in § 5. We describe the local theory of the relative trace formula in § 6. The proof of the main theorems are given in § 7. In § 8, we compute some examples of local constants appearing in the identity for $L(\pi, 1/2)$. The computations are just easy exercises, and the results are used in the translation from adelic language to modular form language of our formula, as well as in a proof in § 7. In § 9, we give a dictionary between the language of representation theory and modular forms. We also give an interpretation for the Kohnen space. In § 10, we prove the Kohnen-Zagier formula without the conditions on the fundamental discriminant D, and give an application in the explicit computation of twisted $L-$values.

Notations and background:

Let F be a totally real number field, $\mathbf{A}$ its adele ring. We will use v to denote places of F. When v is non-archimedean, let $\mathcal{O}_v$ be the ring of integers in F_v, P_v (or P) be its prime ideal, ϖ its uniformizer, and q_v (or q) the size of the residue field $\mathcal{O}_v/P_v$. We use $||_v$ to denote the normalized metric on F_v.

Let $G = GL_2$, $\tilde{G} = \widetilde{GL}_2$ and $G' = \widetilde{SL}_2$. We will use e to denote the identity elements of the groups G, G' and $\tilde{G}$. Let Z be the center of G. Then $PGL_2 = G/Z$. Let B be the subgroup of GL_2 consisting of upper triangular matrices, $\tilde{B}$ be its lifting to $\tilde{G}$.

We will use $(g,\pm 1)$ to denote an element in $\tilde{G}$. Let $[*,*]$ be the Hilbert symbol. The multiplication in $\tilde{G}$ takes the form:

$$(g_1,1)\cdot(g_2,1)=(g_1g_2,[\frac{x(g_1g_2)}{x(g_1)},\frac{x(g_1g_2)}{x(g_2)}\det g_1])$$

where for $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $x(g)=c$ if $c\neq 0$ or d if $c=0$.

Let $n(x)=\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $\tilde{n}(x)=(n(x),1)$. Let $w=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\tilde{w}=(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},1)$. Let $\underline{a}=\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, $\underline{\underline{a}}=(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},1)$.

We fix a nontrivial additive character ψ of $\mathbf{A}/F$. Then $\psi=\prod_v \psi_v$. For $D\in F^*$, let $\psi^D(x)=\psi(Dx)$; let χ_D be the quadratic character of $\mathbf{A}^*/F^*$ associated to the field extension $F(\sqrt{D})$. At a local place v, for $D\in F_v^*$, we let χ_D be the quadratic character of F_v^* associated to the extension $F_v(\sqrt{D})$.

We will fix measures as follows. The choice of additive measure dx on F_v does not matter for the statement of our theorem. We will however fix it to be self dual for the character ψ_v. The multiplicative measure is $d^*a=(1-q^{-1})^{-1}\frac{da}{|a|_v}$, where q is the size of the residue field when v is $p-$adic, and $q=\infty$ when $v=\infty$. We fix the measures for $Z\backslash GL_2$ and $\widetilde{SL}_2$ as in [B-M1] and [B-M2]. Write $g=z(c)n(x)w\underline{a}n(y)$ for $g\in G(F_v)-B(F_v)$, then $dg=|a|_v d^*cd^*adxdy$ is the measure on $G(F_v)$. The measure on $Z(F_v)$ is $dz(c)=d^*c$, and we use the resulting quotient measure on $Z(F_v)\backslash G(F_v)$. For $g\in G'(F_v)-\tilde{B}(F_v)\cap G'(F_v)$ with $g=\tilde{n}(x)\tilde{w}\underline{\underline{a}}\tilde{n}(y)$, we define $dg=|a|_v^2 d^*adxdy$ to be the measure on $G'(F_v)$.

Define the Weil constant $\gamma(a,\psi_v^D)$ over F_v to satisfy:

$$\int\hat{\Phi}(x)\psi_v^D(ax^2)dx=|a|_v^{-1/2}\gamma(a,\psi_v^D)\int\Phi(x)\psi_v(-a^{-1}x^2)dx$$

where

$$\hat{\Phi}(x)=\int\Phi(y)\psi_v^D(-2xy)dy.$$

We let $\tilde{\gamma}(a,\psi_v^D)=\frac{\gamma(a,\psi_v^D)}{\gamma(1,\psi_v^D)}[-1,a]$.

We use π to denote an irreducible cuspidal representation of $G(\mathbf{A})$ with trivial central character, and use $\tilde{\pi}$ to denote an irreducible cuspidal representation of $G'(\mathbf{A})$. π can be considered as a representation of $PGL_2(\mathbf{A})$. We have $\pi = \otimes\pi_v$ and $\tilde{\pi} = \otimes\tilde{\pi}_v$ as the restricted tensor products of representations over local fields. We will use $V_\pi, V_{\tilde{\pi}}, V_{\pi,v}$ and $V_{\tilde{\pi},v}$ to denote the spaces where the representations $\pi, \tilde{\pi}, \pi_v$ and $\tilde{\pi}_v$ act on respectively.

When μ is a character of F_v^*, we will let $\pi(\mu, \mu^{-1})$ denote the principal series representation of $G(F_v)$ induced from μ. It acts by right translation on the space of functions ϕ on $G(F_v)$ that satisfies:

$$\phi(n(x)\underline{a}zg) = \mu(a)|a|_v^{1/2}\phi(g). \tag{1.8}$$

We use $\tilde{\pi}(\mu, \psi_v)$ to denote the principal series representation of $G'(F_v)$ that acts on the space of functions ϕ of $G'(F_v)$ satisfying:

$$\phi(\tilde{n}(x) \cdot \underline{\underline{a}} \cdot g) = \mu(a)\tilde{\gamma}(a, \psi_v)|a|_v\phi(g). \tag{1.9}$$

These representations are unramified if μ and ψ_v are unramified.

The $L-$function $L(\pi, s)$ is defined in [J-L]. So is the factor $\epsilon(\pi, s) = \prod \epsilon(\pi_v, s, \psi_v)$. We note that $\epsilon(\pi_v, 1/2) = \epsilon(\pi_v, 1/2, \psi_v)$ is independent of the choice of ψ_v.

By the well known result on the Shimura-Waldspurger (theta) correspondence ([R-Sc], [Sh1], [W1], [W3]), for the given ψ^D there associates a unique irreducible cuspidal representation $\tilde{\pi}(\pi, D) = \Theta(\pi, \psi^D)$ of $G'(\mathbf{A})$. Here Θ denotes the theta correspondence. Similarly given $\tilde{\pi}$ on G', there is a unique irreducible cuspidal representation $\Theta(\tilde{\pi}, \psi^D)$ on PGL_2. We note that the space of $\Theta(\pi, \psi^D)$ and $\Theta(\tilde{\pi}, \psi^D)$ could be zero dimensional. The theta correspondence is also defined locally, which we again denote by Θ.

We will use S to denote a finite set of local places. We say that S *contains bad places* if it contains all v which is archimedean or has even residue characteristic. For $v \notin S$, the covering $G'(F_v)$ splits over $SL_2(\mathcal{O}_v)$. With this splitting, we consider $SL_2(\mathcal{O}_v)$ a subgroup of $G'(F_v)$. Explicitly the embedding of $SL_2(\mathcal{O}_v)$ in G' is given by $g \mapsto (g, \kappa(g))$, where $\kappa(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = [c, d]$ if $|c|_v < 1$ and $c \neq 0$, $\kappa(g)$ equals 1 when $|c|_v = 1$ or $c = 0$.

We will use $||\varphi||$ to denote the norm of a vector φ: if $(*, *)$ is a Hermitian form on a space V, for $\varphi \in V$, let $||\varphi|| = (\varphi, \varphi)^{1/2}$. We use $\{\delta_i\}$ to denote a set of representatives for the square classes in $F_v^*/(F_v^*)^2$, with $\delta_1 = 1$.

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2. Definition of two constants

2.1. A constant associated to π on G.

We define a constant $d_\pi(S, \psi)$ which can be considered as the Fourier coefficient of π. The constant depends only on the character ψ, the choice of the finite set of places S and the choice of Haar measures. Note that Haar measures are fixed in the introduction.

2.1.1. *Whittaker model on G.* Let π be an irreducible cuspidal automorphic representation of $G(\mathbf{A})$ with trivial central character. Let $V_\pi \subset L^2(Z(\mathbf{A})G(F)\backslash G(\mathbf{A}))$ be the space that π acts on. For $\varphi \in V_\pi$, let

$$W_\varphi(g) = W_\varphi^\psi(g) \int_{\mathbf{A}/F} \varphi(n(u)g)\psi(-u)du. \tag{2.1}$$

Then the space $\{W_\varphi | \varphi \in V_\pi\}$ gives the global Whittaker model of π.

Remark 2.1. When an integral weight form f is considered as a vector φ in the space of V_π, its Fourier coefficients are roughly the values of $W_\varphi(e)$ for various choices of ψ, (see § 9).

For any admissible representation π_v of $G(F_v)$, a ψ_v-Whittaker functional $L_v : V_{\pi,v} \to \mathbb{C}$ on V_{π_v} is a linear functional satisfying:

$$L_v(\pi_v(n(u))v) = \psi_v(u)L_v(v), v \in V_{\pi,v}. \tag{2.2}$$

The space of ψ_v-Whittaker functional is at most one dimensional. For the π_v's appear as local components of π, such a space is one dimensional. We will fix the linear form L_v for any given π_v.

Let S be a finite set of places, and assume that S contains all bad places along with places v where π_v is not unramified. For $v \notin S$, π_v is an unramified representation of $G(F_v)$; let $\varphi_{0,v} \in V_{\pi,v}$ be the unique vector fixed under the action of $G(\mathcal{O}_v)$ such that $L_v(\varphi_{0,v}) = 1$.

We note that

$$L(\varphi) = W_\varphi(e)$$

is a linear form on V_π satisfying (2.2). From the uniqueness of the local Whittaker functional, L can been expressed as a product of local linear forms L_v. There is a constant $c_1(\pi, S, \psi, \{L_v\})$, such that whenever $\varphi = \otimes_{v\in S}\varphi_v \otimes_{v\notin S}\varphi_{0,v}$, (here we fix an identification between V_π and the restricted tensor product $\otimes V_{\pi,v}$ where $V_{\pi,v}$ is the space of the local component π_v)

$$W_\varphi(e) = c_1(\pi, S, \psi, \{L_v\}) \prod_{v\in S} L_v(\varphi_v). \tag{2.3}$$

2.1.2. *Hermitian forms on G.* Define a Hermitian form on V_π by:

$$(\varphi, \varphi') = \int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} \varphi(g)\overline{\varphi'(g)}dg \tag{2.4}$$

Over a local place v, for a unitary representation π_v with a nontrivial Whittaker functional L_v, we can define a G_v-invariant Hermitian form on $V_{\pi,v}$ by:

$$(u, u') = \int_{F_v^*} L_v(\pi_v(\underline{a})u)\overline{L_v(\pi_v(\underline{a})u')}\frac{da}{|a|_v}, \tag{2.5}$$

(see [Go]). From the uniqueness of G_v-invariant Hermitian forms, we get: there is a constant $c_2(\pi, S, \psi, \{L_v\}) > 0$, such that whenever $\varphi = \otimes_{v\in S}\varphi_v \otimes_{v\notin S}$

$\varphi_{0,v}$,

$$||\varphi|| = c_2(\pi, S, \psi, \{L_v\}) \prod_{v\in S} ||\varphi_v||. \tag{2.6}$$

2.1.3. *The constant $d_\pi(S,\psi)$.*

Lemma 2.2. *The constant $d_\pi(S,\psi)$ defined by*

$$d_\pi(S,\psi) = |c_1(\pi, S, \psi, \{L_v\})/c_2(\pi, S, \psi, \{L_v\})|$$

is independent of the choice of the linear forms L_v.

Proof. From the uniqueness of local Whittaker functional, any other choice of linear forms L'_v must have the form $L'_v = a_v L_v$ with a_v some nonzero complex constants. From the definition, we get

$$c_2(\pi, S, \psi, \{L'_v\}) = \prod_{v\in S} |a_v|^{-1} c_2(\pi, S, \psi, \{L_v\}),$$

$$c_1(\pi, S, \psi, \{L'_v\}) = \prod_{v\in S} a_v^{-1} c_2(\pi, S, \psi, \{L_v\}).$$

Thus the constant $d_\pi(S,\psi)$ is independent of the choice of $\{L_v\}$. □

This is the "Fourier coefficient" we associate to π. The constant $d_\pi(S,\psi)$ is well defined once we fix the choice of ψ and the measure on G, (it is easy to check that the constant is independent of the choice of additive measure). One can relate this constant to the value of the partial adjoint $L-$function $L^S(\pi, Ad, 1)$, however it is not important here. Explicitly, for any vector $\varphi = \otimes_{v\in S}\varphi_v \otimes_{v\notin S} \varphi_{0,v}$ with $L_v(\varphi_v) \neq 0$ for $v \in S$,

$$d_\pi(S,\psi) = \frac{|W_\varphi(e)|}{||\varphi||} \prod_{v\in S} \frac{||\varphi_v||}{|L_v(\varphi_v)|}. \tag{2.7}$$

We make an observation on the dependence of $d_\pi(S,\psi)$ on ψ.

Lemma 2.3. *Let $D \in F^*$. If S contains all bad places and $|D|_v = 1$ for all $v \notin S$, then*

$$d_\pi(S,\psi) = d_\pi(S,\psi^D).$$

Proof. Take a vector $\varphi = \otimes_{v\in S}\varphi_v \otimes_{v\notin S} \varphi_{0,v}$ in the space of π. We will let $L_v^D(\varphi_v) = L_v(\pi_v(\underline{D})\varphi_v)$. Then L_v^D is a nontrivial ψ_v^D-Whittaker functional on π_v. Let $||\varphi_v||_D$ be the norm of φ_v defined by (2.5) with L_v replaced by L_v^D. By our assumption on D and on $\varphi_{0,v}$ it follows that when $v \notin S$, $\pi_v(\underline{D})\varphi_{0,v} = \varphi_{0,v}$. Hence $L_v(\pi_v(\underline{D})\varphi_{0,v}) = L_v(\varphi_{0,v}) = 1$. It follows that $\pi(\underline{D})\varphi = \otimes\pi_v(\underline{D})\varphi_v$ satisfies the same assumptions as φ and that we can apply (2.3) to $\pi(\underline{D})\varphi$ to get

$$c_1(\pi, S, \psi, \{L_v\}) = \frac{W_\varphi(e)}{\prod_{v\in S} L_v(\varphi_v)} = \frac{W_{\pi(\underline{D})\varphi}(e)}{\prod_{v\in S} L_v(\pi(\underline{D})\varphi_v)}.$$

Using the above explicit form, we have

$$d_\pi(S, \psi^D) = \frac{|W_\varphi^{\psi^D}(e)|}{||\varphi||} \prod_{v\in S} \frac{||\varphi_v||_D}{|L_v^D(\varphi_v)|}.$$

From a simple change of variable we get $W_\varphi^{\psi^D}(e) = W^\psi_{\pi(\underline{D})\varphi}(e)$. Hence,

$$\frac{|W_\varphi^{\psi^D}(e)|}{\prod_{v\in S}|L_v^D(\varphi_v)|} = \frac{|W^\psi_{\pi(\underline{D})\varphi}(e)|}{\prod_{v\in S}|L_v(\pi(\underline{D})\varphi_v)|} = \frac{|W_\varphi^\psi(e)|}{\prod_{v\in S}|L_v(\varphi_v)|}.$$

Using a change of variable in the integral in (2.5) we get $||\varphi_v||_D = ||\varphi_v||$. Thus we get the equality in the Lemma. $\square$

2.2. **A constant associated to $\tilde{\pi}$ on G'.** Let $\tilde{\pi}$ be an irreducible cuspidal automorphic representation of G'. We associate a constant $d_{\tilde{\pi}}(S, \psi^D)$ to $\tilde{\pi}$ in a similar fashion. Let $V_{\tilde{\pi}}$ be the space of automorphic forms that $\tilde{\pi}$ acts on. For $\tilde{\varphi} \in V_\pi$, let

$$\tilde{W}_{\tilde{\varphi}}^D(g) = W_{\tilde{\varphi}}^{\psi^D} = \int_{\mathbf{A}/F} \tilde{\varphi}(\tilde{n}(x)\cdot g)\psi^D(-x)dx.$$

Then the Fourier coefficients of half integral weight form can be interpreted as some $\tilde{W}_{\tilde{\varphi}}^D(e)$, (see equation (9.2)).

We will assume $\tilde{\pi}$ has a nontrivial ψ^D-Whittaker model, namely $\tilde{W}_{\tilde{\varphi}}^D(g) \neq 0$ for some $\tilde{\varphi} \in V_{\tilde{\pi}}$. Then locally, $\tilde{\pi}_v$ has a nontrivial ψ_v^D-Whittaker model, unique up to a scalar multiple. We will fix the corresponding linear forms

$\tilde{L}_v^D$ satisfying for all $\tilde{\varphi}_v \in V_{\tilde{\pi},v}$,

$$\tilde{L}_v^D(\tilde{\pi}_v(\tilde{n}(x))\tilde{\varphi}_v) = \tilde{L}_v^D(\tilde{\varphi}_v)\psi_v^D(x).$$

Let S be a finite set of places That contains all bad places along with places v where $\tilde{\pi}_v$ is not unramified. When $v \notin S$, $\tilde{\pi}_v$ is unramified and possess a nontrivial ψ_v^D-Whittaker model $\tilde{L}_v^D$; there is a unique vector in $\tilde{\varphi}_{0,v}$ that is fixed under $SL_2(\mathcal{O}_v)$ and satisfying $\tilde{L}_v^D(\tilde{\varphi}_{0,v}) = 1$.

The space $V_{\tilde{\pi}}$ has the Hermitian form

$$(\tilde{\varphi}, \tilde{\varphi}') = \int_{SL_2(F)\backslash G'(\mathbf{A})} \tilde{\varphi}(g)\overline{\tilde{\varphi}'(g)}dg. \tag{2.8}$$

Over the space $V_{\tilde{\pi},v}$, one can define a Hermitian form similar to (2.5), though the definition is more complicated. Let $\{\delta_i\}$ be a set of representatives of $F_v^*/(F_v^*)^2$, with $\delta_1 = 1$. From [B-M1],[B-M2], we see there is a choice of $\psi_v^{D\delta_i}$-Whittaker models $\tilde{L}_v^{D\delta_i}$ (could be trivial) on $V_{\tilde{\pi},v}$, such that

$$(\tilde{\varphi}_v, \tilde{\varphi}'_v) = \sum_{\delta_i} \frac{|2|_v}{2} \int \tilde{L}_v^{D\delta_i}(\tilde{\pi}(\underline{\underline{a}})\tilde{\varphi}_v)\overline{\tilde{L}_v^{D\delta_i}(\tilde{\pi}(\underline{\underline{a}})\tilde{\varphi}'_v)}\frac{da}{|a|_v} \tag{2.9}$$

is a G'_v-invariant Hermitian form. Notice that $\tilde{L}_v^{D\delta_1} = \tilde{L}_v^D$. (We used the factor $\frac{|2|_v}{2}$ to be consistent with [B-M1] and [B-M2]. There we defined the Hermitian form on $\widetilde{GL}_2$ first and restricted the form to $\widetilde{SL}_2$. See section (9.7) of [B-M1]). For some explicit constructions of this form, see § 8.

We can now define the constant $d_{\tilde{\pi}}(S, \psi^D)$. From the uniqueness of Hermitian forms and Whittaker models, there exist constants $\tilde{c}_1(\tilde{\pi}, S, \psi^D, \{\tilde{L}_v^D\})$ and $\tilde{c}_2(\tilde{\pi}, S, \psi^D, \{\tilde{L}_v^D\})$ such that whenever $\tilde{\varphi} = \otimes_{v\in S}\tilde{\varphi}_v \otimes_{v\notin S} \tilde{\varphi}_{0,v}$ (under an identification between $V_{\tilde{\pi}}$ and the restricted tensor product $\otimes V_{\tilde{\pi},v}$):

$$\tilde{W}_{\tilde{\varphi}}^D(e) = \tilde{c}_1(\tilde{\pi}, S, \psi^D, \{\tilde{L}_v^D\}) \prod_{v\in S} \tilde{L}_v^D(\tilde{\varphi}_v),$$

$$||\tilde{\varphi}|| = \tilde{c}_2(\tilde{\pi}, S, \psi^D, \{\tilde{L}_v^D\}) \prod_{v\in S} ||\tilde{\varphi}_v||.$$

As in the case of Lemma 2.2, we have:

Lemma 2.4. *The constant*

$$d_{\tilde{\pi}}(S,\psi^D) = |\tilde{c}_1(\tilde{\pi},S,\psi^D,\{\tilde{L}_v^D\}/\tilde{c}_2(\tilde{\pi},S,\psi^D,\{\tilde{L}_v^D\})|$$

is independent of the choice of the linear forms $\tilde{L}_v^D$*.*

When $\tilde{\pi}$ does not have a nontrivial ψ^D-Whittaker model, we will set $d_{\tilde{\pi}}(S,\psi^D)=0$. The constant $d_{\tilde{\pi}}(S,\psi^D)$ is well defined with our fixed choice of ψ^D and the measure on G', (and again it is independent of the choice of additive measure). Explicitly for any vector $\tilde{\varphi} = \otimes_{v\in S}\tilde{\varphi}_v \otimes_{v\notin S} \tilde{\varphi}_{0,v}$ with $\tilde{L}_v^D(\tilde{\varphi}_v)\neq 0$ for $v\in S$,

$$d_{\tilde{\pi}}(S,\psi^D) = \frac{|\tilde{W}^D_{\tilde{\varphi}}(e)|}{||\tilde{\varphi}||}\prod_{v\in S}\frac{||\tilde{\varphi}_v||}{|\tilde{L}_v^D(\tilde{\varphi}_v)|}. \tag{2.10}$$

3. Results on the theta correspondence

As stated in the introduction, the generalization of the Shimura correspondence between the half integral weight modular forms and integral weight modular forms is the theta correspondence between automorphic representations of $PGL_2(\mathbf{A})$ and $\widetilde{SL}_2(\mathbf{A})$. The theta correspondence is dependent on the choice of the additive character ψ. With a fixed ψ, we denote by $\Theta(\pi,\psi)$ the representation of $\widetilde{SL}_2(\mathbf{A})$ associated to π of $PGL_2(\mathbf{A})$, and $\Theta(\pi_v,\psi_v)$ the representation of $\widetilde{SL}_2(F_v)$ associated to π_v of $PGL_2(F_v)$ under the theta correspondence. Conversely, the theta correspondence associates to $\tilde{\pi}$ of $\widetilde{SL}_2(\mathbf{A})$ and $\tilde{\pi}_v$ of $\widetilde{SL}_2(F_v)$ representations $\Theta(\tilde{\pi},\psi)$ of $PGL_2(\mathbf{A})$ and $\pi_v = \Theta(\tilde{\pi}_v,\psi_v)$ of $PGL_2(F_v)$.

In this paper, of particular importance are the representation $\tilde{\pi}^D = \Theta(\pi\otimes\chi_D,\psi^D)$ and its local counterpart $\tilde{\pi}_v^D = \Theta(\pi_v\otimes\chi_D,\psi_v^D)$. In this section, we recall Waldspurger's beautiful results on these representations. The works in [W1] and [W3] tell us that the set $\{\tilde{\pi}^D\}$ (or $\{\tilde{\pi}_v^D\}$) is finite. Moreover the dependence of $\tilde{\pi}^D$ (or $\tilde{\pi}_v^D$) on D is also given.

We first recall Waldspurger's local theory. Fix a place v of F. Let $P_{0,v}$ be the set of special or supercuspidal representations (or discrete series representations when $F_v = \mathbf{R}$) of $PGL_2(F_v)$. For $D \in F_v^*$, define $(\frac{D}{\pi_v}) \in \pm 1$ by:

$$(\frac{D}{\pi_v}) = \chi_D(-1)\epsilon(\pi_v, 1/2)/\epsilon(\pi_v \otimes \chi_D, 1/2).$$

We then get a partition of $F_v^* = F_v^+(\pi_v) \cup F_v^-(\pi_v)$ where

$$F_v^{\pm}(\pi_v) = \{D \in F_v^* |\, (\frac{D}{\pi_v}) = \pm 1\}.$$

Theorem 3.1. [W3]

When $\pi_v \notin P_{0,v}$, $F_v^+(\pi_v) = F_v^*$ *and* $\tilde{\pi}_v^D = \Theta(\pi_v, \psi_v)$.

When $\pi_v \in P_{0,v}$, *there are two distinct representations* $\tilde{\pi}_v^+$ *and* $\tilde{\pi}_v^-$ *of* G'_v, *such that* $\tilde{\pi}_v^D = \tilde{\pi}_v^+ = \Theta(\pi_v, \psi_v)$ *when* $D \in F_v^+(\pi_v)$, *and* $\tilde{\pi}_v^D = \tilde{\pi}_v^-$ *when* $D \in F_v^-(\pi_v)$.

Moreover $\tilde{\pi}_v^D = \Theta(\pi_v, \psi_v)$ *if and only if* $\Theta(\pi_v, \psi_v)$ *has a nontrivial* ψ_v^D-*Whittaker model.*

We now state the global counterpart of this Theorem. Following the notations in [W3], we let $\tilde{A}_{00}$ be the space of cuspidal automorphic forms on $\widetilde{SL}_2(\mathbf{A})$ that are orthogonal to the theta series generated by quadratic forms of one variable. Let $A_{0,i}$ be the subspace of cuspidal automorphic forms on $PGL_2(\mathbf{A})$, such that for any π subrepresentation of $A_{0,i}$, there is $D \in F^*$, with $L(\pi \otimes \chi_D, 1/2) \neq 0$.

For $\tilde{\pi}_1, \tilde{\pi}_2$ irreducible subrepresentations of $\tilde{A}_{00}$, we will say $\tilde{\pi}_1 \sim \tilde{\pi}_2$ if they are near equivalent, that is at almost all places v, $\tilde{\pi}_{1,v} \cong \tilde{\pi}_{2,v}$. Denote by $\bar{A}_{00}$ the quotient of $\tilde{A}_{00}$ by this relation.

Let $\Sigma = \Sigma(\pi)$ be the set of places v where $\pi_v \in P_{0,v}$. Given $D \in F^*$, let $\epsilon(D, \pi) = (\frac{D_v}{\pi_v})_{v \in \Sigma}$. Then $\epsilon(D, \pi) \in \{\pm 1\}^{|\Sigma|}$. We have

$$\epsilon(\pi \otimes \chi_D, 1/2) = \epsilon(\pi, 1/2) \prod_{v \in \Sigma} (\frac{D_v}{\pi_v}). \tag{3.1}$$

We will use $\epsilon = (\epsilon_v)_{v\in\Sigma}$ to denote an element in $\{\pm 1\}^{|\Sigma|}$, with $\epsilon_v \in \{\pm 1\}$. Given such an ϵ, we will let $F^\epsilon(\pi)$ to be the set of $D \in F^*$ with $\epsilon(D,\pi) = \epsilon$. Then we get a partition $F^* = \cup_{\epsilon\in\{\pm 1\}^{|\Sigma|}} F^\epsilon(\pi)$.

Theorem 3.2. [W3]

1. (Relation with local correspondence) When $\Theta(\tilde{\pi},\psi) \neq 0$, $\Theta(\tilde{\pi},\psi) \cong \otimes_v \Theta(\tilde{\pi}_v,\psi_v)$. When $\Theta(\pi,\psi) \neq 0$, $\Theta(\pi,\psi) \cong \otimes_v \Theta(\pi_v,\psi_v)$.

2. (Nonvanishing of the correspondence) $\Theta(\pi,\psi) \neq 0$ if and only if $L(\pi,1/2) \neq 0$. $\Theta(\tilde{\pi},\psi) \neq 0$ if and only if $\tilde{\pi}$ has a nontrivial ψ-Whittaker model.

3. (Correspondence as a bijection) For $\tilde{\pi}$ an irreducible subrepresentation of $\tilde{A}_{00}$, there is a unique π associated to $\tilde{\pi}$, such that whenever $\Theta(\tilde{\pi},\psi^D) \neq 0$, $\Theta(\tilde{\pi},\psi^D)\otimes\chi_D = \pi$. Denote this association by $\pi = S_\psi(\tilde{\pi})$. This association defines a bijection between $\bar{A}_{00}$ and $A_{0,i}$.

4. (Description of near equivalent class). If $\pi = S_\psi(\tilde{\pi})$, the near equivalence class of $\tilde{\pi}$ consists of all the nonzero $\tilde{\pi}^D$'s.

5. (Dependence of $\tilde{\pi}^D$ on D). Let $\epsilon \in \{\pm 1\}^{|\Sigma|}$. If $\prod_{v\in\Sigma}\epsilon_v \neq \epsilon(\pi,1/2)$, then $\tilde{\pi}^D = 0$ for all $D \in F^\epsilon(\pi)$. If $\prod_{v\in\Sigma}\epsilon_v = \epsilon(\pi,1/2)$, then there is a unique $\tilde{\pi}^\epsilon$ such that for $D \in F^\epsilon(\pi)$, $\tilde{\pi}^D = \tilde{\pi}^\epsilon$ when $L(\pi\otimes\chi_D,1/2) \neq 0$ and $\tilde{\pi}^D = 0$ otherwise.

For convenience, if $\prod_{v\in\Sigma}\epsilon_v \neq \epsilon(\pi,1/2)$, we set $\tilde{\pi}^\epsilon = 0$.

4. Statement of the main results

4.1. **The basic Waldspurger's formula.** The definition of the $L-$function $L(\pi,s)$ and the local $L-$functions $L(\pi_v,s)$ can be found in [J-L]. Fix a finite set of places S, we use $L^S(\pi,s)$ to denote the partial $L-$function $\prod_{v\notin S} L(\pi_v,s)$.

Theorem 4.1. *For an irreducible cuspidal automorphic representations π of $GL_2(\mathbf{A})$ with trivial central character and $L(\pi,1/2) \neq 0$, for $D \in F^*$, let $\tilde{\pi}_D = \Theta(\pi,\psi^D)$. Let S be a finite set of places containing all bad places* *along with all places v where ψ or ψ^D is not unramified, and all places where π_v or $\tilde{\pi}_{D,v}$ is not unramified. Then*

$$|d_\pi(S,\psi)|^2 L^S(\pi,1/2) = |d_{\tilde{\pi}_D}(S,\psi^D)|^2. \tag{4.1}$$

Remark 4.2. We state a more explicit formula using (2.7) and (2.10). Take any vectors $\varphi = \otimes_{v\in S}\varphi_v \otimes_{v\notin S} \varphi_{0,v}$ and $\tilde{\varphi} = \otimes_{v\in S}\tilde{\varphi}_v \otimes_{v\notin S} \tilde{\varphi}_{0,v}$ in V_π and $V_{\tilde{\pi}_D}$ such that $L_v(\varphi_v) \neq 0$ and $\tilde{L}_v(\tilde{\varphi}_v) \neq 0$. Define:

$$e(\varphi_v,\psi_v) = \frac{||\varphi_v||^2}{|L_v(\varphi_v)|^2} \tag{4.2}$$

$$e(\tilde{\varphi}_v,\psi_v^D) = \frac{||\tilde{\varphi}_v||^2}{|\tilde{L}_v^D(\tilde{\varphi}_v)|^2} \tag{4.3}$$

Then as in the proof of the Lemma 2.2, these constants are independent of our choice of L_v and $\tilde{L}_v^D$ and are well defined. From (2.7) and (2.10), we see the identity (4.1) can be stated as follows:

$$\frac{|\tilde{W}_{\tilde{\varphi}}^D(e)|^2}{||\tilde{\varphi}||^2} = \frac{|W_\varphi(e)|^2 L(\pi,1/2)}{||\varphi||^2} \prod_{v\in S} \frac{e(\varphi_v,\psi_v)}{e(\tilde{\varphi}_v,\psi_v^D)L(\pi_v,1/2)} \tag{4.4}$$

4.2. **Formula on twisted $L-$values.** To get a formula on twisted $L-$values, we apply Theorem 4.1 to the case with π replaced by $\pi \otimes \chi_D$. Then $\tilde{\pi}_D = \tilde{\pi}^D = \Theta(\pi\otimes\chi_D,\psi^D)$ and we can apply the results in § 3.

In the rest of the section, we will assume D satisfies that for all *odd* non-archimedean places v, $|D|_v = 1$ or $|D|_v = q_v^{-1}$, and for all even non-archimedean places v, $1 \le |D|_v \le q_v^{-2}$. With a bit abuse of terminology, we call this D a square free integer in F^*.

Recall that from Theorem 3.2, there is a bijection between $\bar{A}_{00}$ and $A_{0,i}$. Let $\pi \in A_{0,i}$ maps to the packet of representations $\{\tilde{\pi}^\epsilon\}$ under this map. Here $\epsilon \in \{\pm 1\}^{|\Sigma|}$, where $\Sigma = \Sigma(\pi)$ as in Theorem 3.2. Then whenever $\tilde{\pi}^\epsilon \neq 0$, $\pi = S_\psi(\tilde{\pi}^\epsilon)$. For $D \in F^\epsilon(\pi)$, $\Theta(\pi\otimes\chi_D,\psi^D) = \tilde{\pi}^\epsilon$ or zero.

Theorem 4.3. *Let π and $\tilde{\pi}^\epsilon$ be as above. Let S be a finite set of place containing all bad places along with the places v where π_v or ψ is not unramified. Then:*

1. For all square free integers $D \in F^*$:

$$\sum_{\epsilon \in \{\pm 1\}^{|\Sigma|}} |d_{\tilde{\pi}^\epsilon}(S, \psi^D)|^2 = |d_\pi(S, \psi)|^2 L^S(\pi \otimes \chi_D, 1/2) \prod_{v \in S} |D|_v^{-1}. \tag{4.5}$$

2. Only one of the terms in the above sum is nonzero. If $D \in F^{\epsilon_0}(\pi)$, then $d_{\tilde{\pi}^\epsilon}(S, \psi^D) = 0$ whenever $\epsilon \neq \epsilon_0$.

3. More explicitly, if $D \in F^{\epsilon_0}(\pi)$ then for $\tilde{\pi} = \tilde{\pi}^{\epsilon_0}$,

$$|d_{\tilde{\pi}}(S, \psi^D)|^2 = |d_\pi(S, \psi)|^2 L^S(\pi \otimes \chi_D, 1/2) \prod_{v \in S} |D|_v^{-1}. \tag{4.6}$$

For $\varphi = \otimes \varphi_v \in V_\pi$ and $\tilde{\varphi} = \otimes \tilde{\varphi}_v \in V_{\tilde{\pi}}$ with $\varphi_v = \varphi_{0,v}$ and $\tilde{\varphi}_v = \tilde{\varphi}_{0,v}$ when $v \notin S$:

$$\frac{|\tilde{W}^D_{\tilde{\varphi}}(e)|^2}{||\tilde{\varphi}||^2} = \frac{|W_\varphi(e)|^2 L(\pi \otimes \chi_D, 1/2)}{||\varphi||^2} \prod_{v \in S} E_v(\varphi_v, \tilde{\varphi}_v, \psi_v, D) \tag{4.7}$$

where

$$E_v(\varphi_v, \tilde{\varphi}_v, \psi_v, D) = \frac{e(\varphi_v, \psi_v)}{e(\tilde{\varphi}_v, \psi_v^D) L(\pi_v \otimes \chi_D, 1/2)|D|_v}.$$

Later we will explicitly compute the local constants $E_v(\varphi_v, \tilde{\varphi}_v, \psi_v, D)$ in many cases, and get from equation (4.7) the generalization of the Kohnen-Zagier formula.

Remark 4.4. 1. The equation (4.5) is in fact a finite set of equations, corresponding to the finite partition of F^* by $F^\epsilon(\pi)$. The conditions on D in (1.1) are precisely the condition $D \in D^{\epsilon_0}(\pi)$ for a given ϵ_0. Thus (1.1) is only one in a set of equations. See § 10 for the whole set of equations.

2. If $\pi \notin A_{0,i}$, then clearly $L(\pi \otimes \chi_D, 1/2) = 0$ for all $D \in F^*$.

4.3. **Ramanujan conjecture and Lindelőf Hypothesis.** The Ramanujan conjecture for the half integral weight cusp form is as follows:

> Let $g(z) = \sum_{n=1}^\infty c(n) e^{2\pi i n z}$ be a cusp form of weight $k+1/2$, $k \in \mathbf{Z}$, such that $g(z)$ is orthogonal to the space generated by the theta series associated to quadratic forms of one variable,

(i.e. $g(z)$ is a vector in $\tilde{A}_{00}$). Then when n is square free, as $n \mapsto \infty$,

$$|c(n)| <<_{g,\alpha} n^{k/2-1/4+\alpha} \tag{4.8}$$

for all $\alpha > 0$. The implied constant depends on $g(z)$ and α only.

To state the generalization of this conjecture to the case of cusp forms over totally real fields, we find it most natural to use the notion $d_{\tilde{\pi}}(S, \psi^D)$ again.

Let S be a finite set of places that contains all bad places along with places v where $\tilde{\pi}_v$ is not unramified. For $S_0 \subset S$, for $\tilde{\varphi}$ a vector in the space of $\tilde{\pi}$, we define

(4.9)
$$d_{\tilde{\pi}}(\tilde{\varphi}, S_0, \psi^D) = \frac{|\tilde{W}^D_{\tilde{\varphi}}(e)|}{||\tilde{\varphi}||} \prod_{v \in S_0} \frac{||\tilde{\varphi}_v||}{|\tilde{L}^D_v(\tilde{\varphi}_v)|} = d_{\tilde{\pi}}(S, \psi^D) \prod_{v \in S - S_0} \frac{|\tilde{L}^D_v(\tilde{\varphi}_v)|}{||\tilde{\varphi}_v||}.$$

As before this constant is well defined and independent of the choice of $\{\tilde{L}^D_v\}$.

Let S_∞ be the collection of archimedean places of F. For $D \in F^*$, define $|D|_{S_\infty} = \prod_{v \in S_\infty} |D|_v$.

Conjecture 4.5. *(Ramanujan conjecture). Let $\tilde{\pi}$ be an irreducible subrepresentation of $\tilde{A}_{00}$. Let $\tilde{\varphi} \in V_{\tilde{\pi}}$. For D a square free integer in F^*, as $|D|_{S_\infty} \mapsto \infty$, for all $\alpha > 0$*

$$|d_{\tilde{\pi}}(\tilde{\varphi}, S_\infty, \psi^D)| <<_{\tilde{\pi},\tilde{\varphi},\alpha} |D|_{S_\infty}^{\alpha-1/2} \tag{4.10}$$

where the implied constant depends only on $\tilde{\pi}, \tilde{\varphi}$ and α.

In the introduction a more explicit inequality is given; it is clear from the definition of $d_{\tilde{\pi}}(\tilde{\varphi}, S_\infty, \psi^D)$ that the two inequalities are equivalent. We will show in § 9 that the above conjecture implies inequality (4.8).

We have stated the (special case of) Lindelőf hypothesis in the introduction. From Theorem 4.3 we can get the following Theorem.

Theorem 4.6. *The inequality (4.10) holds for some $\alpha > 0$ (and for all $\tilde{\pi}$ and $\tilde{\varphi}$ as in Conjecture 4.5) if and only if the inequality (1.7) holds for $\beta = 2\alpha > 0$ (for all π as in Conjecture 1.4). In particular, the Conjecture 4.5 is equivalent to the Conjecture 1.4.*

This equivalence is useful in the work of Cogdell-Piatetski-Shapiro-Sarnak on ternary quadratic forms ([C-PS-S]).

4.4. **Other implications.** The next corollary can be considered as the adelic version of corollary 2 in [W2]. It follows immediately from Theorem 4.3 and gives an identity for the quotient between central values two twisted $L-$functions.

Corollary 4.7. *Let $\pi \in A_{0,i}$ and S, $\Sigma = \Sigma(\pi)$ be as in Theorem 4.3. Fix $\epsilon \in \{\pm 1\}^{|\Sigma|}$. For two square free integers $D_1, D_2 \in F^{\epsilon}(\pi)$,*

(4.11)
$$|d_{\tilde{\pi}^{\epsilon}}(S, \psi^{D_1})|^2 L(\pi \otimes \chi_{D_2}, 1/2) = |d_{\tilde{\pi}^{\epsilon}}(S, \psi^{D_2})|^2 L(\pi \otimes \chi_{D_1}, 1/2) \prod_{v \in S} |\frac{D_2}{D_1}|_v.$$

We also state a result concerning the relation between the $D-$th and $D\Delta^2$-th Fourier coefficients of a half-integral weight form. Note the similarity with Lemma 2.3.

Corollary 4.8. *Let S be as in Theorem 4.3. If $D' = D\Delta^2$ for $\Delta \in F^{\times}$. Let $S_{D,D'}$ be a finite set of places with $|D|_v = |D'|_v = 1$ for all $v \notin S_{D,D'}$. Then*

$$|d_{\tilde{\pi}}(S_{D,D'} \cup S, \psi^D)| = |d_{\tilde{\pi}}(S_{D,D'} \cup S, \psi^{D'})|. \tag{4.12}$$

5. A relative trace formula

In [J1], Jacquet proved some of Waldspurger's results on theta correspondence using a relative trace formula. Our result is based on a local analysis of his trace formula and its variation. We recall some of the results on the trace formula and in the process fix some notations. The main result here is Theorem 5.5.

5.1. **Definition of the global distributions** $I(f,\psi)$ **and** $J(f',\psi^D)$**.** Let $f(g) \in C_c^\infty(Z(\mathbf{A})\backslash G(\mathbf{A}))$ the space of smooth compactly supported functions. Define a kernel function

$$K_f(x,y) = \sum_{\xi \in PGL_2(F)} f(x^{-1}\xi y).$$

Define a distribution $I(f,\psi)$ to be:

$$\int_{\mathbf{A}^*/F^*} \int_{\mathbf{A}/F} K_f(\underline{a}, n(u))\psi(u) du d^*a. \tag{5.1}$$

Let $f'(g) \in C_c^\infty(G'(\mathbf{A}))$, (we use this notation to denote the space of genuine smooth compactly supported functions). Define a kernel function

$$K_{f'}(x,y) = \sum_{\xi \in SL_2(F)} f'(x^{-1} \cdot \xi \cdot y).$$

Here we note that $SL_2(F)$ embeds into $G'(\mathbf{A})$. Define a distribution $J(f',\psi^D)$ to be:

$$\int_{\mathbf{A}/F} \int_{\mathbf{A}/F} K_f(\tilde{n}(x), \tilde{n}(y))\psi^D(-x+y) dx dy. \tag{5.2}$$

The relative trace formula is an identity between the distributions $I(f,\psi)$ and $J(f',\psi^D)$. We will state the result on the distributions $I(f,\psi)$, $J(f',\psi^D)$. The computations are available in [J1] and will not be included.

5.2. **Comparison of orbital integrals.**

Proposition 5.1. [J1] *If* $f = \otimes f_v$ *and* $f' = \otimes f'_v$*, then*

$$I(f,\psi) = \prod I_\psi^+(f_v) + \prod I_\psi^-(f_v) + \sum_{a \in F^*} \prod \mathcal{O}_{f_v}^{\psi_v}(n(a)w).$$

$$J(f',\psi^D) = \prod J_{\psi^D}^+(f'_v) + \prod J_{\psi^D}^-(f'_v) + \sum_{a \in F^*} \prod \mathcal{O}_{f'_v}^{\psi_v^D}(\tilde{w} \cdot \underline{\underline{a}}).$$

In the above equations, $I_\psi^\pm(f_v)$ and $J_{\psi^D}^\pm(f'_v)$ are the so called *singular orbital integrals* of f_v and f'_v, whose precise forms are not important for us, while

$$\mathcal{O}_{f_v}^{\psi_v}(g) = \int f_v(\underline{a} g n(x))\psi_v(x) dx d^*a.$$

$$\mathcal{O}^{\psi_v^D}_{f'_v}(g) = \int f'_v(\tilde{n}(x)\cdot g\cdot \tilde{n}(y))\psi_v^D(x+y)dxdy.$$

Proposition 5.2. [J1] *For each f in $C_c^\infty(G(F_v)/Z(F_v))$, there is $f' \in C_c^\infty(G'(F_v))$ such that for $a \in F_v^*$*

$$\mathcal{O}^{\psi_v}_f(n(\frac{a}{4D})w) = \mathcal{O}^{\psi_v^D}_{f'}(\tilde{w}\cdot \underline{\underline{a}})\psi_v(-\frac{2D}{a})|a|_v^{1/2}\gamma(a^{-1},\psi_v^D)^{-1} \tag{5.3}$$

and $I^\pm_\psi(f) = J^\pm_{\psi^D}(f')$. Conversely, given f', we can find a f satisfying the equations.

We say the two functions f and f' match if the relations in the proposition are satisfied.

Now let v be a non-Archimedean place with odd residue characteristic, and where ψ_v, ψ_v^D have order 0. Recall that the Hecke algebra $\mathcal{H}(G(F_v)/Z(F_v))$ is the algebra of compactly supported functions on $G(F_v)/Z(F_v)$ bi-invariant under the maximal compact group $G(\mathcal{O}_v)$. The Hecke algebra $\mathcal{H}(G'(F_v))$ is similarly defined, except that the functions are genuine, and bi-invariant under $SL_2(\mathcal{O}_v)$ embedded in $G'(F_v)$.

Proposition 5.3. [J1] *There is an algebra isomorphism $\eta_v : \mathcal{H}(G(F_v)/Z(F_v)) \to \mathcal{H}(G'(F_v))$, such that f and $\eta_v(f)$ match.*

From Propositions 5.1, 5.2 and 5.3, we get

Theorem 5.4. *Fix any finite set of places S that contains all bad places and the places where ψ and D are not of order 0. For each place $v \in S$, there is a map $\rho_v : C_c^\infty(G(F_v)/Z(F_v)) \to C_c^\infty(G'(F_v))$, such that*

$$I(\otimes_{v\in S} f_v \otimes_{v\notin S} f_v, \psi) = J(\otimes_{v\in S}\rho_v(f_v)\otimes_{v\notin S}\eta_v(f_v), \psi^D),$$

whenever $f_v \in \mathcal{H}(G(F_v)/Z(F_v))$ for all $v \in S$.

5.3. **Relation with Shimura-Waldspurger correspondence.** For π an irreducible cuspidal automorphic representation of $G(\mathbf{A})$ with trivial central character, define

$$I_\pi(f,\psi) = \sum_{\varphi_i} Z(\pi(f)\varphi_i)\overline{W_{\varphi_i}(e)} \tag{5.4}$$

with φ_i an orthonormal basis of V_π; here for $\varphi \in V_\pi$

$$\pi(f)\varphi = \int_{Z(\mathbf{A})\backslash G(\mathbf{A})} f(g)\pi(g)\varphi dg,$$

$$Z(\varphi) = \int_{\mathbf{A}^*/F^*} \varphi(\underline{a})d^*a.$$

For $\tilde{\pi}$ an irreducible cuspidal representation of $G'(\mathbf{A})$, define

$$J_{\tilde{\pi}}(f',\psi^D) = \sum_{\varphi_i'} \tilde{W}^D_{\tilde{\pi}(f')\tilde{\varphi}_i}(e)\overline{\tilde{W}^D_{\tilde{\varphi}_i}(e)} \tag{5.5}$$

with φ_i' an orthonormal basis of $V_{\tilde{\pi}}$; here for $\tilde{\varphi} \in V_{\tilde{\pi}}$

$$\tilde{\pi}(f')\tilde{\varphi} = \int_{G'(\mathbf{A})} f'(g)\tilde{\pi}(g)\tilde{\varphi}dg.$$

The distributions $I_\pi(f,\psi)$ and $J_{\tilde{\pi}}(f',\psi^D)$ are the contributions from π and $\tilde{\pi}$ to $I(f,\psi)$ and $J(f',\psi^D)$ respectively.

Recall that if π_v is unramified, there exists a vector $\varphi_{0,v}$ that is fixed under the action of $G(\mathcal{O}_v)$. For f_v a Hecke function on $G(F_v)/Z(F_v)$, there is a constant $\hat{f}_v(\pi_v)$ with

$$\pi_v(f_v)\varphi_{0,v} = \hat{f}_v(\pi_v)\varphi_{0,v}. \tag{5.6}$$

Similarly, if $\tilde{\pi}_v$ is unramified, let $\tilde{\varphi}_{0,v}$ be a vector that is fixed under $SL_2(\mathcal{O}_v)$, then for f_v' in the Hecke algebra of $G'(F_v)$, there is a constant $\hat{f'}_v(\tilde{\pi}_v)$ with

$$\tilde{\pi}_v(f_v')\tilde{\varphi}_{0,v} = \hat{f'}_v(\tilde{\pi}_v)\tilde{\varphi}_{0,v}. \tag{5.7}$$

It is standard to derive from the Theorem 5.4 the following result:

Theorem 5.5. [J1] *For any cuspidal representation π of G with trivial central character such that $I_\pi(f,\psi)$ is nontrivial, there is a unique cuspidal representation $\tilde{\pi}$ of G', such that if f and f' match*

$$I_\pi(f,\psi) = J_{\tilde{\pi}}(f',\psi^D). \tag{5.8}$$

Moreover, if S satisfies the condition in Theorem 5.4 and contains all places where π_v or $\tilde{\pi}_v$ is not unramified, for $v \notin S$, if f_v is a Hecke function and $f'_v = \eta_v(f_v)$, then

$$\hat{f}_v(\pi_v) = \hat{f'}_v(\tilde{\pi}_v). \tag{5.9}$$

Remark 5.6. From the definition of $Z(\varphi)$ and the integral representation of $L-$function $L(\pi, s)$, it is clear that $I_\pi(f, \psi)$ is nontrivial if and only if $L(\pi, 1/2) \neq 0$.

Proposition 5.7. *In the above theorem, $\tilde{\pi} = \Theta(\pi, \psi^D)$.*

Proof. From the description of the map η_v of Hecke algebras in [J1] and the equation (5.9), we get that $\tilde{\pi}$ is in the same near equivalence class as $\Theta(\pi, \psi^D)$. It follows from Theorems 3.2 and 3.1 that $\Theta(\pi, \psi^D)$ is the only representation within the near equivalence class with nontrivial ψ^D-Whittaker model. As $J_{\tilde{\pi}}(f', \psi^D) \neq 0$, $\tilde{\pi}$ has ψ^D-Whittaker model from (5.5). Thus $\tilde{\pi} \cong \Theta(\pi, \psi^D)$. From the multiplicity one theorem for $\widetilde{SL}_2$, we get $\tilde{\pi} = \Theta(\pi, \psi^D)$. □

6. The local distributions

Let π and $\tilde{\pi}$ be the cuspidal representations that correspond to each other by Theorem 5.5. Then $\tilde{\pi} = \Theta(\pi, \psi^D)$. Let S be as in Theorem 5.5. Assume $f = \otimes f_v, f' = \otimes f'_v$, where f and f' match, and f_v, f'_v are matching Hecke functions when $v \notin S$. We write $I_\pi(f, \psi)$ and $J_{\tilde{\pi}}(f', \psi^D)$ as products of local distributions over the places in S. We then state the identity between the local distributions, which coupled with Theorem 5.5 gives Theorem 4.1.

6.1. **The distribution $I_{\pi,v}(f_v, \psi_v)$.** We fix a choice of local Whittaker functionals L_v on π_v, and define the Hermitian form on V_{π_v} using this choice of L_v. For $v \in S$, we fix for π_v as above an orthonormal basis of $V_{\pi,v}$, denote it by $\{\varphi_{i,v}\}$. For $v \notin S$, let $\varphi_{0,v}$ be the vector given in § 2. For $\pi = \otimes\pi_v$, from (2.6), the set

$$\{\varphi_I\} = \{c_2(\pi, S, \psi, \{L_v\})^{-1} \otimes_{v\in S} \varphi_{i,v} \otimes_{v\notin S} \varphi_{0,v}\} \tag{6.1}$$

can be extended to an orthogonal basis of V_π. Let $V(\pi, S)$ be the space of vectors generated by the set of $\{\varphi_I\}$. With our choice of f, if is clear that if $\varphi \in V_\pi$ is perpendicular to the space $V(\pi, S)$, then $\pi(f)\varphi = 0$. Thus the expression (5.4) for $I_\pi(f, \psi)$ takes the form:

$$\sum_{\varphi_I} Z(\pi(f)\varphi_I)\overline{W_{\varphi_I}(e)}. \tag{6.2}$$

Using the Hecke theory for GL_2, we show:

Lemma 6.1. *When* $\varphi = \otimes_{v\in S}\varphi_v \otimes_{v\notin S} \varphi_{0,v}$,

$$Z(\varphi) = c_1(\pi, S, \psi, \{L_v\})L(\pi, 1/2) \prod_{v\in S} \lambda_v(\varphi_v) \tag{6.3}$$

where

$$\lambda_v(\varphi_v) = \frac{\int_{F_v^*} L_v(\pi_v(\underline{a})\varphi_v)|a|_v^{s-1/2} d^*a}{L(\pi, s)}|_{s=1/2}.$$

Proof. Since $\varphi(\underline{a}) = \sum_{\delta\in F^*} W_\varphi(\underline{\delta a})$, we get:

$$Z(\varphi) = \int_{\mathbf{A}^*} W_\varphi(\underline{a})|a|_v^{s-1/2} d^*a|_{s=1/2},$$

which by (2.3) equals

$$L(\pi, 1/2)c_1(\pi, S, \psi, \{L_v\}) \prod_v \frac{\int_{F_v^*} L_v(\pi_v(\underline{a})\varphi_v)|a|_v^{s-1/2} d^*a}{L(\pi, s)}|_{s=1/2}.$$

When $v \notin S$, it is well known that the above local factor equals 1, ([Go]). Thus the Lemma. □

Proposition 6.2. *Let S be as in Theorem 5.5. When $f = \otimes f_v$ where f_v is a Hecke function if $v \notin S$:*

$$I_\pi(f, \psi) = L(\pi, 1/2)|d_\pi(S, \psi)|^2 \prod_{v\in S} I_{\pi,v}(f_v, \psi_v) \prod_{v\notin S} \hat{f}_v(\pi_v) \tag{6.4}$$

where

$$I_{\pi,v}(f_v,\psi_v)=\sum_{\varphi_{i,v}}\lambda_v(\pi_v(f_v)\varphi_{i,v})\overline{L_v(\varphi_v)}. \tag{6.5}$$

where the sum is taken over the orthonormal basis of $V_{\pi,v}$.

Proof. Let $\varphi=c_2(\pi,S,\psi,\{L_v\})^{-1}\otimes_{v\in S}\varphi_v\otimes_{v\notin S}\varphi_{0,v}$ be an element in the orthonormal set (6.1). From (2.3), we get

$$W_\varphi(e)=\frac{c_1(\pi,S,\psi,\{L_v\})}{c_2(\pi,S,\psi,\{L_v\})}\prod_{v\in S}L_v(\varphi_v).$$

From (5.6),

$$\pi(f)\varphi=c_2(\pi,S,\psi,\{L_v\})^{-1}\prod_{v\notin S}\hat{f}_v(\pi_v)\otimes_{v\in S}\pi_v(f_v)\varphi_v\otimes_{v\notin S}\varphi_{0,v}.$$

From the above Lemma:

$$Z(\pi(f)\varphi)\overline{W_\varphi(e)}=|\frac{c_1(\pi,S,\psi,\{L_v\})}{c_2(\pi,S,\psi,\{L_v\})}|^2L(\pi,1/2)\prod_{v\notin S}\hat{f}_v(\pi_v)\prod_{v\in S}\lambda_v(\pi_v(f_v)\varphi_v)\overline{L_v(\varphi_v)}.$$

The proposition follows from (6.2) and the definition of $d_\pi(S,\psi)$. □

Remark 6.3. The expression $I_{\pi,v}(f_v,\psi_v)$ is well defined and independent of the linear form L_v we choose, as a change in L_v will result in a change in the Hermitian form, thus the orthonormal basis of V_{π_v}, leaving $I_{\pi,v}(f_v,\psi_v)$ unchanged.

6.2. **The distribution** $J_{\tilde{\pi},v}(f'_v,\psi_v^D)$**.** We can apply the above argument also to $J_{\tilde{\pi}}(f',\psi_v^D)$. Similarly we have:

Proposition 6.4. *Let S be as in Theorem 5.5. When $f'=\otimes f'_v$ where f'_v is a Hecke function if $v\notin S$:*

$$J_{\tilde{\pi}}(f',\psi^D)=|d_{\tilde{\pi}}(S,\psi^D)|^2\prod_{v\in S}J_{\tilde{\pi},v}(f'_v,\psi_v^D)\prod_{v\notin S}\hat{f'}_v(\tilde{\pi}_v) \tag{6.6}$$

where

$$J_{\tilde{\pi},v}(f'_v,\psi_v^D)=\sum_{\tilde{\varphi}_{j,v}}\tilde{L}_v^D(\tilde{\pi}_v(f'_v)\tilde{\varphi}_{j,v})\overline{\tilde{L}_v^D(\tilde{\varphi}_{j,v})} \tag{6.7}$$

where the sum is taken over the orthonormal basis $\{\tilde{\varphi}_{j,v}\}$ of $V_{\tilde{\pi},v}$.

Remark 6.5. Again one can show that the expression $J_{\tilde{\pi},v}(f'_v, \psi_v^D)$ is well defined and independent of the linear form $\tilde{L}_v^D$ we choose.

6.3. **Statement of the local identity.**

Theorem 6.6. *Fix a place v, when f_v, f'_v match, when π_v is a local component of an irreducible cuspidal automorphic representation π of $PGL_2(\mathbf{A})$ with $L(\pi, 1/2) \neq 0$, let $\tilde{\pi}_v = \Theta(\pi_v, \psi_v^D)$, then*

$$J_{\tilde{\pi},v}(f'_v, \psi_v^D) = |2D|_v \epsilon(\pi_v, 1/2) L(\pi_v, 1/2) I_{\pi,v}(f_v, \psi_v) \tag{6.8}$$

The proof of this Theorem is quite technical. It is done in [B-M1],[B-M2]. In [B-M1] we established the identity when v is nonarchimedean. In [B-M2], the identity is proved when $v = \mathbf{R}$. In this case, the identity follows from the identities between classical Bessel functions. We established the Theorem in a bit more generality, as we do not assume π_v is a local component of an automorphic representation. The proof of the Theorem stated as above is easier, as from Theorem 5.5, we have $J_{\tilde{\pi},v}(f'_v, \psi_v^D) = c(\pi_v, \tilde{\pi}_v) I_{\pi,v}(f_v, \psi_v)$ for some constant $c(\pi_v, \tilde{\pi}_v)$ independent of f_v and f'_v. One then only needs to determine this constant.

7. Proof of the main results

We now prove the results stated in § 4.

Proof of Theorem 4.1: Let π and $\tilde{\pi} = \tilde{\pi}_D$ be as in the Theorem. From Theorem 5.5 and Proposition 5.7, we see $I_\pi(f, \psi) = J_{\tilde{\pi}}(f', \psi^D)$ when f and f' match. Assume $f = \otimes f_v$ with f_v a Hecke function if $v \notin S$, and $f' = \otimes f'_v$ with $f'_v = \eta_v(f_v)$ when $v \notin S$ and f'_v matches f_v elsewhere. From Theorem 5.5 and Propositions 6.2 and 6.4, we get:

$$L(\pi, 1/2)|d_\pi(S, \psi)|^2 \prod_{v \in S} I_{\pi,v}(f_v, \psi_v) = |d_{\tilde{\pi}}(S, \psi^D)|^2 \prod_{v \in S} J_{\tilde{\pi},v}(f'_v, \psi_v^D). \tag{7.1}$$

From Theorem 6.6, we get:

$$\prod_{v \in S} J_{\tilde{\pi},v}(f'_v, \psi_v^D) = \prod_{v \in S} |2D|_v \epsilon(\pi_v, 1/2) L(\pi_v, 1/2) I_{\pi,v}(f_v, \psi_v). \tag{7.2}$$

Combine the above two equations, we get:

$$L(\pi,1/2)|d_\pi(S,\psi)|^2 = |d_{\tilde\pi}(S,\psi^D)|^2 \prod_{v\in S} |2D|_v \epsilon(\pi_v,1/2) L(\pi_v,1/2).$$

As $|2D|_v = 1$ for $v \notin S$, we get $\prod_{v\in S} |2D|_v = 1$. As $\epsilon(\pi,1/2) = 1$ and for $v \notin S$, $\epsilon(\pi_v,1/2) = 1$, we get $\prod_{v\in S} \epsilon(\pi_v,1/2) = 1$. Thus we get the identity (4.1). $\square$

Proof of Theorem 4.3: We first prove part 2 of the Theorem. Assume $D \in F^{\epsilon_0}(\pi)$ and $\epsilon_0 \neq \epsilon$. Then for some $v \in \Sigma$, $(\frac{D_v}{\pi_v}) = \epsilon_v \neq \epsilon_v$; thus $\Theta(\pi_v \otimes \chi_D, \psi_v^D) \neq \tilde\pi_v^\epsilon$, and by Theorem 3.1, $\tilde\pi_v^\epsilon$ does not have a nontrivial ψ_v^D-Whittaker functional. Therefore $\tilde\pi^\epsilon$ does not have a nontrivial ψ^D-Whittaker model, and $d_{\tilde\pi^\epsilon}(S,\psi^D) = 0$.

We now prove part 3; part 1 is the immediate consequence of parts 2 and 3. Assume $D \in F^{\epsilon_0}(\pi)$. Let S_D be a finite set of places, such that $|D|_v = 1$ when $v \notin S_D$. Let $S_1 = S \cup S_D$, $S_2 = S_1 - S$. Let $\tilde\pi^D = \Theta(\pi \otimes \chi_D, \psi^D)$. Then by Theorem 3.2, we have $\tilde\pi^D = \tilde\pi^{\epsilon_0} = \tilde\pi$ or 0.

When $\tilde\pi^D = 0$, $L(\pi \otimes \chi_D, 1/2) = 0$ from Theorem 3.2. Meanwhile from Propsition 30 of [W1], $\tilde\pi$ does not have a ψ^D-Whittaker model and $d_{\tilde\pi}(S,\psi^D) = 0$ by definition. The equation (4.6) holds in this case.

Next we consider the case $\tilde\pi^D = \tilde\pi$. As $\pi_v \otimes \chi_D$ and $\tilde\pi^D$ are clearly unramified for $v \notin S_1$, we can apply Theorem 4.1 to get:

$$|d_{\pi\otimes\chi_D}(S_1,\psi)|^2 L^{S_1}(\pi\otimes\chi_D,1/2) = |d_{\tilde\pi}(S_1,\psi^D)|^2. \tag{7.3}$$

We first observe:

Lemma 7.1. *When $S_1 = S \cup S_D$,*

$$d_{\pi\otimes\chi_D}(S_1,\psi) = d_\pi(S_1,\psi). \tag{7.4}$$

Proof. Globally the space $V_{\pi\otimes\chi_D}$ consists of the automorphic forms $\varphi\chi_D(g) = \varphi(g)\chi_D(\det(g))$ where $\varphi \in V_\pi$. From (2.1) and (2.4), we see $W_\varphi(e) = W_{\varphi\chi_D}(e)$ and $||\varphi|| = ||\varphi\chi_D||$.

Locally $\pi_v \otimes \chi_D$ acts on the same space V_{π_v} of π_v with the action being $\pi_v \otimes \chi_D(g)\varphi_v = \chi_D(\det(g))\pi_v(g)\varphi_v$. It is easy to check a Whittaker functional L_v on V_{π_v} also defines a Whittaker functional L'_v on $V_{\pi_v \otimes \chi_D}$ under this identification. Thus locally $L_v(\varphi_v) = L'_v(\varphi_v)$; and from (2.5) we have $||\varphi_v||_L = ||\varphi_v||_{L'}$ where the two norms are defined for the representation π_v and $\pi_v \otimes \chi_D$ respectively. Also clear is that for $v \notin S_1$, the unramified vector $\varphi_{0,v}$ in V_{π_v} is also unramified as a vector in $V_{\pi_v \otimes \chi_D}$.

With the above local and global identities, (7.4) follows from the explicit formula (2.7). □

We can now derive the equation (4.6) from (7.3). Take a vector $\tilde{\varphi}$ in the space of $\tilde{\pi}$ so that $\tilde{\varphi} = \otimes \tilde{\varphi}_v$ with $\tilde{\varphi}_v = \tilde{\varphi}_{0,v}$ when $v \notin S$. Using the equation (2.10), we get:

$$|d_{\tilde{\pi}}(S_1, \psi^D)|^2 = |d_{\tilde{\pi}}(S, \psi^D)|^2 \prod_{v \in S_2} e(\tilde{\varphi}_{0,v}, \psi_v^D). \tag{7.5}$$

Similarly as π_v is unramified for $v \notin S$, we have

$$|d_\pi(S_1, \psi)|^2 = |d_\pi(S, \psi)|^2 \prod_{v \in S_2} e(\varphi_{0,v}, \psi_v). \tag{7.6}$$

From equations (7.3), (7.4), (7.5) and (7.6), we get:

$$|d_\pi(S, \psi)|^2 L^{S_1}(\pi \otimes \chi_D, 1/2) = |d_{\tilde{\pi}}(S, \psi^D)|^2 \prod_{v \in S_2} \frac{e(\tilde{\varphi}_{0,v}, \psi_v^D)}{e(\varphi_{0,v}, \psi_v)}. \tag{7.7}$$

For $v \in S_2$, π_v is unramified and unitary, $\tilde{\pi}_v = \Theta(\pi_v, \psi_v)$, the quotient $\frac{e(\tilde{\varphi}_{0,v}, \psi_v^D)}{e(\varphi_{0,v}, \psi_v)}$ is given in Propositions 8.1 and 8.2; it equals $(|D|_v L(\pi_v \otimes \chi_D, 1/2))^{-1}$. (This is the only place we use the fact that D is a square free integer). As $S_1 = S \cup S_2$, we get from (7.7):

$$|d_\pi(S, \psi)|^2 L^S(\pi \otimes \chi_D, 1/2) = |d_{\tilde{\pi}}(S, \psi^D)|^2 \prod_{v \in S_2} |D|_v^{-1}.$$

Since D is in F^* and $|D|_v = 1$ when $D \notin S_1$, $\prod_{v \in S_1} |D|_v = 1$; thus $\prod_{v \in S_2} |D|_v^{-1} = \prod_{v \in S} |D|_v$. We get (4.6) and the Theorem. □

Proof of Theorem 4.6: First note that for D a square free integer, for any fixed finite set of places S, the value of $\prod_{v \in S - S_\infty} |D|_v$ lies in a finite set of positive numbers. We will use $a \sim b$ to denote the equivalence relation that a/b lies in a finite set of positive numbers. Thus $\prod_{v\in S-S_\infty} |D|_v \sim 1$.

Assume (4.10) holds for some $\alpha > 0$. Given any π irreducible cuspidal representation of $PGL_2(\mathbf{A})$, we prove (1.7). Let S be a large enough set, so that equation (4.5) holds.

Recall from the definition, when $\tilde{\varphi} \in V_{\tilde{\pi}}$ such that $\tilde{W}^D_{\tilde{\varphi}}(e) \neq 0$,

$$|d_{\tilde{\pi}}(S, \psi^D)| = |d_{\tilde{\pi}}(\tilde{\varphi}, S_\infty, \psi^D)| \prod_{v\in S-S_\infty} \frac{||\tilde{\varphi}_v||}{|\tilde{L}^D_v(\tilde{\varphi}_v)|}. \tag{7.8}$$

For a given v, the value of $\frac{||\tilde{\varphi}_v||}{|\tilde{L}^D_v(\tilde{\varphi}_v)|}$ does depend on D (as it depends on ψ^D_v). We put the dependence on D in the notation and denote the value as $\frac{||\tilde{\varphi}_v||_D}{|\tilde{L}^D_v(\tilde{\varphi}_v)|}$.

Lemma 7.2. *Let $\tilde{\varphi}$ be as above. For a fixed $v \in S - S_\infty$ and fixed $\tilde{\varphi}_v$, there are only finitely many possible values of $\frac{||\tilde{\varphi}_v||_D}{|\tilde{L}^D_v(\tilde{\varphi}_v)|}$ when D changes over the set of all square free integers. Thus*

$$|d_{\tilde{\pi}}(S, \psi^D)| \sim_{\tilde{\varphi}} |d_{\tilde{\pi}}(\tilde{\varphi}, S_\infty, \psi^D)|.$$

Here $\sim_{\tilde{\varphi}}$ indicates that the finite set of positive number in the definition of $\sim$ depends only on $\tilde{\varphi}$.

Proof. The subset of D_v with $|D|_v = 1$ or $|D|_v = q^{-1}$ consists of finitely many cosets of $(\mathcal{O}_v{}^*)^2$. Write $D = D_0\alpha^2$ with $\alpha \in \mathcal{O}_v{}^*$, then we can let $\tilde{L}^{D\delta_i}_v(\tilde{\varphi}_v) = \tilde{L}^{D_0\delta_i}_v(\tilde{\pi}_v(\underline{\underline{\alpha}})\tilde{\varphi}_v)$ where δ_i are representatives of square classes of F^*_v. From (2.9), we get $||\tilde{\varphi}_v||_D = ||\tilde{\varphi}_v||_{D_0}$. Meanwhile $\tilde{L}^D_v(\tilde{\varphi}_v) = \tilde{L}^{D_0}_v(\tilde{\pi}_v(\underline{\underline{\alpha}})\tilde{\varphi}_v)$. As $\tilde{\pi}_v$ is admissible, the set of $\{\tilde{\pi}_v(\underline{\underline{\alpha}})\tilde{\varphi}_v | \alpha \in \mathcal{O}_v{}^*\}$ is finite. There are only finitely many possible values of $\tilde{L}^D_v(\tilde{\varphi}_v)$ when $D = D_0\alpha^2$, thus only finitely many possible values of the quotient $\frac{||\tilde{\varphi}_v||_D}{|\tilde{L}^D_v(\tilde{\varphi}_v)|}$. $\square$

From the Lemma and our assumption, we get for all $\tilde{\pi}$

$$|d_{\tilde{\pi}}(S, \psi^D)| << _{\tilde{\pi},\alpha} |D|^{\alpha-1/2}_{S_\infty}.$$

From equation (4.5) we get (as π determines $\tilde{\pi}^\epsilon$)

$$L^{S_\infty}(\pi \otimes \chi_D, 1/2) \sim \sum_{\epsilon \in \{\pm 1\}^{|\Sigma|}} |d_{\tilde{\pi}^\epsilon}(S, \psi^D)|^2 |D|_{S_\infty} << _{\pi,\alpha} |D|_{S_\infty}^{2\alpha}.$$

We get immediately the inequality (1.7) with $\beta = 2\alpha$.

Conversely, assume the inequality (1.7) holds for some $\beta = 2\alpha > 0$, take any $\tilde{\pi} \in \tilde{A}_{00}$ and $\tilde{\varphi} \in V_{\tilde{\pi}}$, we prove (4.10). We may as well assume $\tilde{\varphi} = \otimes_{v \in S} \tilde{\varphi}_v \otimes_{v \notin S} \tilde{\varphi}_{0,v}$, where S is a large enough finite set of places. Let $\pi = S_\psi(\tilde{\pi})$. From (4.5) and our assumption, we get

$$|d_{\tilde{\pi}}(S, \psi^D)| << _{\pi,\alpha} |D|_{S_\infty}^{\alpha - 1/2}.$$

From Lemma 7.2, we get

$$|d_{\tilde{\pi}}(\tilde{\varphi}, S_\infty, \psi^D)| << _{\tilde{\varphi},\pi,\alpha} |D|_{S_\infty}^{\alpha - 1/2}.$$

As π is determined by $\tilde{\pi}$, the implied constant only depends on α, $\tilde{\pi}$ and $\tilde{\varphi}$. $\square$

Proof of Corollary 4.8: Let $\pi = S_\psi(\tilde{\pi})$ as in Theorem 3.2. We note that $\epsilon(D, \pi) = \epsilon(D', \pi)$. Assume $\tilde{\pi} = \tilde{\pi}^{\epsilon_0}$ for some ϵ_0. When $\epsilon(D, \pi) \neq \epsilon_0$, from Theorem 4.3, $d_{\tilde{\pi}}(S \cup S_{D,D'}, \psi^D) = d_{\tilde{\pi}}(S \cup S_{D,D'}, \psi^{D'}) = 0$. When $\epsilon(D, \pi) = \epsilon_0$, we follow the proof of Theorem 4.3, replacing S_1 by $S \cup S_{D,D'}$ in the argument. We get (see (7.3)):

$$|d_{\pi \otimes \chi_D}(S \cup S_{D,D'}, \psi)|^2 L^{S \cup S_{D,D'}}(\pi \otimes \chi_D, 1/2) = |d_{\tilde{\pi}}(S \cup S_{D,D'}, \psi^D)|^2,$$

$$|d_{\pi \otimes \chi_{D'}}(S \cup S_{D,D'}, \psi)|^2 L^{S \cup S_{D,D'}}(\pi \otimes \chi_{D'}, 1/2) = |d_{\tilde{\pi}}(S \cup S_{D,D'}, \psi^{D'})|^2.$$

The equation (4.12) follows from the fact that $\chi_{D'} = \chi_D$ and Lemma 7.1. (The equation can also be established directly as in the proof of Lemma 2.3). $\square$

8. Local factors: some examples

In this section, we compute the local factors $e(\varphi_v, \psi)$ and $e(\tilde{\varphi}_v, \psi^D)$ in equation (4.4) for some specific choices of the vectors φ_v and $\tilde{\varphi}_v$. The computation here is standard and fairly easy. The result has already been used in the proof of Theorem 4.3. It is also used when we translate our formula into more explicit results about cusp forms, (see the proof of Theorem 10.1).

In subsections 8.1–8.3, we assume v is a nonarchimedean place, with odd residue characteristic. For simplicity, we will assume ψ_v has order 0, (and denote it simply by ψ), and D is either a unit or generates the prime ideal in $\mathcal{O}_v$ at nonarchimedean places v.

The cases we consider are the following:

1. When π_v is an unramified unitary representation of $G(F_v)$ where v is an odd non-archimedean place. Then $\tilde{\pi}_v^D = \Theta(\pi_v, \psi)$ for all $D \in F_v^*$ and is an unramified unitary representation of $G'(F_v)$. We take φ_v and $\tilde{\varphi}_v$ to be the unramified vectors in $V_{\pi,v}$ and $V_{\tilde{\pi},v}$ respectively.

2. When π_v is a holomorphic discrete series representation of $G(\mathbb{R})$. Then by Theorem 3.1 $\tilde{\pi}_v^D$ is either a holomorphic discrete series or an anti-holomorphic discrete series representation of $G'(\mathbb{R})$. We will only consider the case when $\tilde{\pi}_v$ is the corresponding holomorphic discrete series. We take φ_v and $\tilde{\varphi}_v$ to be the minimal weight vectors in $V_{\pi,v}$ and $V_{\tilde{\pi},v}$ respectively.

3. When π_v is a special representation of $G(F_v)$ where v is a non-archimedean place. Then by Theorem 3.1 $\tilde{\pi}_v^D$ could be either $\tilde{\pi}_v^+$ or $\tilde{\pi}_v^-$. $\tilde{\pi}_v^+$ is a special representation of $G'(F_v)$ while $\tilde{\pi}_v^-$ is a supercuspidal representation. We consider both cases. The vectors φ_v and $\tilde{\varphi}_v$ will be described in subsection 8.3.

When we look at the cuspidal representations corresponding to the integral weight forms of level N, the local components at infinite places and odd non-archimedean places are of the form π_v considered above. The situation at the even place is more subtle and will be considered in the next section.

8.1. **Some principal series at nonarchimedean places.** Let $\pi_v = \pi(\mu, \mu^{-1})$ be a unitary representation with $\mu(x) = |x|_v^s$, $is \in \mathbb{R}$. Let $\tilde{\pi}_v = \Theta(\pi_v, \psi)$, it is $\tilde{\pi}(\mu, \psi)$ by Proposition 4 of [W3]. Note that $\tilde{\pi}(\mu, \psi)$ is unramified. We will let $\tilde{\varphi}_v$ be $\tilde{\varphi}_{0,v}$ the unramified vector in $V_{\tilde{\pi},v}$, and let φ_v be $\varphi_{0,v}$ the unramified vector in $V_{\pi,v}$. Then $\tilde{\varphi}_v$ and φ_v are respectively $SL_2(\mathcal{O}_v)$ and $G(\mathcal{O}_v)$ invariant functions in $\tilde{\pi}(\mu,\psi)$ and $\pi(\mu,\mu^{-1})$.

Proposition 8.1. *With above choices,*

$$\begin{aligned} e(\varphi_v,\psi) &= \frac{1+q^{-1}}{|1-q^{-2s-1}|^2}, \\ e(\tilde{\varphi}_v,\psi^D) &= \begin{cases} |D|_v^{-1}\frac{1+q^{-1}}{|1+q^{-1/2-s}\chi_D(\varpi)|^2} & \text{when } |D|_v=1, \\ |D|_v^{-1}\frac{1+q^{-1}}{|1-q^{-2s-1}|^2} & \text{when } |D|_v=q^{-1}. \end{cases} \end{aligned}$$

Therefore $\frac{e(\varphi_v,\psi)}{e(\tilde{\varphi}_v,\psi^D)} = |D|_v L(\pi_v\otimes\chi_D,1/2)$.

Proof. We will use the Whittaker functional on $V_{\pi,v}$:

$$L_v(\phi_v) = \int \phi_v(wn(x))\psi(-x)dx,\ \ \phi_v\in V_{\pi,v} \tag{8.1}$$

The formula for spherical Whittaker function is well known, see [Ca-Sl]. The formula for our case is available in [Go]. We have $L_v(\pi_v(\underline{a})\varphi_v)=0$ if $|a|_v>1$; it equals

$$q^{-m/2}\mu^{-1}(a)\frac{(1-q^{-2s-1})(1-q^{-2(m+1)s})}{1-q^{-2s}}\varphi_v(e)$$

if $|a|_v=q^{-m}$ with $m\geq 0$. It is easy to show from (2.5) that $||\varphi_v||^2$ equals

$$|\frac{1-q^{-2s-1}}{1-q^{-2s}}|^2|\varphi_v(e)|^2\sum_{m=0}^{\infty}q^{-m}(1-q^{-1})|1-q^{-2(m+1)s}|^2 = (1+q^{-1})|\varphi_v(e)|^2, \tag{8.2}$$

and thus the result on $e(\varphi_v,\psi)$.

To compute $e(\tilde{\varphi}_v,\psi^D)$, we use the ψ^D-Whittaker functional

$$\tilde{L}_v^D(\phi) = \int \phi(\tilde{w}\cdot\tilde{n}(x))\psi^D(-x)dx \tag{8.3}$$

The formula for spherical Whittaker functions on the metaplectic groups is given in [Bu-F-H]. In our case it is an easy exercise to show when $|D|_v=1$, $\tilde{L}_v^D(\tilde{\varphi}_{0,v}) = \tilde{\varphi}_{0,v}(e)(1+q^{-1/2-s}\chi_D(\varpi))$; when $|D|_v=q^{-1}$, it equals $\tilde{\varphi}_{0,v}(e)(1-q^{-1-2s})$.

The Hermitian form on $V_{\tilde{\pi},v}$ takes the form ([B-M1] (9.19)):

$$(8.4) \qquad (\phi,\phi') = \sum_{\delta_i \in F_v^*/(F_v^*)^2} 1/2 \int \tilde{L}_v^{D\delta_i}(\tilde{\pi}(\underline{\underline{a}})\phi)\overline{\tilde{L}_v^{D\delta_i}(\tilde{\pi}(\underline{\underline{a}})\phi')}|\delta_i|_v \frac{da}{|a|_v}.$$

The above form in turn equals: ([B-M1] (9.18))

$$(8.5) \qquad \int |D|_v^{-1}\phi(\tilde{w}\cdot\tilde{n}(x))\overline{\phi'(\tilde{w}\cdot\tilde{n}(x))}dx.$$

Use the second formula for Hermitian form to compute $||\tilde{\varphi}_{0,v}||^2$. From Iwasawa decomposition we get

$$(8.6) \qquad ||\tilde{\varphi}_{0,v}||^2 = |D|_v^{-1}[\int_{|x|_v\le 1} |\tilde{\varphi}_{0,v}(e)|^2dx + \int_{|x|_v>1} |x|_v^{-2}|\tilde{\varphi}_{0,v}(e)|^2dx]$$

which gives

$$||\tilde{\varphi}_{0,v}||^2 = |D|_v^{-1}(1+q^{-1})|\tilde{\varphi}_{0,v}(e)|^2$$

This gives the result on $e(\tilde{\varphi}_{0,v},\psi^D)$. The result on the quotient $\frac{e(\varphi_v,\psi)}{e(\tilde{\varphi}_{0,v},\psi^D)}$ follows from the table on $L-$functions in [Go]. $\square$

8.2. **Complementary series at nonarchimedean places.** Let π_v be as in subsection 8.1, except that now $\mu(x) = |x|_v^s\chi_\tau(x)$ with τ a unit in $\mathcal{O}_v$, and $|s| < 1/2, s\in\mathbb{R}$. Let $\tilde{\pi}_v = \Theta(\pi_v,\psi)$. Then as before $\tilde{\pi}_v = \tilde{\pi}(\mu,\psi) = \tilde{\pi}(||_v^s,\psi^\tau)$. We will choose the vectors φ_v and $\tilde{\varphi}_v$ as in subsection 8.1.

Proposition 8.2. *With above choices,*

$$\begin{aligned} e(\varphi_v,\psi) &= \frac{1+q^{-1}}{(1-q^{-2s-1})(1-q^{2s-1})}, \\ e(\tilde{\varphi}_v,\psi^D) &= \begin{cases} |D|_v^{-1}\frac{1+q^{-1}}{(1+q^{-1/2-s}\chi_{D\tau}(\varpi))(1+q^{s-1/2}\chi_{D\tau}(\varpi))} & \text{when } |D|_v = 1, \\ |D|_v^{-1}\frac{1+q^{-1}}{(1-q^{2s-1})(1-q^{-1-2s})} & \text{when } |D|_v = q^{-1}. \end{cases} \end{aligned}$$

Therefore $\frac{e(\varphi_v,\psi)}{e(\tilde{\varphi}_v,\psi^D)} = |D|_vL(\pi_v\otimes\chi_D,1/2)$.

Proof. Retain the notations in the proof of Proposition 8.1. The formula for $L_v(\pi_v(\underline{a})\varphi_v)$ still holds. From (2.5), one gets

$$||\varphi_v||^2 = \frac{(1-q^{-1-2s})^2(1+q^{-1})}{1-q^{2s-1}}|\varphi_v(e)|^2.$$

Thus we have the formula for $e(\varphi_v,\psi)$.

The formula for $\tilde{L}_v^D(\tilde{\varphi}_{0,v})$ in the proof of Proposition 8.1 remains valid. The Hermitian form however takes a more complicated form. If $z = \Delta^2\delta$, let

$$\begin{aligned}\lambda(z) &= |\Delta|_v^{-2s-2}[(1-q^{-2s})^{-1}(1-q^{-1}) + q^{s-1/2}\chi_{\delta\tau}(\varpi)], \text{ if } |\delta|_v = 1;\\ &= |\Delta|_v^{-2s-2}q[(1-q^{-2s})^{-1}(1-q^{2s-1})], \text{ if } |\delta|_v = q^{-1}.\end{aligned}$$

Then $\lambda(z) = |z|_v^{s-1}\Delta(\psi,\tau,v)(z)$ where $\Delta(\psi,\tau,v)(z)$ is defined in [B-M1] Proposition 9.8. From equation (9.22) of [B-M1], the Hermitian form is:

$$(8.7)\qquad (\phi,\phi') = \sum_{\delta_i\in F_v^*/(F_v^*)^2} \frac{1}{2}\int \tilde{L}_v^{D\delta_i}(\tilde{\pi}_v(\underline{\underline{a}})\phi)\overline{\tilde{L}_v^{D\delta_i}(\tilde{\pi}_v(\underline{\underline{a}})\phi')}\frac{\lambda(z)}{\lambda(D)}\frac{da}{|a|_v}.$$

From equation (9.21) of [B-M1], we see this form can also be written as:

$$(8.8)\qquad (\phi,\phi') = \lambda(D)^{-1}\int A\phi(\tilde{w}\cdot\tilde{n}(x))\overline{\phi'(\tilde{w}\cdot\tilde{n}(x))}dx.$$

where

$$A\phi(g) = \int \phi(\tilde{w}\cdot\tilde{n}(y)\cdot g)dy$$

Then $A\tilde{\varphi}_{0,v}$ is the unique vector in the space of $\tilde{\pi}(\mu^{-1}\chi_{-1},\psi)$ fixed under $SL_2(\mathcal{O}_v)$, with $A\tilde{\varphi}_{0,v}(e) = \frac{1-q^{-1-2s}}{1-q^{-2s}}\tilde{\varphi}_{0,v}(e)$. Using the Iwasawa decomposition, we see

$$||\tilde{\varphi}_{0,v}||^2 = (1+q^{-1})\lambda(D)^{-1}\frac{1-q^{-1-2s}}{1-q^{-2s}}|\tilde{\varphi}_{0,v}(e)|^2.$$

This gives the formula for $e(\tilde{\varphi}_{0,v},\psi^D)$. The result on the quotient $\frac{e(\varphi_v,\psi)}{e(\tilde{\varphi}_{0,v},\psi^D)}$ follows from the table on $L-$functions in [Go]. □

8.3. **Special representations.**

8.3.1. *Description of* $\tilde{\pi}_v^+$ *and* $\tilde{\pi}_v^-$. Let $\mu_\tau(x) = |x|_v^{1/2}\chi_\tau(x)$, where τ is in F_v^*. Let $\sigma^\tau = \sigma(\mu_\tau,\mu_\tau^{-1})$ be the special representation associated to the character μ_τ. We will only consider the case when $|\tau|_v = 1$. The space of σ^τ is the subspace of $\pi(\mu_\tau,\mu_\tau^{-1})$ consisting of functions ϕ with

$$(8.9)\qquad \int \phi(wn(x))dx = 0$$

From Theorem 3.1, the set $\{\Theta(\sigma_v^\tau \otimes \chi_D, \psi^D)\}$ consists of two elements. These two elements are described in [W3]. When $D\tau$ is not a square, $\Theta(\sigma_v^\tau \otimes \chi_D, \psi^D) = \tilde{\pi}^+$ is the special representation $\tilde{\sigma}^\tau(\psi)$. The space of this representation is the subspace of $\tilde{\pi}(\mu_\tau, \psi)$ consisting of functions ϕ satisfying

$$(8.10) \qquad \tilde{L}_v^{\tau\Delta^2}(\phi) = \int \phi(\tilde{w}\cdot\tilde{n}(x))\psi(-\tau\Delta^2 x)dx = 0, \text{ for all } \Delta.$$

On the other hand, when $D\tau$ is a square, $\Theta(\sigma_v^\tau \otimes \chi_D, \psi^D) = \tilde{\pi}^-$ is a supercuspidal representation of $G'(F_v)$. More precisely, it is the odd component of the Weil representation, denoted $r_{\psi^\tau}^-$. The space of $r_{\psi^\tau}^-$ is the subspace of odd functions of $C_c^\infty(F_v)$, with the action being: for $\Phi(z) \in C_c^\infty(F_v)$, $\Phi(z) = -\Phi(-z)$,

$$(8.11) \qquad r_{\psi^\tau}^-(\tilde{n}(x))\Phi(z) = \psi^\tau(xz^2)\Phi(z),$$

$$(8.12) \qquad r_{\psi^\tau}^-(\underline{\underline{a}})\Phi(z) = |a|_v^{1/2}\gamma(1,\psi^\tau)/\gamma(a,\psi^\tau)\Phi(az),$$

$$(8.13) \qquad r_{\psi^\tau}^-(\tilde{w})\Phi(z) = \gamma(1,\psi^\tau)^2/\gamma(-1,\psi^\tau)\int \Phi(y)\psi^\tau(-2yz)dy.$$

8.3.2. *The case of* $\tilde{\pi}_v^+$. When τD is not a square, $\Theta(\sigma_v^\tau \otimes \chi_D, \psi^D)$ is the special representation $\tilde{\pi}_v^+ = \tilde{\sigma}^\tau(\psi)$.

Define the Iwahori subgroup

$$K_0 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid |a|_v = |d|_v = 1, |c|_v < 1, |b|_v \le 1\}.$$

Recall in the introduction we defined an embedding of $SL_2(\mathcal{O}_v)$ in G' given by $g \mapsto (g, \kappa(g))$. Let K_0' be the image in G' of the restriction of the splitting to $K_0 \cap SL_2$.

Denote by char$(G(\mathcal{O}_v))$ and Denote by char(K_0) the characteristic functions if $G(\mathcal{O}_v)$ and K_0 respectively. Denote by char$(G'(\mathcal{O}_v))$ a function on G' with char$(G'(\mathcal{O}_v))((g,\xi))$ equals 0 if $g \notin SL_2(\mathcal{O}_v)$, and equals $\xi\kappa(g)$ otherwise. Let char$(K_0')(g,\xi)$ be the genuine function on G' that is 0 if $g \notin K_0$, and equals char$(G'(\mathcal{O}_v))(g,\xi)$ if $g \in K_0$.

Lemma 8.3. *Let φ_v be a function in $\pi(\mu_\tau, \mu_\tau^{-1})$ such that it equals char$(G(\mathcal{O}_v))-(q+1)$char(K_0) over $G(\mathcal{O}_v)$, then φ_v is in $V_{\sigma^\tau,v}$ and is fixed under K_0.*

Let $\tilde{\varphi}_v$ be a function in $\tilde{\pi}(\mu_\tau, \psi)$ such that it equals char$(G'(\mathcal{O}_v)) - (q+1)$char$(K_0')$ over $G'(\mathcal{O}_v)$, then $\tilde{\varphi}_v$ lies in the space of $\tilde{\sigma}^\tau(\psi)$ and is fixed under K_0'.

The spaces of K_0 fixed vectors in σ^τ and K_0' fixed vectors in $\tilde{\sigma}^\tau(\psi)$ are one dimensional.

Proof. We can consider the vectors in $\pi(\mu_\tau, \mu_\tau^{-1})$ and $\tilde{\pi}(\mu_\tau, \psi)$ as functions on $G(\mathcal{O}_v)$ and $SL_2(\mathcal{O}_v)$ respectively. Since $B\backslash G/K_0$ and $\tilde{B} \cap G'\backslash G'/K_0'$ both have two elements, the space of vectors in $\pi(\mu_\tau, \mu_\tau^{-1})$ fixed by K_0 is two dimensional, with basis $\{\text{char}(G(\mathcal{O}_v)), \text{char}(K_0)\}$; the space of vectors in $\tilde{\pi}(\mu_\tau, \psi)$ fixed by K_0' is two dimensional, with basis $\{\text{char}(G'(\mathcal{O}_v)), \text{char}(K_0')\}$.

When ϕ_1 is the vector corresponding to char$(G(\mathcal{O}_v))$, the integral $\int \phi_1(wn(x))dx$ equals $1 + q^{-1}$. When ϕ_2 is the vector corresponding to char$(G(K_0))$, the integral $\int \phi_2(wn(x))dx$ equals q^{-1}. Thus φ_v satisfies the condition (8.9) and generates the one dimensional space fixed by K_0 in σ^τ.

Next we compute $\tilde{L}_v^z(\tilde{\varphi}_v)$ with $z \in F_v^*$. Let ϕ_1' be the vector corresponding to char$(G'(\mathcal{O}_v))$, then from (8.3) and the Iwasawa decomposition, we get

$$\tilde{L}_v^z(\phi_1') = \int_{|x|_v \le 1} \psi(-zx)dx + \sum_{r=1}^{\infty} \int_{|x|_v = q^r} \tilde{\gamma}(x,\psi)|x|_v^{-3/2}\chi_\tau(x)[-1,x]\psi(-zx)dx.$$

When $|x|_v = q^r$ with r even, with our assumption on τ being a unit,

$$\tilde{\gamma}(x,\psi)\chi_\tau(x)[-1,x] = 1.$$

When $|x|_v = q^r$ with r odd,

$$\tilde{\gamma}(x,\psi)\chi_\tau(x)[-1,x] = \gamma(x,\psi)[\tau,\varpi].$$

It is a simple calculation to get the following result: write $z = \delta\Delta^2$ with $|\delta|_v = 1$ or q^{-1},

$$\tilde{L}_v^{\delta\Delta^2}(\phi'_1) = \begin{cases} 0 & \text{when } |\Delta|_v > 1, \\ 1 + q^{-1} + |\Delta|_v(q^{-1}[\tau\delta, \varpi] - q^{-1}) & \text{when } |\delta|_v = 1, |\Delta|_v \le 1, \\ q^{-1} - |\Delta|_v(q^{-1} + q^{-2}) & \text{when } |\delta|_v = q^{-1}, |\Delta|_v \le 1. \end{cases}$$

Let ϕ'_2 be the vector corresponding to $\mathrm{char}(K'_0)$. Then

$$\tilde{L}_v^z(\phi'_1 - \phi'_2) = \int_{|x|_v \le 1} \psi(-zx)dx.$$

We get

$$\tilde{L}_v^{\delta\Delta^2}(\phi'_1) = \begin{cases} 0 & \text{when } |\Delta|_v > 1, \\ q^{-1} + |\Delta|_v(q^{-1}[\tau\delta, \varpi] - q^{-1}) & \text{when } |\delta|_v = 1, |\Delta|_v \le 1, \\ q^{-1} - |\Delta|_v(q^{-1} + q^{-2}) & \text{when } |\delta|_v = q^{-1}, |\Delta|_v \le 1. \end{cases}$$

The formula for $\tilde{L}_v^z(\tilde{\varphi}_v)$ is

(8.14)

$$\tilde{L}_v^{\delta\Delta^2}(\tilde{\varphi}_v) = \begin{cases} 0 & \text{when } |\Delta|_v > 1, \\ 2|\Delta|_v & \text{when } |\delta|_v = 1, |\Delta|_v \le 1, \delta\tau \text{ is not a square}, \\ 0 & \text{when } |\delta|_v = 1, |\Delta|_v \le 1, \delta\tau \text{ is a square}, \\ |\Delta|_v(q^{-1} + 1) & \text{when } |\delta|_v = q^{-1}, |\Delta|_v \le 1. \end{cases}$$

It is now clear that $\tilde{L}_v^z(\tilde{\varphi}_v)$ satisfies the condition (8.10). The vector $\tilde{\varphi}_v$ generates the space of K'_0 fixed vectors in $\tilde{\sigma}^\tau(\psi)$ which from the above formulas is clearly one dimensional. □

Proposition 8.4. *Assume $D\tau$ is not a square. Let φ_v and $\tilde{\varphi}_v$ be the vectors in Lemma 8.3. Then*

$$\begin{aligned} e(\varphi_v, \psi) &= \frac{1}{1 + q^{-1}}, \\ e(\tilde{\varphi}_v, \psi^D) &= \begin{cases} 1/2 & \textit{when } |D|_v = 1, \\ q/(1 + q^{-1}) & \textit{when } |D|_v = q^{-1}. \end{cases} \end{aligned}$$

Therefore

$$\frac{e(\varphi_v,\psi)}{e(\tilde{\varphi}_v,\psi^D)}=\begin{cases}2L(\pi_v\otimes\chi_D,1/2)|D|_v & \textit{when } |D|_v=1,\\ L(\pi_v\otimes\chi_D,1/2)|D|_v & \textit{when } |D|_v=q^{-1}.\end{cases}$$

Proof. We can use the Iwasawa decomposition to compute $L_v(\sigma^\tau(\underline{a})\varphi_v)$ where L_v is defined as in (8.1). We will skip the details. One gets

$$L_v(\sigma^\tau(\underline{a})\varphi_v)=\begin{cases}0 & \text{when } |a|_v>1,\\ (1+q^{-1})|a|_v\chi_\tau(a) & \text{when } |a|_v\le 1.\end{cases}$$

Thus $||\varphi_v||^2=(1+q^{-1})$ from (2.5) and we get the value of $e(\varphi_v,\psi)$.

Assume $|D|_v=1$. From (8.14) $\tilde{L}_v^D(\tilde{\varphi}_v)=2$. To find $||\tilde{\varphi}_v||$, we use the Hermitian form (9.23) in [B-M1]:

$$(8.15)\qquad (\phi,\phi')=\sum_{b=D,D\varpi,\tau\varpi}c_b/2\int \tilde{L}_v^b(\tilde{\sigma}(\underline{a})\phi)\overline{\tilde{L}_v^b(\tilde{\sigma}(\underline{a})\phi')}\frac{da}{|a|_v}$$

where $\tilde{\sigma}=\tilde{\sigma}^\tau(\psi)$ and $c_b=1$ when $b=D$, and $c_b=2q^{-1}/(1+q^{-1})$ when $b=D\varpi,\tau\varpi$. Using this formula, the formula (8.14) for $\tilde{L}_v^z(\tilde{\varphi})$ and the fact that

$$|\tilde{L}_v^b(\tilde{\sigma}(\underline{a})\tilde{\varphi})|=|a|_v^{1/2}|\tilde{L}_v^{a^2b}(\tilde{\varphi})|,$$

we get

$$||\tilde{\varphi}||^2=\int_{|a|_v\le 1}2|a|_v^3\frac{da}{|a|_v}+\frac{2q^{-1}}{1+q^{-1}}\int_{|a|_v\le 1}(1+q^{-1})^2|a|_v^3\frac{da}{|a|_v}=2.$$

Thus we get the formula for $e(\tilde{\varphi}_v,\psi^D)$ when $|D|_v=1$

When $|D|_v=q^{-1}$, from (8.14) $\tilde{L}_v^D(\tilde{\varphi}_v)=1+q^{-1}$. The computation of $||\tilde{\varphi}_v||$ goes as before except that the Hermitian form changes from (8.15) to

$$(\phi,\phi')=\sum_{b=D,D\tau,\delta}c'_b/2\int \tilde{L}_v^b(\tilde{\sigma}(\underline{a})\phi)\overline{\tilde{L}_v^b(\tilde{\sigma}(\underline{a})\phi')}\frac{da}{|a|_v}$$

where δ is a unit in $\mathcal{O}_v$ such that $\delta\tau$ is not a square. Here $c'_D=c'_{D\tau}=1$ and $c'_\delta=\frac{1+q^{-1}}{2q^{-1}}$. We get $||\tilde{\varphi}_v||=(1+q^{-1})/q^{-1}$ and the formula for $e(\tilde{\varphi}_v,\psi^D)$.

The claim on the quotient follows from the following formulas for $L-$values (see [Go]). When $|D|_v=1$ with τD not a unit, $L(\pi_v\otimes\chi_D,1/2)=L(\sigma^{\tau D},1/2)=(1+q^{-1})^{-1}$. When $|D|_v=q^{-1}$, $L(\pi_v\otimes\chi_D,1/2)=L(\sigma^{\tau D},1/2)=1$. □

8.3.3. *The case of* $\tilde{\pi}_v^-$. When τD is a square, $\Theta(\sigma_v^\tau \otimes \chi_D, \psi^D)$ is the supercuspidal representation $\tilde{\pi}_v^- = r_{\psi^\tau}^-$. Recall by our assumption τ and D are both units in $\mathcal{O}_v$. We will let φ_v to be the vector defined in Lemma 8.3. Next we describe a vector $\tilde{\varphi}_v$ in the space of $r_{\psi^\tau}^-$.

Let

$$K_{00} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid |a|_v = |d|_v = 1, |c|_v < q^{-1}, |b|_v \leq 1\}.$$

Let K_{00}' be the image of $K_{00} \cap SL_2$ embedded in G'. Then $K_{00}' = \{(\sigma, \kappa(\sigma)) | \sigma \in K_{00}\}$. Let χ be any odd character of $\mathcal{O}_v^*$ that is trivial on $1+P$, ($\chi(-1) = -1$), then χ defines a character on K_{00}' by:

$$\chi(\sigma, \kappa(\sigma)) \mapsto \chi(d),\ \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{00}.$$

Let $char(X)$ denote the characteristic function of a subset X in F_v.

Proposition 8.5. *The space of vectors* Φ *in* $r_{\psi^\tau}^-$ *satisfying* $r_{\psi^\tau}^-(k)\Phi = \chi(k)\Phi$, $k \in K_{00}'$ *is one dimensional. It is generated by the element*

$$\Phi_\chi(z) = \sum_{b \in \mathcal{O}_v^*/1+P} \chi^{-1}(b)\, char(b+P)(z).$$

Proof. Let Φ be a vector satisfying the relation in the Proposition. With our assumptions on ψ, τ and the place v, the equation (8.12) gives

$$r_{\psi^\tau}^-(\underline{\underline{z}})\Phi(1) = \Phi(z) = \chi(z^{-1})\Phi(1),\ z \in \mathcal{O}_v^*.$$

Thus $\Phi_0 = \Phi - \Phi(1)\Phi_\chi$ vanishes over the set $\mathcal{O}_v^*$. Next if $z \notin \mathcal{O}_v$, from (8.11), for $x \in \mathcal{O}_v$:

$$r_{\psi^\tau}^-(\tilde{n}(x))\Phi(z) = \psi(\tau z x^2)\Phi(z) = \Phi(z),$$

thus $\Phi(z) = 0$. We get Φ_0 is supported on P.

We now show Φ_χ satisfies the relation in the Proposition.

Lemma 8.6. *For $z \in F_v$, $x \in \mathcal{O}_v$,*

$$r^-_{\psi^\tau}(\begin{pmatrix} 1 & \\ x\varpi^2 & 1 \end{pmatrix}, 1)char(z+P) = char(z+P). \tag{8.16}$$

Proof. We use the fact that

$$(\begin{pmatrix} 1 & \\ x\varpi^2 & 1 \end{pmatrix}, 1) = \tilde{w} \cdot (-e, 1) \cdot \tilde{n}(-x\varpi^2) \cdot \tilde{w}.$$

From (8.11), (8.12) and (8.13) the left hand side of (8.16) is

$$r^-_{\psi^\tau}(\begin{pmatrix} 1 & \\ x\varpi^2 & 1 \end{pmatrix}, 1)\mathrm{char}(z+P)(a) = \int\int \mathrm{char}(z+P)(u)\psi^\tau(xy^2\varpi^2+2uy-2ay)dudy.$$

The integral over u is nonzero only when $y \in P^{-1}$, in which case $\psi^\tau(xy^2\varpi^2) = 1$ and from the Fourier inversion formula, the above integral just equals $\mathrm{char}(z+P)(a)$. □

It is easy to check using (8.11) and (8.12) that $r^-_{\psi^\tau}(k))\Phi_\chi = \chi(k)\Phi_\chi$ when $k \in K'_{00} \cap \tilde{B}$. From the above Lemma we get for $x \in \mathcal{O}_v$,

$$r^-_{\psi^\tau}(\begin{pmatrix} 1 & \\ x\varpi^2 & 1 \end{pmatrix}, 1)\Phi_\chi = \Phi_\chi.$$

Since these group K'_{00} is generated by the elements in $K'_{00} \cap \tilde{B}$ and $\{(\begin{pmatrix} 1 & \\ x\varpi^2 & 1 \end{pmatrix}, 1) | x \in \mathcal{O}_v\}$, we see Φ_χ satisfies the relation in the Proposition.

Thus $\Phi_0 = \Phi - \Phi(1)\Phi_\chi$ satisfies the relation in the Proposition and is supported over P. To finish the proof, we need to show such a function is identically 0. From the proof of Lemma 8.6, we get:

$$\int_{x \in \mathcal{O}_v} r^-_{\psi^\tau}(\begin{pmatrix} 1 & \\ x\varpi^2 & 1 \end{pmatrix}, 1)\Phi_0(a) = \Phi_0(a)$$

$$= \int_{x \in \mathcal{O}_v} \int\int \Phi_0(u)\psi^\tau(xy^2\varpi^2 + 2uy - 2ay)dudydx.$$

For the integration over x to be nonzero, $y \in P^{-1}$, in which case $\Phi_0(u)\psi^\tau(2uy) = \Phi_0(u)$. Thus the above integral equals

$$\int_{x\in\mathcal{O}_v}\int_{y\in P^{-1}}\int \Phi_0(u)\psi^\tau(-2ay)dudydx$$

which equals a constant times char(P). Thus for $a \in P$, $\Phi_0(a) = \Phi_0(0)$. Since Φ_0 is an odd function, Φ_0 vanishes over P, thus vanishes identically.

□

The representation $r^-_{\psi^\tau}$ is a *distinguished* representation, in the sense that it only has nontrivial Whittaker functional for ψ^δ with δ in the same square class as τ. Assume $D = \tau\alpha^2$, we can define $\tilde{L}_v^D$ by setting

$$\tilde{L}_v^D(\Phi) = \Phi(\alpha).$$

Then the Hermitian form is just:

$$(\Phi_1, \Phi_2) = 1/2 \int \tilde{L}_v^D(r^-_{\psi^\tau}(\underline{\underline{a}})\Phi_1)\overline{\tilde{L}_v^D(r^-_{\psi^\tau}(\underline{\underline{a}})\Phi_1)}\frac{da}{|a|_v}.$$

Clearly $\tilde{L}_v^D(\Phi_\chi) = \chi(\alpha^{-1})$ and by (8.12),

$$|\tilde{L}_v^D(r^-_{\psi^D}(\underline{\underline{a}})\Phi_\chi)| = |a|_v^{1/2}|\Phi_\chi(a\alpha)|$$

which equals 1 when $|a|_v = 1$ and 0 otherwise. Thus

$$||\Phi_\chi||^2 = 1/2\int_{|a|_v=1}\frac{da}{|a|_v} = (1-q^{-1})/2.$$

Thus $e(\Phi_\chi, \psi^D) = (1-q^{-1})/2$. Note that from [Go]

$$L(\pi_v \otimes \chi_D, 1/2) = L(\sigma^{\tau D}, 1/2) = L(\sigma^1, 1/2) = (1-q^{-1})^{-1}.$$

We have

Proposition 8.7. *Assume $D\tau$ is a square. Let φ_v be a vector in $\pi_v = \sigma^\tau$ given by Lemma 8.3. Let $\tilde{\varphi}_v$ be Φ_χ as in Proposition 8.5. Then $e(\varphi_v, \psi) = (1+q^{-1})^{-1}$, $e(\tilde{\varphi}_v, \psi^D) = (1-q^{-1})/2$ and*

$$\frac{e(\varphi_v, \psi)}{e(\tilde{\varphi}_v, \psi^D)} = 2(1+q^{-1})^{-1}L(\pi_v \otimes \chi_D, 1/2).$$

8.4. **Holomorphic discrete series.** Let $F_v = \mathbb{R}$. Let π_v be the discrete series $\sigma(\mu, \mu^{-1})$ ([W3]) where $\mu(x) = |x|^{s/2}(\text{sgn } x)^{(s+1)/2}$, $k = \frac{s+1}{2}$ being a positive integer. Then according to Theorem 3.1, $\Theta(\pi \otimes \chi_D, \psi^D)$ can be one of the following two representations: $\tilde{\pi}_v = \tilde{\pi}_v^+ = \Theta(\pi_v, \psi)$ and $\tilde{\pi}_v^- = \Theta(\pi_v \otimes \text{ sgn}, \psi^{-1})$. We note in this case $\pi_v \cong \pi_v \otimes \text{sgn}$.

We now assume $\psi(x) = e^{2\pi i n x}$ with n a positive integer. Then $\tilde{\pi}_v$ is a holomorphic discrete series $\tilde{\sigma}(\mu)$ while $\tilde{\pi}_v^- = \tilde{\sigma}(\mu \text{ sgn})$ is an antiholomorphic discrete series, ([W3]).

Let φ_v and $\tilde{\varphi}_v$ be a vector of minimal weight in $V_{\pi,v}$ and $V_{\tilde{\pi},v}$ respectively. These vectors are determined up to a scalar. We have:

Proposition 8.8. *Let D be a positive integer. With the above notations:*

$$\begin{aligned} e(\varphi_v, \psi) &= e^{4\pi n}(4\pi n)^{-2k}\Gamma(2k), \\ e(\tilde{\varphi}_v, \psi^D) &= 2e^{4\pi n D}(4\pi n D)^{-(1/2+k)}\Gamma(1/2+k). \end{aligned}$$

Therefore $\frac{e(\varphi_v,\psi)}{e(\tilde{\varphi}_v,\psi^D)} = \frac{1}{2}e^{4\pi n(1-D)}D^{1/2+k}n^{1/2-k}\pi^{-k}(k-1)!$.

Proof. From [Go], we see the Whittaker model for φ_v has the form:

$$L_v(\pi_v(\underline{a})\varphi_v) = \begin{cases} \alpha a^k e^{-2\pi n a} & a > 0 \\ 0 & a < 0 \end{cases}$$

where α is some nonzero constant which we may as well fix to be 1. With this model, we get from (2.5)

$$||\varphi_v||^2 = \int_{a>0} a^{2k} e^{-4\pi n a} d^* a$$

which equals $(4\pi n)^{-(2k)}\Gamma(2k)$. This gives the result for $e(\varphi_v, \psi)$.

From [W1] p.24, we see the Whittaker model for $\tilde{\varphi}_v$ with respect to ψ^D has the form:

$$\tilde{L}_v^D(\tilde{\pi}_v(\underline{\underline{a}})\tilde{\varphi}_v) = \alpha\omega(sgn(a))|a|^{1/2+k}e^{-2\pi n D a^2}$$

where ω is the central character of $\tilde{\sigma}(\mu)$. We will again let $\alpha = 1$. Since in this case, $\tilde{\pi}_v$ is *distinguished*, i.e., the Whittaker functional for ψ^z is always trivial when $z < 0$, the Hermitian form in (2.9) simplifies to:

$$(\phi, \phi') = \int \tilde{L}_v^D(\tilde{\pi}_v(\underline{a})\phi)\overline{\tilde{L}_v^D(\tilde{\pi}_v(\underline{a})\phi')}d^*a$$

Apply the formula to compute $||\tilde{\varphi}_v||$, we find that

$$||\tilde{\varphi}_v||^2 = 2(4\pi nD)^{-(1/2+k)}\Gamma(1/2+k).$$

The result for $e(\tilde{\varphi}_v, \psi^D)$ follows. The assertion on the quotient follows from the formula (k an integer):

$$\Gamma(2k) = \pi^{-1/2}2^{2k-1}\Gamma(k)\Gamma(k+1/2),\ \Gamma(k) = (k-1)!.$$

□

9. Cusp forms over $\mathbb{Q}$

We will apply the results in § 4 to the case of holomorphic cusp forms over $\mathbb{Q}$. Fix the additive character ψ as follows: if $x \in \mathbb{R}$, $\psi(x) = e^{2\pi i x}$; at a rational prime p, if $x \in \mathbb{Q}_p$, choose $\hat{x} \in \mathbb{Q}$ so that $|x - \hat{x}|_p \leq 1$, and set $\psi(x) = e^{-2\pi i \hat{x}}$. Denote by $\tilde{\gamma}(x)$ the number $\tilde{\gamma}(x, \psi)$. We denote by $|D|_v$ the metric at a place v, and $|D|$ the absolute value of D which equals $|D|_\infty$.

We first recall the correspondence between the cusp forms and automorphic representations. Our reference is [W2] section III. The main result in this section is the choice of a one dimensional subspace in a two dimensional subspace of $V_{\tilde{\pi}_2}$. This choice is closely related to the definition of the Kohnen space of half-integral weight forms.

9.1. **Integral weight forms.** Let $\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | c \equiv 0(N)\}$. Let $S_{2k}(N)$ be the space of cusp form of weight $2k$ on $\Gamma_0(N)$ (of level N), and with trivial character. Assume from now on that N is odd and square free. Let $f \in S_{2k}(N)$ be a new form. Then f determines a vector in the space of automorphic forms on $GL_2(\mathbf{A}_\mathbb{Q})$ by $f \mapsto \varphi = s(f)$. The map

$s(f)$ is defined as follows. For $g_\infty = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$, let

$$f|_{g_\infty}(z) = f(\frac{az+b}{cz+d})(cz+d)^{-2k}$$

Consider g_∞ as an element $(g_\infty, e, e, \ldots)$ in $GL_2(\mathbf{A}_\mathbb{Q})$, then $\varphi(g_\infty) = f|_{g_\infty}(i)$, and $\varphi(\gamma g k) = \varphi(g)$ whenever $\gamma \in GL_2(\mathbb{Q})Z(\mathbf{A}_\mathbb{Q})$, and $k \in \prod_{p\nmid N} GL_2(\mathcal{O}_p)\prod_{p|N} K_{0,p}$.

Then φ is a vector in the space of an irreducible cuspidal representation π of $GL_2(\mathbf{A}_\mathbb{Q})$, with trivial central character. The representation $\pi = \otimes\pi_v$, and $\varphi = \otimes_v\varphi_v$ can be described as follows:

(9.1.1). When $v = \infty$, π_v is the discrete series $\sigma(\mu_\infty, \mu_\infty^{-1})$ as in subsection 8.4, with $\mu_\infty(x) = |x|_v^{k-1/2}(\text{sgn } x)^k$. The vector φ_∞ is a minimal weight vector.

(9.1.2). When v is $p-$adic, p not dividing N, then $\pi_v = \pi(\mu_v, \mu_v^{-1})$ with μ_v an unramified character, and φ_v is an unramified vector.

(9.1.3). When v is $p-$adic, $p|N$, then π_v is a special representation σ^{τ_v} as in subsection 8.3, where τ_v is a unit in $\mathbb{Z}_p$. Then φ_v is the vector described in Lemma 8.3.

Conversely, given an irreducible cuspidal automorphic representation $\pi = \otimes\pi_v$ with local components as described in (9.1.1)–(9.1.3), pick φ as above (which is unique up to scalar multiple), then φ is a scalar multiple of $s(f)$ for some new form f in $S_{2k}(N)$.

If $f(z) = \sum_{n=1}^\infty a(n)e^{2\pi i n z}$, then $a(1) = e^{2\pi}W_{s(f)}(e)$. As we assume $a(1) = 1$, for $\varphi = s(f)$, $W_\varphi(e) = e^{-2\pi}$.

9.2. **Half integral weight forms.** Assume now that k is a nonnegative integer. Let N be a positive odd integer. Let χ be a Dirichlet character mod $4N$ such that $\chi(-1) = 1$. Assume $4N = \prod_{p|4N} p^{v(p)}$, then $(\mathbb{Z}/4N)^* \cong \prod_{p|N}(\mathbb{Z}/p^{v(p)})^*$, and χ can be decomposed into a product of characters $\chi_{(p)}$ of $(\mathbb{Z}/p^{v(p)})^*$ under this isomorphism. We can trivially extend $\chi_{(p)}$ to a character of $\mathbb{Z}_p^*$.

Let $S'_{k+1/2}(4N,\chi)$ be the space of holomorphic cusp forms of weight $k+1/2$, level $4N$ and character χ. The functions in the space satisfies: [W2]

$$(9.1)\quad g(\frac{az+b}{cz+d}) = j(\sigma,z)^{2k+1}\chi(d)g(z),\ \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),\ 4N|c.$$

Here

$$j(\sigma,z) = \theta(\frac{az+b}{cz+d})/\theta(z),\ \ \theta(z) = \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 z}.$$

Let $\tilde{A}'_{k+1/2}(4N,\chi)$ be the space generated by vectors $\tilde{\varphi} = \otimes_v \tilde{\varphi}_v$ in the space of cuspidal automorphic forms on $\tilde{SL}_2(\mathbf{A})$ satisfying:

(9.2.1) When $v=\infty$, $\tilde{\varphi}_v$ is a minimal weight vector in the space of a holomorphic discrete series $\tilde{\sigma}(\mu_\infty)$ where $\mu_\infty(x) = |x|^{k-1/2}(\mathrm{sgn}\ x)^k$.

(9.2.2) When v is $p-$adic, p not dividing $2N$, then $\tilde{\varphi}_p$ is the unramified vector in the space of $\tilde{\pi}(\mu_v,\psi)$ where μ is an unramified character.

(9.2.3) When v is $p-$adic, $p\neq 2$, and $p|N$, then $\tilde{\varphi}_p$ is a vector in the space of some $\tilde{\pi}_p$ such that $\tilde{\pi}_p(\sigma,\kappa(\sigma))\tilde{\varphi}_p = \chi_{(p)}(d)\tilde{\varphi}_v$ whenever $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}_p)$ with $|c|_p \le |N|_p$.

(9.2.4) When v is $p-$adic with $p=2$, $\tilde{\varphi}_2$ is a vector in the space of some $\tilde{\pi}_2$ such that $\tilde{\pi}_2(\sigma,1)\tilde{\varphi}_2 = \tilde{\epsilon}_2(\sigma)\chi_{(2)}\chi_{-1}^k(d)\tilde{\varphi}_2$ whenever $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}_2)$ with $|c|_2 \le |4N|_2$. Here

$$\tilde{\epsilon}_2(\sigma) = \begin{cases} \tilde{\gamma}(d)[c,d] & \text{when } c\neq 0 \\ \tilde{\gamma}(d)^{-1} & \text{when } c=0. \end{cases}$$

The Proposition 3 of [W2] establishes a bijection from $S'_{k+1/2}(4N,\chi)$ to $\tilde{A}'_{k+1/2}(4N,\chi)$. The bijection is given by $g(z)\mapsto \tilde{\varphi} = t(g)$, where $t(g)$ is the unique function on $\tilde{SL}_2(\mathbf{A})$ that is continuous and left invariant under $SL_2(\mathbb{Q})$ and satisfies:

$$t(g)(\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},1) = y^{k/2+1/4}e^{i(k+1/2)\theta}g(x+yi),$$

where $y>0, x\in\mathbb{R}$ and $-\pi<\theta\le\pi$.

The relation between the Whittaker functionals of $t(g)$ and the Fourier coefficients of $g(z)$ is given by follows: From Lemma 3 of [W2], we get when $g(z) = \sum_{n=1}^{\infty} c(n)e^{2\pi inz}$,

$$c(n) = e^{2\pi n}\tilde{W}^n_{t(g)}(e). \tag{9.2}$$

Remark on Peterson norm: If f is a cusp form of weight $k \in \frac{1}{2}\mathbb{Z}$ for a subgroup Γ of finite index in $\Gamma_1 = SL_2(\mathbb{Z})$, we define as usual the norm of f to be

$$< f, f >= \frac{1}{[\Gamma(1) : \Gamma]} \int_{\Gamma\backslash\mathcal{H}} |f(z)|^2 y^{k-2} dxdy$$

where $z = x + iy$ and $\mathcal{H}$ is the upper half plane. Then

Lemma 9.1. *For $\varphi = s(f)$ and $\tilde{\varphi} = t(g)$ as above:*

$$\frac{||\varphi||^2}{< f, f >} = \frac{|\tilde{\varphi}||^2}{< g, g >}. \tag{9.3}$$

Proof. It is well known that $||\varphi||^2 =< f, f >$ and $||\tilde{\varphi}||^2 =< g, g >$ when we use the following Haar measures d' on GL_2 and SL_2 instead of the one given in the introduction. When v is a non-archimedean place, choose the measure d' on GL_2 so that $G(\mathcal{O}_v)$ has volume 1; choose the measure d' on SL_2 so that $SL_2(\mathcal{O}_v)$ has volume 1. When v is the infinite place, let $k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. From the Iwasawa decomposition any $g \in GL_2^+(\mathbb{R})$ (the subgroup with positive determinant) can be written uniquely as $g = z(c)n(x)\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}k(\theta)$ with $c \in \mathbb{R}^*, x \in \mathbb{R}, a > 0$ and $0 \le \theta < \pi$. Let $d'g = \frac{1}{2\pi}|a|^{-2}d^*cd^*adxd\theta$ be the measure on $GL_2^+(\mathbb{R})$ and thus on $GL_2(\mathbb{R})$. Similarly any $g \in SL_2(\mathbb{R})$ can be written uniquely as $g = n(x)\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}k(\theta)$ with $x \in \mathbb{R}, a > 0$ and $0 \le \theta < 2\pi$. Let $d'g = \frac{1}{2\pi}|a|^{-2}d^*adxd\theta$ be the measure on $SL_2(\mathbb{R})$.

We now compare the measures dg and $d'g$ on GL_2 and SL_2 respectively. When v is a nonarchimedean place for the rational prime p, we have $dg =$

$(1+p^{-1})d'g$ in both case GL_2 and SL_2. When v is archimedean, we note the measures d' defined on GL_2 and SL_2 induce the same quotient measure on $Z(\mathbb{R})\backslash GL_2(\mathbb{R}) \cong Z \cap SL_2(\mathbb{R})\backslash SL_2(\mathbb{R})$, and the measures d defined in the introduction also induce the same quotient measure. Thus our change of the measures is consistent and the equation (9.3) still holds. $\square$

9.3. **Ramanujan conjecture.** We show here that the conjecture (4.10) implies the conjecture (4.8).

Let $g(z)$ be as in (4.8). Then $\tilde{\varphi} = t(g)$ is a linear combination of vectors in the space $\tilde{A}_{00} \cap \tilde{A}'(4N, \chi)$ for some N and χ. We may as well assume $g(z)$ correspond to a vector $\tilde{\varphi} = t(g)$ in a sub-representation $\tilde{\pi}$ of $\tilde{A}_{00}$. From (9.2), $c(n) = e^{2\pi n}\tilde{W}^n_{\tilde{\varphi}}(e)$. From the definition,

$$|d_{\tilde{\pi}}(\tilde{\varphi}, S_\infty, \psi^n)| = \frac{|\tilde{W}^n_{\tilde{\varphi}}(e)|}{||\tilde{\varphi}||} e(\tilde{\varphi}_\infty, \psi^n)^{1/2}.$$

From Proposition 8.8

$$|d_{\tilde{\pi}}(\tilde{\varphi}, S_\infty, \psi^n)| = e^{-2\pi n}|c(n)|[\frac{1}{2}e^{4\pi n}(4\pi n)^{-(1/2+k)}\Gamma(1/2+k)]^{1/2}/||\tilde{\varphi}||.$$

Thus $|d_{\tilde{\pi}}(\tilde{\varphi}, S_\infty, \psi^n)| = \delta(\tilde{\varphi})|c(n)|n^{-1/4-k/2}$ where $\delta(\tilde{\varphi})$ is a positive constant depending only on $\tilde{\varphi}$. From (4.10), for n a square free integer

$$|d_{\tilde{\pi}}(\tilde{\varphi}, S_\infty, \psi^n)| <<_{\tilde{\varphi},\alpha} n^{\alpha-1/2}.$$

Thus we get $|c(n)| <<_{\tilde{\varphi},\alpha} n^{k/2-1/4+\alpha}$ and (4.8).

9.4. **Choice of $\tilde{\varphi}_2$.** Let $\tilde{\pi}$ be a cuspidal representation such that the space of $\tilde{\pi}$ has nontrivial intersection with $\tilde{A}'_{k+1/2}(4N, \chi)$. The condition (9.2.4) puts a restriction on $\tilde{\pi}_2$ (the component at place $v = 2$ of $\tilde{\pi}$). The representation $\tilde{\pi}_2$ must be of the form $\tilde{\pi}(\mu\chi^{k'}_{-1}, \psi)$ where μ is an unramified character of $\mathbb{Z}_2^*$, and $k' = k$ if $\chi_{(2)}(-1) = 1$ and $k' = k+1$ if $\chi_{(2)}(-1) = -1$. The space of vectors in $\tilde{\pi}(\mu\chi^{k'}_{-1}, \psi)$ satisfying (9.2.4) is then two dimensional. It is spanned by two vectors $F[2,1]$ and $F[2,2^2]$ ([W2] Proposition 12). We recall their definitions ([W2] p.415, 427). They are the unique functions in the space of $\tilde{\pi}(\mu\chi_{-1}^{k'}, \psi)$ satisfying:

$$F[2,1](\tilde{w}) = 1,\ F[2,1](\begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}, 1) = 0,\ c \in 2\mathbb{Z}_2.$$

$$F[2,2^2](\tilde{w}) = 0,\ F[2,2^2](\begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}, 1) = \mathrm{char}(\mathbb{Z}_2)(2^{-2}c).$$

We make a choice of a vector $\tilde{\varphi}_2$ in the above two dimensional space. Define the linear combination

$$\tilde{\varphi}_2 = \mu(2^2)\frac{1+(-1)^{k'}i}{4}F[2,1] + F[2,2^2]. \tag{9.4}$$

The reason for this choice is explained by the following Proposition. Recall the definition of the Whittaker functional $\tilde{L}_v^z$ by equation (8.3).

Proposition 9.2. *When* $(-1)^{k'}z \equiv 2,3$ *mod* 4*,* $\tilde{L}_2^z(\tilde{\varphi}_2) = 0$.

This is a direct consequence of the following computation of $\tilde{L}_2^z(F[2,1])$ and $\tilde{L}_2^z(F[2,2^2])$.

Lemma 9.3. *With above definitions,* $\tilde{L}_2^z(F[2,1]) = char(\mathbb{Z}_2)(z)$ *and*

$$\tilde{L}_2^z(F[2,2^2]) = \begin{cases} 0 & |z|_2 > 1, \\ (\mu(2^2)+\sqrt{2}\mu(2^3))\frac{1+(-1)^{k'}i}{4} & z \in (-1)^{k'} + P^2, \\ -\mu(2^2)\frac{1+(-1)^{k'}i}{4} & z \in ((-1)^{k'+1}+P^2) \cup (2+P^2), \\ (\mu(2^2)-\mu(2^4))\frac{1+(-1)^{k'}i}{4} & (-1)^{k'}\frac{z}{4} \in (2+P^2) \cup (-1+P^2). \end{cases}$$

Proof. The claim for $F[2,1]$ is easy to verify. For $F[2,2^2]$, using the Iwasawa decomposition, we see:

$$\tilde{L}_2^z(F[2,2^2]) = \int_{|x|_2 \geq 2^2} \mu(x^{-1})|x|_2^{-1}\tilde{\gamma}(x)\chi_{-1}(x)^{k'+1}e^{2\pi izx}dx$$

Consider the integral

$$T(z,i) = \int_{|x|_2 = 2^i} \mu(x^{-1})|x|_2^{-1}\tilde{\gamma}(x)\chi_{-1}(x)^{k'+1}e^{2\pi izx}dx$$

Then

$$\tilde{L}_2^z(F[2,2^2]) = \sum_{i=2}^{\infty} T(z,i). \tag{9.5}$$

If $l = 2m$ is even, then a change of variable $x \mapsto x2^{-l}$ gives $T(z,l) = \mu(2^l)T(2^{-l}z,0)$. Over $|x|_2 = 1$, we have ([W2], p. 382)

$$\tilde{\gamma}(x) = 1/2(1 - i + (1+i)\chi_{-1}(x))$$

Define $\eta(\nu, t)$ to be the Gauss sum: ([W2], p.382)

$$\int_{|u|_2=1} \nu(x)e^{-2\pi itu}d^*u$$

Then

$$T(2^{-l}z,0) = (1-2^{-1})^{-1}[\frac{1-i}{2}\eta(\chi_{-1}^{k'+1}, -2^{-l}z) + \frac{1+i}{2}\eta(\chi_{-1}^{k'}, -2^{-l}z)].$$

Thus

$$T(z,2m) = 2\mu(2^{2m})[\frac{1-i}{2}\eta(\chi_{-1}^{k'+1}, -2^{-2m}z) + \frac{1+i}{2}\eta(\chi_{-1}^{k'}, -2^{-2m}z)]. \tag{9.6}$$

If $l = 2m+1$ is odd, then using the formula $\tilde{\gamma}(2^{-1}x) = \chi_2(x)\tilde{\gamma}(x)$ and make a change of variable $x \to 2^{-1}x$, we get

$$T(z,2m+1) = \mu(2)\int_{|x|_2=2^{2m}} \mu(x^{-1})|x|_2^{-1}\tilde{\gamma}(x)\chi_2(x)\chi_{-1}(x)^{k'+1}e^{\pi izx}dx$$

which by above argument becomes:

(9.7)

$$T(z,2m+1) = 2\mu(2^{2m+1})[\frac{1-i}{2}\eta(\chi_{-1}^{k'+1}\chi_2, -2^{-2m-1}z) + \frac{1+i}{2}\eta(\chi_{-1}^{k'}\chi_2, -2^{-2m-1}z)].$$

Note that Gauss sum $\eta(\nu,t)$ vanish if the conductor of ν is nonzero and not equal to $-v(t)$, or if ν is unramified and $|t|_2 > 2$. Observe that χ_{-1} is of conductor 2, and χ_2 is of conductor 3. Thus

$$\tilde{L}_2^z(F[2,2^2]) = \begin{cases} 0 & |z|_2 > 1, \\ T(z,2)+T(z,3) & |z|_2 = 1, \\ T(z,2) & |z|_2 = 2^{-1}, \\ T(z,2)+T(z,4)+T(z,5) & |z|_2 = 2^{-2}, \\ T(z,2)+T(z,4) & |z|_2 = 2^{-3}. \end{cases}$$

We can use the following formulas for η ([W2], p.383) to finish the computation: $\eta(\chi_2, 2^{-3}) = \frac{1}{\sqrt{2}}$, $\eta(\chi_{-2}, 2^{-3}) = \frac{-i}{\sqrt{2}}$, and $\eta(\chi_{-1}, 2^{-2}) = -i$ (there is a typo in [W2] for this value). Note also that $\chi_{-1}(\pm 1 + P^2) = \pm 1$ and $\eta(\nu, tt') = \eta(\nu, t)\nu^{-1}(t')$ when $|t'|_2 = 1$. Our assertion follows the formulas (9.6) and (9.7). $\square$

9.5. **Kohnen space.** Kohnen introduced a subspace $S^+_{k+1/2}(4N, \chi)$ in $S'_{k+1/2}(4N, \chi)$ in [K2], (we note the notation in [K2] is different from ours). It consists of $g(z) = \sum_{n=1}^{\infty} c(n)e^{2\pi i n z}$ with the Fourier coefficients $c(n)$ satisfying:

$$c(n) = 0, \quad \text{when } \chi_{(2)}(-1)(-1)^k n \equiv 2, 3 \bmod 4. \tag{9.8}$$

With our definition of $\tilde{\varphi}_2$, the Kohnen space has a natural interpretation in the representation language. Let $\tilde{A}^+_{k+1/2}(4N, \chi)$ be the space generated by vectors $\tilde{\varphi} = \otimes_v \tilde{\varphi}_v$ in the space of cuspidal automorphic forms on $\tilde{SL}_2(\mathbf{A})$ satisfying (9.2.1)–(9.2.4) and with $\tilde{\varphi}_2$ being the vector defined in (9.4).

Corollary 9.4. *The bijection $g(z) \mapsto t(g) = \tilde{\varphi}$ restricts to a bijection between the Kohnen space $S^+_{k+1/2}(4N, \chi)$ and $\tilde{A}^+_{k+1/2}(4N, \chi)$.*

Proof. Assume $\tilde{\varphi} = t(g)$ with $g(z) = \sum_{n=1}^{\infty} c(n)e^{2\pi i n z}$. From (9.2), $c(n) = 0$ if and only if $\tilde{W}^n_{\tilde{\varphi}}(e) = 0$.

Let $\tilde{\varphi}$ be a vector as above with $\tilde{\varphi}_2$ satisfying (9.4). Let n be a positive integer such that $\chi_{(2)}(-1)(-1)^k n \equiv 2, 3 \bmod 4$. Since $(-1)^{k'} = \chi_{(2)}(-1)(-1)^k$, at $v = 2$, $\tilde{L}^n_2(\tilde{\varphi}_2) = 0$. From the uniqueness of the local Whittaker functional, $\tilde{W}^n_{\tilde{\varphi}}(e)$ vanishes when $\tilde{L}^n_v(\tilde{\varphi}_v)$ vanish for any place v. We get $\tilde{W}^n_{\tilde{\varphi}}(e) = 0$ for such n. Thus $g(z) = t^{-1}(\tilde{\varphi})$ lies in the Kohnen space, and $\tilde{A}^+_{k+1/2}(4N, \chi) \subset t(S^+_{k+1/2}(4N, \chi))$.

In Proposition 1 of [K2], Kohnen defined an operator Q on $S'_{k+1/2}(4N, \chi)$. The operator has two different eigenvalues on this space and $S^+_{k+1/2}(4N, \chi)$ is the eigenspace of one eigenvalue (denoted α). The operator Q induces an operator Q' on the spaces $V_{\tilde{\pi}} \cap \tilde{A}'_{k+1/2}(4N, \chi)$. We have a factorization $V_{\tilde{\pi}} \cap \tilde{A}'_{k+1/2}(4N, \chi) = \otimes V'_{\tilde{\pi},v}$ with $V'_{\tilde{\pi},2}$ a two dimensional space. Then $Q' = \otimes Q'_v$.

In fact Q'_v are all trivial actions for $v \neq 2$. Clearly $\tilde{\varphi}_2$ in (9.4) is the eigenvector of Q'_2 with eigenvalue α as the vector $\tilde{\varphi} = \otimes\tilde{\varphi}_v$ with local component $\tilde{\varphi}_2$ lies in $\tilde{A}^+_{k+1/2}(4N,\chi)$. Let $\tilde{A}^-_{k+1/2}(4N,\chi)$ be the subspace of $\tilde{A}'_{k+1/2}(4N,\chi)$ generated by $\tilde{\varphi}' = \otimes\tilde{\varphi}'_v$ where $\tilde{\varphi}'_2$ is the eigenvector for the other eigenvalue. Then $\tilde{A}'_{k+1/2}(4N,\chi) = \tilde{A}^+_{k+1/2}(4N,\chi) \oplus \tilde{A}^-_{k+1/2}(4N,\chi)$. As $\tilde{A}^-_{k+1/2}(4N,\chi) \cap t(S^+_{k+1/2}(4N,\chi)) = 0$ and $t(S^+_{k+1/2}(4N,\chi)) \subset \tilde{A}'_{k+1/2}(4N,\chi)$, we get the corollary. $\square$

9.6. **Local computation at the place 2.** We compute the local factor as in § 8. The vector φ_2 is the unramified vector chosen as in subsection 8.1.

Proposition 9.5. *Assume $\mu_2(x) = |x|_2^{ir}$, with $r \in \mathbb{R}$. Then*

$$e(\varphi_2,\psi) = 3/2|1 - q^{-1-2ir}|^{-2},$$

$$e(\tilde{\varphi}_2,\psi^{|D|}) = \begin{cases} 3/4|1+2^{-1/2-ir}|^{-2} & D \in 1+P^2, \\ 3/4|1-2^{-1-2ir}|^{-2}|D|_2^{-1} & \frac{D}{4} \in (2+P^2)\cup(-1+P^2). \end{cases}$$

Therefore, when $D \in 1+P^2$,

$$\frac{e(\varphi_2,\psi)}{e(\tilde{\varphi}_2,\psi^{|D|})} = \begin{cases} 2|D|_2 L(\pi_2,1/2) & D \in 1+P^2, \\ 2|D|_2 & \frac{D}{4} \in (2+P^2)\cup(-1+P^2). \end{cases}$$

Proof. The formula for $e(\varphi_2,\psi)$ is given in Proposition 8.1. One can use (8.5) to compute $||\tilde{\varphi}_2||$ when $\mu_2(x) = |x|^{ir}$ where $r \in \mathbb{R}$. Since $F[2,1]$ and $F[2,2^2]$ are perpendicular, we get

$$||\tilde{\varphi}_2||^2 = 1/8||F[2,1]||^2 + ||F[2,2^2]||^2$$

Use the Iwasawa decomposition it is easy to show

$$||\tilde{\varphi}_2||^2 = (1/4+1/8)|D|_2^{-1} = 3/8|D|_2^{-1}.$$

From Lemma 9.3, we get the claim for $e(\tilde{\varphi}_2,\psi^{|D|})$. From the formula on local $L-$factor in [Go], we get the last statement of the Proposition. $\square$

10. A generalization of the Kohnen-Zagier formula

10.1. **Statement of the Theorem.** Let $f(z)$ be a cusp form as in (1.1) with square free level N (odd) and weight $2k$. Let S_N be the set of primes $p|N$. Let S be a (possibly empty) subset of S_N. Define $\mathcal{D}_S$ to be the set of fundamental discriminants D such that $(\frac{D}{p}) = -w_p$ if $p \in S$ and $(\frac{D}{p}) \neq -w_p$ if $p \in S_N - S$. Then the set of fundamental discriminants is the disjoint union $\cup_{S \subset S_N} \mathcal{D}_S$. For D a fundamental discriminant, let $T(D)$ be the set of $p|N$ that also divides D. Let $sgn(D) = D/|D|$. The character ψ is defined as in § 9.

Theorem 10.1. *Let $S \subset S_N$ and s be the size of S. Let $N' = \prod_{p \in S} p$, let $\chi = \prod_{p|2N} \chi_{(p)}$ be any Dirichlet character of $(\mathbb{Z}/4NN')^*$ such that $\chi_{(p)} \equiv 1$ when $pN'|N$, $\chi_{(p)}(-1) = -1$ when $p|N'$ and $\chi(-1) = 1$. There exists a unique (up to scalar multiple) cusp form $g_S(z)$ in $S'_{k+1/2}(4NN', \chi)$, such that the following is true:*

(1) $g_S(z)$ is a Shimura lift of $f(z)$.

(2) $g_S(z)$ lies in the Kohnen space, i.e. if $g_S(z) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n z}$, then $c(n) = 0$ when $(-1)^{s+k} n \equiv 2, 3$ mod 4.

(3) $c(D) = 0$ if $(-1)^{s+k} D$ is a fundamental discriminant that is not in $\mathcal{D}_S$.

Moreover for this $g_S(z)$ and for $D \in \mathcal{D}_S$, if $(-1)^{s+k} \neq sgn(D)$, then $L(f, D, k) = 0$; if $(-1)^{s+k} = sgn(D)$, then

$$\frac{|c(|D|)|^2}{< g_S, g_S >} = \frac{L(f, D, k)}{< f, f >} |D|^{k-1/2} \frac{(k-1)!}{\pi^k} 2^{\nu(N)-t} \prod_{p \in S} \frac{p}{p+1}. \tag{10.1}$$

Here t is the size of $T(D)$.

Remark 10.2. From subsection 9.1, the new form $f(z)$ determines an irreducible cuspidal representation π of $GL_2(\mathbf{A}_{\mathbb{Q}})$ with trivial central character. Here we say $g(z)$ is a Shimura lift of $f(z)$ if $\tilde{\varphi} = t(g)$ lies in a space $V_{\tilde{\pi}}$ where $\tilde{\pi}$ satisfies $\pi = S_\psi(\tilde{\pi})$ or $\pi = S_{\psi^{-1}}(\tilde{\pi})$ (see Theorem 3.2 for the notion S_ψ).

The proof of the Theorem involves translating Theorem 4.3 into the language of cusp forms. We will set up the translation in subsections (10.2)–(10.5).

10.2. **A Lemma on Atkin-Lehner involution.** Let $\hat{w}_p = \begin{pmatrix} p & a \\ N & pb \end{pmatrix}$ with $a, b \in \mathbb{Z}$ satisfying $\det \hat{w}_p = p$. Let $\tilde{f}(z) = f|_{p^{-1/2}\hat{w}_p}(z)$ be the Atkin-Lehner involution. Let $\hat{\varphi} = s(\tilde{f})$. For lack of reference, we give a proof of the following well known result.

Lemma 10.3. *When $p|N$ the local component π_p is a special representation $\sigma^\tau = \sigma(\mu, \mu^{-1})$ with $\mu(x) = |x|^{1/2}\chi_\tau(x)$, τ a unit in $\mathbb{Z}_p$. We have $\tilde{f} = w_p f$, with $w_p = 1$ when τ is not a square, and $w_p = -1$ when τ is a square. Moreover $\epsilon(\pi_p, 1/2) = w_p$*

Proof. The first result follows from [G]. Since $\hat{\varphi}(g_\infty) = \varphi(\hat{w}_p g_\infty)$, from left $GL_2(\mathbb{Q})Z(\mathbf{A}_\mathbb{Q})$ invariance,

$$\hat{\varphi}(g_\infty) = \varphi(g_\infty \prod_{v \neq \infty} \hat{w}_{p,v}^{-1}) = \pi(\prod_{v \neq \infty} \hat{w}_{p,v}^{-1})\varphi(g_\infty)$$

As $\varphi = \otimes \varphi_v$, we get $\hat{\varphi} = \varphi_\infty \otimes_{v \neq \infty} \pi_v(\hat{w}_{p,v}^{-1})\varphi_v$. When v does not divide N, $\hat{w}_{p,v}^{-1} \in GL_2(\mathbb{Z}_v)$, thus fixes φ_v. When $v|N$ but $v \neq p$, $\hat{w}_{p,v}^{-1} \in K_{0,v}$, thus fixes φ_v. When $v = p$, $\hat{w}_{p,p}(w\underline{p})^{-1} \in K_{0,p}$, thus $\pi_p(\hat{w}_{p,p}^{-1})\varphi_p = \pi_p(w\underline{p})^{-1}\varphi_p$. As $w\underline{p}K_{0,p}(w\underline{p})^{-1} \in K_{0,p}$, the vector $\pi_p(w\underline{p})^{-1}\varphi_p$ is again fixed by $K_{0,p}$, thus is a scalar multiple of φ_p. Denote this multiple by w_p. Then $\hat{\varphi} = w_p\varphi$, thus $\tilde{f} = w_p f$. To find the multiple, we only need to evaluate $\pi_p(w\underline{p})^{-1}\varphi_p(e)$ which clearly equals $p\chi_\tau(p)$. Since $\varphi_p(e) = -p$, we get $w_p = -\chi_\tau(p)$ which gives the claim in the Lemma. The computation of $\epsilon(\pi_p, 1/2)$ is given in [Go]. $\square$

10.3. **Definition of $\epsilon(S)$ and $\tilde{\pi}^{\epsilon(S)}$.** Let f and π be as before. Since π_v is unramified for all places v where $v \neq \infty$ and $|N|_v = 1$, we can let Σ in Theorem 3.2 to be the set $\{\infty\} \cup S_N$. Let S be a set as in the Theorem. Then it determines an $\epsilon(S) \in \{\pm 1\}^{|\Sigma|}$, where the component $\epsilon(S)_p$ at $p \in S$

is $-w_p$, at $p|N$ and $p \notin S$ is w_p, and at ∞ is $(-1)^{s+k}$. (The reason for this definition should be clear from Corollary 10.7 below).

Lemma 10.4. *We have* $\epsilon(\pi, 1/2) = \prod_{v\in\Sigma} \epsilon(S)_v$.

Proof. The product on the right is $(-1)^k \prod_{p|N} w_p$. Since $w_p = \epsilon(\pi_p, 1/2)$ by Lemma 10.3 and $(-1)^k = \epsilon(\pi_\infty, 1/2)$ from [Go], we get the claim. □

From Theorem 3.2, associated to π and the character $\psi_S(x) = \psi((-1)^{k+s}x)$ is the Shimura lift $\tilde{\pi}^{\epsilon(S)}$. Here $\tilde{\pi}^{\epsilon(S)} = \Theta(\pi\otimes\chi_D, \psi_S^D)$ for some $D \in \mathbb{Q}^{\epsilon(S)}(\pi)$. (The use of ψ_S is to make sure there is a holomorphic form in the space of $\tilde{\pi}^{\epsilon(S)}$).

10.4. **Relation between $\mathcal{D}_S$ and $\mathbb{Q}^{\epsilon(S)}(\pi)$.** Given D a fundamental discriminant, let $\epsilon_v(D) = (\frac{D}{\pi_v})$ for $v \in \Sigma$.

Lemma 10.5. *When* $v = \infty$, π_∞ *is a discrete series,* $\epsilon_\infty(D) = sgn(D)$.

When $p|N$, π_p *is a special representation* σ^τ *as in Lemma 10.3. Then* $\epsilon_p(D) = w_p$ *if* $p|D$; $\epsilon_p(D) = (\frac{D}{p})$ *when* D *is a unit in* $\mathbb{Z}_p$.

Proof. At $v = \infty$, $\pi_\infty \cong \pi_\infty\chi_D$, thus $\epsilon_\infty(D) = \chi_D(-1) = sgn(D)$. When $p|N$, $\Theta(\pi_p, \psi)$ is either a special representation (when τ is not a square) or an odd Weil representation (when τ is a square). Note that $w_p = 1$ if and only if τ is not a square (see Lemma 10.3). In the case τ is not a square, $\Theta(\pi_p, \psi)$ has a nontrivial ψ^D-Whittaker model when D is not a non-square unit. In the case τ is a square, then only when D is a square does $\Theta(\pi_p, \psi)$ has a nontrivial ψ^D-Whittaker model. From Theorem 3.1, we get the result. □

Lemma 10.6. *When* $D \in \mathcal{D}_S$, $\epsilon_p(D) = \epsilon(S)_p$ *for all* $p|N$.

Proof. When $p \in S$, then D is a unit in $\mathbb{Z}_p$, and $\epsilon_p(D) = (\frac{D}{p}) = -w_p = \epsilon(S)_p$. When $p \in S_N - S$, then either $p|D$ in which case $\epsilon_p(D) = w_p = \epsilon(S)_p$ or D is a unit in $\mathbb{Z}_p$, in which case $\epsilon_p(D) = (\frac{D}{p}) = w_p = \epsilon(S)_p$. □

As the set $\mathbb{Q}^{\epsilon(S)}(\pi)$ consists of D with $\epsilon_v(D) = \epsilon(S)_v$ for $v \in \Sigma$, from the above lemma and the formula for $\epsilon_\infty(D)$, we get

Corollary 10.7. *A fundamental discriminant D lies in $\mathbb{Q}^{\epsilon(S)}(\pi)$ if and only if $D \in \mathcal{D}_S$ and $(-1)^{s+k} = sgn(D)$.*

10.5. **Description of g_S.** The cusp forms g_S in the Theorem is taken to be the inverse image $t^{-1}(\tilde{\varphi}_S)$ of some vector $\tilde{\varphi}_S$ in the space of $\tilde{\pi}^{\epsilon(S)}$. We describe the choice of $\tilde{\varphi}_S = \otimes\tilde{\varphi}_v$.

Using the explicit description of theta correspondence in [W3], we get the following description on the local components of $\tilde{\pi}^{\epsilon(S)} = \otimes\tilde{\pi}_v^{\epsilon(S)}$. Note that $\tilde{\pi}_v^{\epsilon(S)} = \Theta(\pi_v \otimes \chi_D, \psi_S^D)$ for some $D \in \mathbb{Q}^{\epsilon(S)}(\pi)$.

Recall that the description of $\pi = \otimes\pi_v$ is given in (9.1.1)–(9.1.3), along with a choice of the vector $\varphi = \otimes\varphi_v$ such that $\varphi = s(f)$. Below is the description of π_v, $\tilde{\pi}_v^{\epsilon(S)}$ and the choice of the vectors $\tilde{\varphi}_v$.

(10.5.1) When $v = \infty$, π_v is the discrete series $\sigma(\mu_\infty, \mu_\infty^{-1})$, with $\mu_\infty(x) = |x|_v^{k-1/2}(\text{sgn } x)^k$. When $\epsilon_\infty(D) = \epsilon(S)_\infty$, we get $sgn(D) = (-1)^{s+k}$ thus $\psi_S^D = \psi^{|D|}$. Thus $\tilde{\pi}_\infty^{\epsilon(S)} = \Theta(\pi_\infty \otimes \chi_D, \psi^{|D|})$. As $\pi_\infty \otimes \chi_{|D|} \cong \pi_\infty \otimes \chi_D$ and $|D|$ and 1 are in the same square class, we get $\tilde{\pi}_\infty^{\epsilon(S)} = \Theta(\pi_\infty, \psi) = \tilde{\sigma}_\infty(\mu_\infty)$, ([W3]). We take $\tilde{\varphi}_\infty$ to be the vector with minimal weight.

(10.5.2) At $p \notin \Sigma$, $\pi_p = \pi(\mu_p, \mu_p^{-1})$ with μ_p an unramified character. From a well known result of Deligne, the unramified characters μ_p has the form $\mu_p(x) = |s|^{ir}$ with $r \in \mathbb{R}$, (this is the Ramanujun conjecture for the integral weight forms). Then $\tilde{\pi}_p^{\epsilon(S)} = \Theta(\pi_p, \psi_S) = \tilde{\pi}(\mu_p\chi_{-1}^{s+k}, \psi)$. We take $\tilde{\varphi}_p$ to be the unramified vector in this unramified representation when $p \neq 2$. We let $\tilde{\varphi}_2$ be the vector defined by (9.4) with $k' = k + s$.

(10.5.3) At $p \in S_N - S$, $\pi_p = \sigma^\tau$ with $\tau \in \mathbb{Z}_p^*$. Let D be a unit in $\mathbb{Z}_p$ such that τD is not a square, then $\epsilon_p(D) = w_p = \epsilon(S)_p$. Thus $\tilde{\pi}_p^{\epsilon(S)} = \Theta(\sigma^{\tau D}, \psi_S^D) = \tilde{\sigma}^\delta(\psi_S^D)$; here δ is any non-square unit. We take $\tilde{\varphi}_p$ to be the vector in Lemma 8.3.

(10.5.4) When $p \in S$, again $\pi_p = \sigma^\tau$ with $\tau \in \mathbb{Z}_p^*$. Let D be a unit in $\mathbb{Z}_p$ such that τD is a square, then $\epsilon_p(D) = -w_p = \epsilon(S)_p$. Thus $\tilde{\pi}_p^{\epsilon(S)} = \Theta(\sigma^1, \psi_S^D)$ which is $r_{\psi_S^D}^-$ from subsection 8.3. Let $\chi_{(p)}$ be the character on

$\mathbb{Z}_p^*$ defined in the Theorem, we let $\tilde{\varphi}_p = \Phi_{\chi_{(p)}}$ where $\Phi_{\chi_{(p)}}$ is defined in Lemma 8.5.

Each of the choices of $\tilde{\varphi}_p$ is determined unique up to a scalar multiple. Let $\tilde{\varphi} = \otimes\tilde{\varphi}_v$. We define the cusp form $g_S(z)$ to be $t^{-1}(\tilde{\varphi})$.

10.6. **Proof of the Theorem.**

Proof. As $S_{\psi_S}(\tilde{\pi}^{\epsilon(S)}) = \pi$, we see $g_S(z)$ is a Shimura lift of f. We can check $\tilde{\varphi} = \otimes\tilde{\varphi}_v \in \tilde{A}'_{k+1/2}(4NN', \chi)$. Thus $g_S(z) \in S'_{k+1/2}(4NN', \chi)$. It lies in the Kohnen space because of our choice of $\tilde{\varphi}_2$. If D is a fundamental discriminant with $\pm D \notin \mathcal{D}_S$, then $\pm D \notin \mathbb{Q}^{\epsilon(S)}(\pi)$. By Theorem 4.3, we get $d_{\tilde{\pi}^{\epsilon(S)}}(\Sigma \cup \{2\}, \psi_S^{\pm D}) = 0$. As $\psi^{|D|}$ is one of $\psi_S^{\pm D}$, we get $c(|D|) = 0$ from the consideration in § 9.

Next we show the uniqueness. If $g(z) \neq 0$ is a Shimura lift of f satisfying the conditions (2) and (3) in the Theorem. Assume $t(g)$ lies in the space of $\tilde{\pi}$, we show $\tilde{\pi} = \tilde{\pi}^{\epsilon(S)}$. As $\tilde{\pi}_\infty$ is a holomorphic discrete series, by Theorem 3.2 $\tilde{\pi} = \Theta(\pi \otimes \chi_{D_1}, \psi^{|D_1|})$ for some D_1. As $g(z) \neq 0$, $c((-1)^{s+k}D_2) \neq 0$ for some $D_2 \in \mathcal{D}_S$. The condition implies that $D_2 \in \mathbb{Q}^{\epsilon(S)}(\pi)$ and $\tilde{\pi}_v$ has nontrivial $\psi^{|D_2|}$-Whittaker model at all places v. From Theorem 3.1, we see $\tilde{\pi} = \Theta(\pi \otimes \chi_{\alpha D_2}, \psi^{|D_2|})$, where $\alpha = \frac{|D_1 D_2|}{D_1 D_2} = \pm 1$. Examine the component $\tilde{\pi}_2$. As $\chi_{(2)}(-1) = (-1)^s$ under our assumptions, we get $\tilde{\pi}_2 = \Theta(\pi \otimes \chi_{-1}^{s+k}, \psi)$, thus we see $\alpha = 1$ and $\tilde{\pi} = \Theta(\pi \otimes \chi_{D_2}, \psi^{|D_2|}) = \tilde{\pi}^{\epsilon(S)}$. From Corollary 9.4 and the fact that $\tilde{A}^+_{k+1/2}(4N, \chi) \cap V_{\tilde{\pi}^{\epsilon(S)}}$ is one dimensional, we get $g(z)$ must be a multiple of $g_S(z)$.

We now prove the identity (10.1). Let $D \in \mathcal{D}_S$. If $(-1)^{s+k} \neq sgn(D)$, then from equation (3.1), Lemma 10.4 and 10.6, we get:

$$\epsilon(\pi \otimes \chi_D, 1/2) = \epsilon(\pi, 1/2) \prod_{v \in \Sigma} \epsilon_v(D) = -\epsilon(\pi, 1/2) \prod_{v \in \Sigma} \epsilon(S)_v = -1.$$

Thus $L(\pi \otimes \chi_D, 1/2) = 0$, i.e. $L(f, D, k) = 0$.

When $(-1)^{s+k} = sgn(D)$, then $D \in \mathbb{Q}^{\epsilon(S)}(\pi)$, thus we can apply Theorem 4.3 to get:

$$L^{\Sigma\cup\{2\}}(\pi \otimes \chi_D, 1/2) = \frac{|d_{\tilde{\pi}^{\epsilon(S)}}(\Sigma \cup \{2\}, \psi_S^D)|^2}{|d_\pi(\Sigma \cup \{2\}, \psi_S)|^2} \prod_{v\in\Sigma\cup\{2\}} |D|_v. \tag{10.2}$$

From Lemma 2.3, $d_\pi(\Sigma\cup\{2\}, \psi_S) = d_\pi(\Sigma\cup\{2\}, \psi)$. Observe also $\psi_S^D = \psi^{|D|}$. From the explicit description of $d_{\tilde{\pi}^{\epsilon(S)}}(\Sigma \cup \{2\}, \psi^{|D|})$ and $d_\pi(\Sigma \cup \{2\}, \psi)$, we get:

$$\frac{|d_{\tilde{\pi}^{\epsilon(S)}}(\Sigma \cup \{2\}, \psi^{|D|})|^2}{|d_\pi(\Sigma \cup \{2\}, \psi)|^2} = \frac{|\tilde{W}_{\tilde{\varphi}}^{|D|}(e)|^2||\varphi||^2}{|W_\varphi(e)|^2||\tilde{\varphi}_S||^2} \prod_{v\in\Sigma\cup\{2\}} \frac{e(\tilde{\varphi}_v, \psi^{|D|})}{e(\varphi_v, \psi)}. \tag{10.3}$$

Recall the results on the local factors from Propositions 8.8, 8.4, 8.7 and 9.5:

(10.4)

$$\frac{e(\varphi_p, \psi)}{e(\tilde{\varphi}_p, \psi^{|D|})} = \begin{cases} \frac{1}{2}e^{4\pi(1-|D|)}|D|^{1/2+k}\pi^{-k}(k-1)! & p = \infty, \\ 2L(\pi_p \otimes \chi_D, 1/2)|D|_p & p \in S_N - S, p \not| D, \\ L(\pi_p \otimes \chi_D, 1/2)|D|_p & p \in S_N - S, p|D, \\ 2(1+p^{-1})^{-1}L(\pi_p \otimes \chi_D, 1/2) & p \in S, \\ 2|D|_2 L(\pi_2, 1/2) & p = 2, D \in 1 + P^2, \\ 2|D|_2 & p = 2, \frac{D}{4} \in (2 + P^2) \cup (-1 + P^2). \end{cases}$$

From subsection 9.1, $W_\varphi(e) = e^{-2\pi}$, and from (9.2), $\tilde{W}_{\tilde{\varphi}}^{|D|}(e) = e^{-2\pi|D|}c(|D|)$. Thus we get

(10.5)

$$L^{\infty\cup\{2\}}(\pi\otimes\chi_D, 1/2)l_2 = (1/2)^{\nu(N)-t}|D|^{-k+1/2}\frac{\pi^k}{(k-1)!}\frac{|c(|D|)|^2||\varphi||^2}{||\tilde{\varphi}_S||^2}\prod_{p\in S}(1+p^{-1}).$$

Here we set l_2 to be $L(\pi_2, 1/2)$ when $D \equiv 1 \bmod 4$ and to be 1 when $D \equiv 0 \bmod 4$.

We notice some differences between $L(\pi \otimes \chi_D, s)$ and $L(f, D, s')$. First $L(f, D, s')$ does not have factor at ∞. Secondly, because $\chi_D(2)$ over $v = 2$ is not the same as $(\frac{D}{2})$, we need to correct the factor at the place $v = 2$. We have actually

$$L(f, D, k) = L^{\infty\cup 2}(\pi \otimes \chi_D, 1/2)l_2.$$

From the above remark, Lemma 9.1 and (10.5), we get:

$$L(f,D,k)=(1/2)^{\nu(N)-t}|D|^{-k+1/2}\frac{\pi^k}{(k-1)!}\frac{|c(|D|)|^2<f,f>}{<g_S,g_S>}. \tag{10.6}$$

Therefore we get (10.1). □

10.7. **Some examples.** Example 1: When S is empty, $g_S(z)$ is the $g(z)$ in (1.1). If $T(D)$ is also empty, we recover (1.1). If $T(D)$ is nonempty, (1.1) should be revised to:

$$\frac{|c(|D|)|^2}{<g,g>}=|D|^{k-1/2}\frac{(k-1)!}{\pi^k}2^{\nu(N)-t}\frac{L(f,D,k)}{<f,f>}.$$

This fact is observed in [Gr] when $\nu(N)=1$.

Example 2: Look at the case when $f(z)$ is the new form of weight 2 and level 11. Such a form exists and is unique, with $w_{11}=-1$. This cusp form corresponds to the unique elliptic curve over $\mathbb{Q}$ of conductor 11. In [Gr], Gross constructed the form $g(z)$ corresponding to the case when S is the empty set in Theorem 10.1. The theorem says there is a cusp form $g_{\{11\}}(z)$ in $S'_{3/2}(484,\chi)$, where χ satisfies the condition in the Theorem, (for example $\chi(x)=(-1)^{(x-1)/2}(\frac{x}{11})$), such that $c(n)=0$ whenever $n\equiv 2,3 \bmod 4$, and when the fundamental discriminant $D>0$ is such that $(\frac{D}{11})=1$,

$$\frac{|c(D)|^2}{<g_{\{11\}},g_{\{11\}}>}=|D|^{1/2}\frac{11}{6\pi}\frac{L(f,D,1)}{<f,f>}.$$

The form $g_{\{11\}}(z)$ can be constructed as follows. Let $F_{11}=\mathbb{Z}/11$ be the finite field with 11 elements. Let $F_{11^2}=F_{11}[i]$ where $i^2=-1$. Define a function $\mu: F_{11}\times F_{11}\mapsto\{1,-1,0\}$ as follows: for $(a,b)\in F_{11}\times F_{11}$, let $c=a+bi$ and $\bar{c}=a-bi$; if $c\bar{c}=0$ or is not a square in F_{11}, $\mu(a,b)=0$; when $c\bar{c}=1$, $\mu(a,b)=1$ if c is a square in F_{11^2} and $\mu(a,b)=-1$ if not; extend this definition to the case $c\bar{c}$ is a square in F_{11}^* by the equalities $\mu(a,b)=\mu(ad,bd)$ for $d\in F_{11}^*$. This definition of μ extends naturally to $\mathbb{Z}\times\mathbb{Z}$.

Define (with $a, b, c \in \mathbb{Z}$)

$$h_1(q) = \frac{1}{2} \sum_{c \equiv b \bmod 2} \mu(a,b) q^{a^2+b^2+11c^2},$$

$$h_2(q) = \frac{1}{2} \sum_{c \equiv -2a \bmod 9, b \equiv c \bmod 2} \mu(4a,b) q^{(a^2+11c^2)/9+b^2}.$$

Then $g_{\{11\}}(z) = h_1(q) - h_2(q)$ with $q = e^{2\pi i z}$. With this construction one can compute $L(f, D, 1)$ using the above equation.

This is the most efficient algorithm to compute the values $L(f, D, 1)$. B. Conrey and M. Rubenstein provided us the values of $L(f, D, 1)$ for fundamental discriminants $D < 1000$ using a different algorithm. Their data matches ours and we thank them for helping us verify our formula.

We can observe that $3h_1(q) + 2h_2(q) = 3\sum_{n=1}^{\infty} (\frac{n}{11}) q^{n^2}$. Thus from the Theorem we get the following vanishing criterion

Corollary 10.8. *Let $f(z)$ be the unique new form of weight 2 and level 11, let $D > 0$ be a fundamental discriminant, then $L(f, D, 1) = 0$ if and only if the D-th Fourier coefficient of $h_1(q)$ equals 0, i.e:*

$$\sum_{c \equiv b \ mod\ 2, a^2+b^2+11c^2=D} \mu(a,b) = 0.$$

To save space, we leave the more detailed discussion of the above construction to a separate paper.

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