BESSEL FUNCTIONS FOR $GL(n)$ OVER A p-ADIC FIELD

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Abstract. We attach Bessel functions to generic representations of $GL_n(F)$ where F is a p-adic field and show that they are given locally by orbital integrals.

1. Introduction and Main Results

Let F be a non-archimedean local field. In [3] we attached Bessel functions to every generic representation of a quasi-split reductive group over F using a distribution approach similar to Harish-Chandra's approach for the character functions. In the present paper we attach Bessel functions to generic representations of $GL_n(F)$ using a Whittaker integral method similar to the one in [5],[12],[1] and generalizing the results in [4]. As in [4] we show that these Bessel functions are given locally by orbital integrals. Hence it follows from [11] that they have an asymptotic expansion in terms of the Jacquet-Ye germs.

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1.1. **Main results.** We state here our main theorems. We shall only consider here the main Bessel function of a representation which is the one attached to the open Bruhat cell. Other Bessel functions are described in Section 8.

Let $G = GL_n(F)$ and let B be the Borel subgroup of upper triangular matrices, A the subgroup of diagonal matrices and N the subgroup of upper unipotent matrices. Let ψ be a non-degenerate character of N. Let $\mathbb{W} = N(A)/A$ be the Weyl group where $N(A)$ is the normalizer of A. We identify $\mathbb{W}$ with the set of permutation matrices in $N(A)$. This set is also identified

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with S_n, the symmetric group on n-letters in a natural way. Let

$$w_0 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot & & \\ 1 & & & \end{pmatrix} \tag{1.1}$$

be the longest Weyl element in $\mathbb{W}$. Let (π, V) be an irreducible admissible representation of G. We say that π is generic if there exists a nonzero functional $L : V \to C$ such that

$$L(\pi(n)v) = \psi(n)L(v) \quad n \in N,\ v \in V.$$

It is well known that such a functional is unique up to scalar multiples. We call this functional a ψ Whittaker functional. Now define

$$W_v(g) = L(\pi(g)v) \quad v \in V,\ g \in G. \tag{1.2}$$

and let G act on the space of these functions by right translations. That is, if $g_1 \in G$ and W is a function on G then we define

$$(\rho(g_1)W)(g) = W(gg_1), \quad g \in G. \tag{1.3}$$

Then the map $v \to W_v$ gives a realization of π on a space of Whittaker functions satisfying

$$W(ng) = \psi(n)W(g) \quad n \in N,\ g \in G.$$

We denote this space by $\mathcal{W}(\pi, \psi)$. In Section 3 we define the subspace $\mathcal{W}^0(\pi, \psi)$ of $\mathcal{W}(\pi, \psi)$. In the case where π is supercuspidal we have that $\mathcal{W}^0(\pi, \psi) = \mathcal{W}(\pi, \psi)$. Let $\alpha_1, ..., \alpha_{n-1}$ be the positive roots realized as functions on A (See (2.13)). Let $M > 0$ be a constant and let

$$A^M = A^M(w_0) = \{a \in A : \ \alpha_i(a) < M, i = 1, 2, ..., n-1\}. \tag{1.4}$$

Our first main theorem is the following.

Theorem 1.1. *Let $W \in \mathcal{W}^0(\pi, \psi)$ and M a positive constant. Then the function*

$$(a, n) \mapsto W(aw_0n)$$

defined on the set $A^M \times N$ is compactly supported in N. That is, if $W(aw_0n) \neq 0$ and $a \in A^M, n \in N$ then n is in some compact set independent of a.

Since A^M cover A as $M \to \infty$ we get the following Corollary.

Corollary 1.2. *Let π be a supercuspidal representation of G and $W \in \mathcal{W}(\pi, \psi)$ a Whittaker function associated to π. Fix $g \in Bw_0B$. Then the function*

$$n \mapsto W(gn)$$

is compactly supported in N.

Proof. Write $g = n_1 a w_0 n_2$ and choose M large enough such that $a \in A^M$. Since $W(gn) = \psi(n_1)W(aw_0n_2n)$ the result follows from Theorem 1.1 □

This result allows us to define Bessel functions for supercuspidal representations (See Section 6). In order to treat all irreducible admissible representations we will need the following result which allows us to move from $\mathcal{W}(\pi,\psi)$ to $\mathcal{W}^0(\pi,\psi)$.

Theorem 1.3. *Let $W \in \mathcal{W}(\pi,\psi)$. There exists a compact open subgroup $N_0 = N_0(W)$ of N such that the function $W_{N_0,\psi} \in \mathcal{W}^0(\pi,\psi)$.*
Here $W_{N_0,\psi}$ is defined by

$$W_{N_0,\psi}(g) = \int_{N_0} W(gn)\psi^{-1}(n)dn$$

Corollary 1.4.

$$\mathcal{W}^0(\pi,\psi) \neq \{0\}$$

Proof. Let $W \in \mathcal{W}(\pi,\psi)$ be such that $W(e) \neq 0$. Then $W_{N_0,\psi}(e) \neq 0$ for every compact open subgroup N_0 in N. □

Let $N_1 \subset N_2 \subset N_3 \subset ...$ be a filtration of N with compact open subgroups N_i, $i = 1, 2, ...$, such that $N = \bigcup_{i=1}^{\infty} N_i$. We denote this filtration by $\mathcal{N}$. Let $f : N \to \mathbb{C}$ be a locally constant function.

Definition 1.5.

$$\int_N^{\mathcal{N}} f(n)dn = \lim_{m\to\infty} \int_{N_m} f(n)dn$$

if this limit exists.

Corollary 1.6. *Let $\mathcal{N} = \{N_i, i > 0\}$ be a filtration of N as above. Let $g \in Bw_0B$ and $W \in \mathcal{W}(\pi,\psi)$. Then*

$$\int_N^{\mathcal{N}} W(gn)\psi^{-1}(n)dn$$

is convergent, and the value is independent of the choice of filtration $\mathcal{N}$.

Proof. Let $N_0 = N_0(W)$ be a compact open subgroup of N as in Theorem 1.3. There exists an integer M such that $N_0 \subset N_m$ for all $m > M$. Let $m > M$. We have

$$\begin{aligned} (1.5) \qquad & \frac{1}{\mathrm{vol}(N_0)} \int_{N_m} W_{N_0,\psi}(gn)\psi^{-1}(n)dn \\ &= \frac{1}{\mathrm{vol}(N_0)} \int_{N_m}\int_{N_0} W(gn_1n_2)\psi(n_1n_2)dn_1dn_2 \end{aligned}$$

Applying Fubini and changing variables $n = n_1n_2$ we get that the last integral is the same as

$$\int_{N_m} W(gn)\psi^{-1}(n)dn$$

By Theorem 1.1 the function $n \mapsto W_{N_0,\psi}(gn)$ is compactly supported in N, hence we can take the limit $m \to \infty$ in (1.5) to get the value

$$\frac{1}{\mathrm{vol}(N_0)} \int_N W_{N_0,\psi}(gn)\psi^{-1}(n)dn.$$

It is clear that this value is independent of the filtration $\mathcal{N}$. □

Let $g \in Bw_0B$ and define the linear functional $L_g : V \to \mathbb{C}$ by

$$L_g(v) = \int_N^{\mathcal{N}} W_v(gn)\psi^{-1}(n)dn$$

It is easy to see that L_g is a Whittaker functional, hence it follows from the uniqueness of Whittaker functionals that there exists a scalar $j_{\pi,\psi}(g)$ such that

$$L_g(v) = j_{\pi,\psi}(g)L(v) \tag{1.6}$$

for all $v \in V$. We call $j_\pi = j_{\pi,\psi}$ the Bessel function of π. (See Section 5 for other Bessel functions). The Bessel function j_π is defined on the set Bw_0B and we will prove that it is locally constant there. It is clear that $j_\pi(g)$ satisfies

$$j_\pi(n_1gn_2) = \psi(n_1n_2)j_\pi(g), \quad n_1, n_2 \in N,\, g \in Bw_0B,$$

hence it is determined by its values on the set Aw_0. As is the case with the character of the representation [6], the Bessel function j_π is expected to be locally integrable on G. Harish-Chandra's proof of the local integrability of the character depended on certain relations between the asymptotics of the character and certain orbital integrals. In this paper we establish that the asymptotics of j_π are the same as the asymptotics of certain orbital integrals which were studied by Jacquet and Ye [11]. We now describe the relation between the Bessel functions and orbital integrals.

Let $C_c^\infty(G)$ be the space of locally constant and compactly supported functions on G. Let Z be the center of G and let ω be a quasi character on Z.

For $\phi \in C_c^\infty(G)$ and $g \in Bw_0B$ we define the orbital integral (see [11] (6))

$$J_{\psi,\omega}(g,\phi) = \int_{N\times Z\times N} \phi(n_1zgn_2)\psi^{-1}(n_1n_2)\omega^{-1}(z)dn_1dn_2dz$$

It follows from [11] that this integral converges absolutely and defines a locally constant function on Bw_0B.

Theorem 1.7. *Let π be an irreducible admissible representation of G with central character ω_π. Let $x \in G$. There exists a neighborhood U_x of x in G and a function $\phi \in C_c^\infty(G)$ such that*

$$j_{\pi,\psi}(g) = J_{\psi,\omega_\pi}(g,\phi)$$

for all $g \in U_x$.

Remark 1.8. Since $j_{\pi,\psi}$ and J_{ψ,ω_π} are only defined on Bw_0B we are really asserting the equality on $Bw_0B \cap U_x$. Another option is to define these functions as having value zero outside of Bw_0B in which case the equality above does hold. In any case, the equality is true up to a set of measure zero.

Corollary 1.9. *If $g \mapsto J_{\psi,\omega_\pi}(g,\phi)$ is locally integrable as a function on G for every $\phi \in C_c^\infty(G)$ then $j_{\pi,\psi}$ is locally integrable on G.*

Hence the question of local integrability of the Bessel function reduces to the question of the local integrability of the orbital integral.

Our paper is divided as follows. In Section 2 we introduce some notations including roots, weights and Bruhat ordering. In Section 3 we study some cones and dual bases in a Euclidean space. These will be applied later for different bases of roots and weights. In Section 4 we describe the method of proof used for our main results and prove a result which is needed later using this method. In Section 5 we prove a more general version of Theorem 1.1. In Section 6 we define Bessel functions for supercuspidal representations. In Section 7 we prove Theorem 1.3. In Section 8 we define Bessel functions for general generic representations, including Bessel functions attached to other Weyl elements. In Section 9 we prove Theorem 1.7 and in Section 10 we indicate how to generalize our results to simply laced groups.

2. Notations and Preliminaries

Let F be a non-archimedean local field. Let O be the ring of integers in F and let P be the maximal ideal in F. Let ϖ be a generator of P. We denote by $|x|$ the normalized absolute value of $x \in F$. Let $q = |O/P|$ be the order of the residue field of F. Then $|\varpi| = q^{-1}$. Let $G = GL_n(F)$ and let A be the group of diagonal matrices in G consisting of matrices of the form

$$d(a_1, a_2, ..., a_n) = \begin{pmatrix} a_1 & & & & \\ & a_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & a_n \end{pmatrix}$$

We let

$$Z = Z(G) = \{d(a, a, ..., a) : a \in F^*\}. \tag{2.1}$$

Let $X(A) = \mathrm{Hom}_F(A, F)$ be the group of rational homomorphisms. Then each $\alpha \in X(A)$ is of the form

$$\alpha(d(a_1, a_2, ..., a_n)) = a_1^{k_1} a_2^{k_2} ... a_n^{k_n}$$

with $k_1, k_2, ..., k_n \in \mathbf{Z}$. We view $X(A)$ as a group under addition where the addition is defined by

$$(\alpha + \beta)(a) = \alpha(a)\beta(a), \quad \alpha, \beta \in X(A), a \in A \tag{2.2}$$

We let $|X| = X(A) \otimes \mathbf{R}$. Then we shall identify $|X|$ with the vector space of functions $|\alpha| = |\alpha|_{\lambda_1, \lambda_2, ..., \lambda_n}$ from A to $\mathbf{R}$ of the form

$$|\alpha|(d(a_1, a_2, ..., a_n)) = |a_1|^{\lambda_1} |a_2|^{\lambda_2} ... |a_n|^{\lambda_n} \tag{2.3}$$

where $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbf{R}$. Here addition is defined as in (2.2) and scalar multiplication is defined by

$$(\lambda|\alpha|)(a) = (|\alpha|(a))^{\lambda}, \quad |\alpha| \in |X|, a \in A, \lambda \in \mathbf{R}.$$

We define an inner product on $|X|$ by

$$< \alpha_{\lambda_1, \lambda_2, ..., \lambda_n}, \alpha_{\gamma_1, \gamma_2, ..., \gamma_n} >= \sum_{i=1}^{n} \lambda_i \gamma_i \tag{2.4}$$

For $i, j \in \{1, 2, ..., n\}$, $i \neq j$ we let $\alpha_{i,j} : A \to F$ be the functions defined by

$$\alpha_{i,j}(d(a_1, a_2, ..., a_n)) = \frac{a_i}{a_j}$$

and $|\alpha|_{i,j}(a) = |\alpha_{i,j}(a)|$. Let $\Phi = \{\alpha_{i,j}\}$. Then $|\Phi| = \{|\alpha|_{i,j}\}$ is a root system in $|X|$. We have that $\Phi = \Phi^+ \cup \Phi^-$ where $\Phi^+ = \{\alpha_{i,j} : i < j\}$ is the set of positive roots and Φ^- is the set of negative roots. Let $E_{i,j}$ be the matrix whose (i,j)th entry is 1 and all other entries are zero. For $\alpha = \alpha_{i,j} \in \Phi$ and for $b \in F$ we let

$$x_\alpha(b) = x_{i,j}(b) = I + bE_{i,j}$$
$$h_\alpha(b) = h_{i,j}(b) = bE_{i,i} - b^{-1}E_{j,j}$$

For each $\alpha \in \Phi$ we let $N_\alpha = \{x_\alpha(b) : b \in F\}$. Let $\mathbb{W} = N(A)/A$ be the Weyl group of G. We shall identify $\mathbb{W}$ with S_n, the symmetric group. In particular if $\sigma \in S_n$ then we let w_σ be the associated permutation matrix. In particular, $w_{(i,j)}$ is the permutation matrix having 1s in the (i,j) and (j,i) entries and in the (k,k) entries for $k \neq i, j$. $\mathbb{W}$ acts on Φ and $|\Phi|$ in a natural way. We have that if $i \neq j$ then

$$a x_\alpha(b) a^{-1} = x_\alpha(\alpha(a)b), \quad a \in A, b \in F. \tag{2.5}$$

$$x_\alpha(b) x_{-\alpha}(-b^{-1}) x_\alpha(b) = w_\alpha h_\alpha(b), \quad b \in F. \tag{2.6}$$

$$w x_\alpha(b) w^{-1} = x_{w(\alpha)}(b), \quad w \in \mathbb{W}, b \in F \tag{2.7}$$

Also, if $\alpha, \beta \in \Phi$ and $\alpha \neq \pm\beta$ then

$$x_\alpha(b_1) x_\beta(b_2) = x_{\alpha+\beta}(\epsilon b_1 b_2) x_\beta(b_2) x_\alpha(b_1) \tag{2.8}$$

where $\epsilon = \pm 1$ and $x_{\alpha+\beta}(r) = e$ if $\alpha + \beta \notin \Phi$. Let N be the subgroup of upper unipotent matrices. Then every $n \in N$ can be written uniquely in the form

$$n = \prod_{i>j} x_{i,j}(b_{i,j})$$

where $b_{i,j} \in F$ and the order of the product is fixed. (Any fixed order is fine).

2.1. **Roots, and Weights.** The root system $|\Phi|$ spans a subspace $|V|$ of $|X|$ given by

$$|V| = \{|\alpha|_{\lambda_1,\lambda_2,\ldots,\lambda_n} : \lambda_1 + \lambda_2 + \ldots + \lambda_n = 0\}. \tag{2.9}$$

Then $\Delta = \{|\alpha|_{i,i+1} : i = 1, \ldots, n-1\}$ is a basis for $|V|$ consisting of simple roots. If $\mathcal{B}$ is a basis for $|V|$ then we denote by $\mathcal{B}^*$ the dual basis (up to positive scalars) with respect to (2.4). In particular, the fundamental weights $\lambda_1, \ldots, \lambda_{n-1}$ give Δ^*, the basis dual to Δ where
(2.10)

$$\lambda_1 = |\alpha|_{n-1,-1,-1,\ldots,-1}, \quad \lambda_2 = |\alpha|_{n-2,n-2,-2,-2,\ldots,-2}, \quad \lambda_{n-1} = |\alpha|_{1,1,\ldots,1,1-n}$$

We write $\alpha_i = |\ \alpha|_{i,i+1}$. Then

$$\Delta = \{\alpha_1, \ldots, \alpha_{n-1}\} \tag{2.11}$$

and

$$\Delta^* = \{\lambda_1, \ldots, \lambda_{n-1}\}. \tag{2.12}$$

Notice that we have chosen λ_i so that $< \alpha_i, \lambda_j >= 0$ if $i \neq j$ and that $< \alpha_i, \lambda_i >> 0$. We now recall the three different notation that we have for the simple roots for future reference:

$$\alpha_i = |\alpha|_{i,i+1} = |\alpha|_{0,\ldots,0,1,-1,0,\ldots,0}. \tag{2.13}$$

Remark 2.1. It is easy to see that if $\alpha \in \Phi$ is a negative root and if $r \in F$ is such that $|r| \geq D$ for some constant $D > 0$ then there exists a constant $C = C_D > 0$ such that

$$\lambda(h_\alpha(r)) < C \tag{2.14}$$

for all $\lambda \in \Delta^*$. Moreover, if $|r| \geq 1$ then

$$\lambda(h_\alpha(r)) \geq 1 \tag{2.15}$$

Each positive root $\alpha \in \Phi$ can be written as a positive integral combination of simple roots, that is,

$$\alpha = \sum_{i=1}^{n-1} c_i \alpha_{i,i+1}$$

with c_i is a non negative integer. We define the height of the positive root α to be

$$\text{height}(\alpha) = \sum_{k=1}^{n-1} c_k.$$

If α is a negative root thhen we define $\text{height}(\alpha) = \text{height}(-\alpha)$. It is easy to check that for $j > i$

$$\text{height}(\alpha_{i,j}) = j - i.$$

2.2. **Bruhat ordering.** For each $\alpha \in \Phi$ we let $w_\alpha \in \mathbb{W}$ be the reflection associated with α. That is, $w_{\alpha_{i,j}} = w_{(i,j)}$ Since $\mathbb{W}$ is generated by the simple reflections $w_{(i,i+1)}$ it follows that each $w \in \mathbb{W}$ can be written as a product of simple reflections. Let $w \in \mathbb{W}$ and write

$$w = w_{\beta_1} w_{\beta_2} \cdots w_{\beta_l}, \ \ \beta_i \in \Delta, \ i = 1, ..., k. \tag{2.16}$$

If (2.16) is a minimal expression for w, then we define

$$l(w) = l \tag{2.17}$$

$$S(w) = \{\beta_1, ..., \beta_l\} \subseteq \Delta \tag{2.18}$$

It is well known (see [9]) that $l(w)$ and $S(w)$ are independent of the the minimal decomposition (2.16) We define the Bruhat partial order on $\mathbb{W}$ by $w' \leq w \iff w'$ can be written as a subexpression of w, i.e,

$$w' = w_{\beta_{i_1}} w_{\beta_{i_2}} \cdots w_{\beta_{i_l}} \ \ 1 \leq i_1 < i_2 < ... < i_t \leq l$$

This Bruhat ordering does not depend on the choice of minimal expression in (2.16) (see [9]).

It is clear that if $w_1 \leq w_2$ then $S(w_1) \subseteq S(w_2)$. It is well known that w_0 is the longest Weyl element in $\mathbb{W}$ and that $w_0 \geq w$ for all $w \in \mathbb{W}$. Also, by ([9] 5.9, example 3) we have that

$$w_1 \leq w_2 \ \Leftrightarrow w_1 w_0 \geq w_2 w_0.$$

It will be convenient to use the following notation:

Definition 2.2.

$$S^0(w) = S(ww_0)$$

It follows from the above discussion that

$$w_1 \leq w_2 \ \Rightarrow \ S^0(w_1) \supseteqq S^0(w_2) \tag{2.19}$$

We define

$$S^-(w) = \{\alpha \in \Phi^+ \, : \, w(\alpha) < 0\}, \quad S^+(w) = \{\alpha \in \Phi^+ \, : \, w(\alpha) > 0\}. \tag{2.20}$$

Let S be a subset of simple roots, that is, $S \subset \Delta$. Let $\Phi(S) \subset \Phi$ be the set of roots in Φ which are in the span of S. It is well known that $\Phi(S)$ is itself a root system. We say that a root in $\Phi(S)$ has support in S. We let $\mathbb{W}_S$ be the Weyl group associated with S and we identify $\mathbb{W}_S$ as the subgroup of

$\mathbb{W}$ generated by the simple reflections w_α, $\alpha \in S$. We let w_S be the longest Weyl element in $\mathbb{W}_S$.

Let $w = w_{i,i+1}$ be a simple reflection. It is easy to see that w sends $\alpha = \alpha_{i,i+1}$ to $-\alpha$ and that α permutes all the other positive roots. The following lemma is well known.

Lemma 2.3.
(a) w permutes the positive roots which do not have support in $S(w)$.
(b) $S^-(w) \subset \Phi(S(w))$.
(c) If $\alpha \in \Phi(S(w))$ then $w(\alpha) \in \Phi(S(w))$.

Proof. (a) Write w as a minimal product of simple reflections. It is clear from the above remark on the simple reflections $w_{i,i+1}$ that each simple reflection in the decomposition of w permutes the positive roots with support not in S. Hence w also permutes the positive roots with support not in S.
(b) If α is positive and $w(\alpha)$ is negative then it follows from part (a) that α is supported on S.
(c) Since $w \in \mathbb{W}_S$ and $\alpha \in \Phi(S)$ it is clear that $w(\alpha) \in \Phi(S)$. □

Let $S^0(w)$ be defined as in (2.2).

Corollary 2.4. *Let $\alpha \in \Phi^+$ be such that $w(\alpha) > 0$ then $w(\alpha) \in \Phi(S^0(w))$.*

Proof. Let $w' = ww_0$. Since $w_0^2 = e$ we have $w'(w_0(\alpha)) = ww_0w_0(\alpha) = w(\alpha) > 0$. Since $w_0(\alpha) < 0$, it follows from Lemma 2.3 (b) that $w_0(\alpha) \in \Phi(S(w'))$. It follows from Lemma 2.3 (c) that $w'(w_0(\alpha)) \in \Phi(S(w'))$. Since $w'w_0 = w$ we get that $w(\alpha) \in \Phi(S(w')) = \Phi(S(ww_0)) = \Phi(S^0(w))$. □

Corollary 2.5. *Let $\alpha \in \Phi^+$ be such that $w(\alpha) < 0$. Let $w_1 = ww_\alpha$ then $w(\alpha) \in \Phi(S^0(w_1))$*

Proof. We have that $w_1(\alpha) = -w(\alpha) > 0$. Since $w_1(\alpha) > 0$ it follows from Corollary 2.4 that $w_1(\alpha) \in \Phi(S^0(w_1))$. □

2.3. **Bruhat decomposition.** We define

$$N_w^- = \prod_{\alpha \in S^-(w)} N_\alpha, \quad N_w^+ = \prod_{\alpha \in S^+(w)} N_\alpha. \tag{2.21}$$

It is well known that $|S^-(w)| = l(w)$ and that $N = N_w^+ N_w^-$. The Bruhat decomposition of G is given by

$$G = \bigcup_{w \in \mathbb{W}} BwB$$

Moreover, we have, $BwB = NAwN_w^-$ with uniqueness of expression. That is, every $g \in BwB$ can be uniquely written in the form $g = n_1awn_2$ with

$n_1 \in N$, $a \in A$ and $n_2 \in N_w^-$. Hence, if $S^-(w) = \{\alpha_1, ..., \alpha_l\}$ then every $g \in BwB$ can be written uniquely in the form

$$g = nawx_{\alpha_1}(r_1) \cdots x_{\alpha_l}(r_l)$$

with $a \in A$,$n \in N$ and $r_1, ..., r_l \in F$. The following lemma will play a crucial role in the proofs of our main results in this paper. It supplies us with a tool to move from smaller Bruhat cells to larger Bruhat cells and to cover G in an inductive way.

Lemma 2.6. *Let $w \in \mathbb{W}$ and assume that $S^-(w) = \{\alpha_1, ..., \alpha_l\}$. Assume also that height$(\alpha_i) \geq$ height(α_{i+1}) for $i = 1, ..., l-1$. Let $g \in BwB$ and assume that*

$$g = nawx_{\alpha_1}(r_1) \cdots x_{\alpha_t}(r_t)$$

with $t \leq l$. Assume also that $r_t \neq 0$. Let

$$g_1 = gx_{-\alpha_t}(-1/r_t)$$

Then $g_1 \in Bw_1B$ with $w_1 = ww_{\alpha_t}$ and in particular $w_1 < w$. Moreover, if $g_1 = n_1a_1w_1n_2$ is the unique decomposition of g_1 with $n_1 \in N$, $a_1 \in A$ and $n_2 \in N_{w_1}^-$ then

$$a_1 = ah_{w(\alpha_t)}(r_t)$$

Proof. By (2.6) we have $x_{\alpha_t}(r_t)x_{-\alpha_t}(-1/r_t) = w_{\alpha_t}h_{\alpha_t}(r_t)x_{\alpha_t}(-r_t)$. Hence

$$g_1 = nawx_{\alpha_1}(r_1) \cdots x_{\alpha_{t-1}}(r_{t-1})w_{\alpha_t}h_{\alpha_t}(r_t)x_{\alpha_t}(-r_t) \tag{2.22}$$

Since $w_{\alpha_t} = w_{\alpha_t}^{-1}$ it follows from (2.7) that

$$w_{\alpha_t}^{-1}x_{\alpha_i}(r_i)w_{\alpha_t} = x_{w_{\alpha_t}(\alpha_i)}(r_i).$$

Since height$(\alpha_i) \geq$ height(α_t) for $i < t$ we have that $w_{\alpha_t}(\alpha_i) > 0$. (To see this, write $\alpha_t = \alpha_{a,b}$, $\alpha_i = \alpha_{c,d}$ and $w_{\alpha_t} = w_{a,b}$. Since height$(\alpha_i) \geq$ height(α_t) it follows that $d - c > b - a$. The claim now follows from a case by case computation.) Hence by conjugating $w_{\alpha_t}h_{\alpha_t}$ across the expression in (2.22) we get that

$$g = na_1w_1n_3$$

with a_1 and w_1 as defined above and $n_3 \in N$. Hence it is clear that $g \in Bw_1B$. To get the unique form of g we decompose $n_3 = n_3^+n_3^-$ with $n_3^+ \in N_{w_1}^+$ and $n_3^- \in N_{w_1}^-$. We can move n_3^+ across a_1w_1 by conjugating to get the unique form of g_1. It is clear that a_1 gives the required torus part. ☐

3. Bases and Cones in a Eucledean space

We recall some facts about dual bases and cones in a Eucledean space. We shall apply these facts to the base Δ of $|V|$ defined in Section 2. Let E be an m dimensional vector space over $\mathbb{R}$ equipped with an inner product $< u, v >$. If $S = \{v_1, ..., v_r\}$ is a set of linearly independent vectors in E then we let $S^* = \{v_1^S, ..., v_r^S\}$ where v_i^S, $i = 1, ...r$, is in the linear subspace spanned by S and is determined by the following equations

$$< v_i^S, v_j >= \delta_{i,j}, \quad j = 1, ..., r. \tag{3.1}$$

Remark 3.1. In most cases we will be satisfied by finding a vector w_i in the linear span of S satisfying $< w_i, v_j >= 0$ for $i \neq j$ and $< w_i, v_i >> 0$. It is clear that w_i is a positive multiple of v_i^S and we will not bother normalizing w_i.

Remark 3.2. ([8], pg. 72 ex.7 and ex.8) Write $v_i^S = \sum c_{i,j} v_j$. Then $c_{i,i} > 0$. Moreover, if S is an obtuse set of vectors, that is, $< v_i, v_j > \leq 0$ for $i \neq j$ then $c_{i,j} \geq 0$ and $< v_i^S, v_j^S > \geq 0$ for every i and j.

Let $\Delta = \{v_1, ..., v_m\}$ be a fixed base of E. For $v \in \Delta$ we denote $v^* = v^\Delta$. Let $S \subset \Delta$.

Definition 3.3. *We denote by $\Delta(S)$ the set of m vectors where we replaced v with v^* when $\alpha \notin S$, That is, $\Delta(S) = \{w_1, w_2, ..., w_m\}$ is given by*

$$w_i = \begin{cases} v_i, & \text{if } v_i \in S \\ v_i^* & \text{if } v_i \notin S. \end{cases}$$

Lemma 3.4.

(a) *$\Delta(S)$ is a basis for E.*
(b) *The dual basis $\Delta(S)^* = \{u_1, ..., u_m\}$ is given up to positive scalar multiplications (see Remark 3.1) by*

$$u_i = \begin{cases} v_i^S, & \text{if } v_i \in S; \\ v_i^{S \cup \{v_i\}} & \text{if } v_i \notin S. \end{cases} \tag{3.2}$$

Proof. (a) Assume that $w = \sum c_i w_i = 0$ where $c_i \in \mathbf{R}$. Write $w = x_1 + x_2$ where $x_1 = \sum_{v_i \in S} c_i v_i$ and $x_2 = \sum_{v_j \notin S} c_j v_j^*$. It is clear that $< x_1, x_2 >= 0$ and since $w = 0$ we have $< x_1, w >=< x_2, w >= 0$. Hence $< x_1, x_1 >= 0$ and $< x_2, x_2 >= 0$ so $x_1 = x_2 = 0$. Since both Δ and Δ^* form a basis we get that $c_i = 0$ for all i.

(b) Assume $v_i \in S$ and let $u_i = v_i^S$. Then u_i is in the linear span of S hence $(u_i, v_j^*) = 0$ for all $v_j \notin S$. By definition $(u_i, v_j) = 0$ for all $v_j \in S$, $v_j \neq v_i$ and $(u_i, v_i) = 1$

If $v_i \notin S$ then $u_i = v_i^{S \cup \{v_i\}}$ is in the linear span of $S \cup \{v_i\}$. So $(u_i, v_j^*) = 0$ for all $v_j \notin S \cup \{v_i\}$. If $v_j \in S$ then by definition $(u_i, v_j) = 0$.

Since $\Delta(S)$ is a basis, and $u_i \neq 0$ we must have $< u_i, v_i^* > \neq 0$. If we write $u_i = cv_i + \sum_{v_j \in S} c_j v_j$ for $c, c_j \in \mathbf{R}$ then we have that $< u_i, v_i^* >= c$ and by Remark 3.2, $c > 0$. □

3.1. **Polyhedral cones.** Let S be a finite set of vectors in E. We define the cones

$$C(S) = \{\sum_{v \in S} c_v v : \ c_v \geq 0, \},$$
$$C^*(S) = \{u \in E : < u, v > \geq 0, v \in S\}.$$

If S is minimal then S is called a basis for $C(S)$. It is a well known theorem that these two representations of polyhedral cones are equivalent, that is, for every S there exist finite sets $T_1, T_2 \subset E$ such that $C(S) = C^*(T_1)$ and $C^*(S) = C(T_2)$. When $S = \Delta$ is a basis of E, this theorem is easy to prove and is summarized in the following Lemma:

Lemma 3.5.

$$C(\Delta) = C^*(\Delta^*), \quad C^*(\Delta) = C(\Delta^*).$$

We now assume that $\Delta = \{v_1, ..., v_m\}$ is an obtuse base of E, that is,

$$< v_i, v_j > \leq 0, \quad i \neq j.$$

Notice that our base Δ of $|V|$ of simple roots defined in (2.11) is obtuse.

By Remark 3.2 it follows that if Δ is an obtuse base then Δ^* is an acute base and

$$C(\Delta^*) \subseteq C(\Delta) \tag{3.3}$$

We will also need the following Lemma:

Lemma 3.6. *Let Δ be an obtuse base of E and let $S \subset \Delta$. Then*

$$C^*(S \cup \Delta^*) = C^*(\Delta(S)) = C(\Delta(S)^*)$$

Proof. The second equality follows from the fact that $\Delta(S)$ is a basis for E (see Lemma 3.4 (a)) and from Lemma 3.5. Here we do not need Δ to be an obtuse basis.

since $S \cup \Delta^* \supseteq \Delta(S)$ it follows that

$$C^*(S \cup \Delta^*) \subseteq C^*(\Delta(S))$$

To finish the proof will show that

$$C(\Delta(S)^*) \subseteq C^*(S \cup \Delta^*).$$

Let $\Delta = \{v_1, ..., v_m\}$ and $\Delta(S)^* = \{u_1, ..., u_m\}$ where by (3.2)

$$u_i = \begin{cases} v_i^S, & \text{if } v_i \in S; \\ v_i^{S \bigcup \{v_i\}} & \text{if } v_i \notin S. \end{cases}$$

Let $u \in C(\Delta(S)^*)$. Then $u = \sum c_i u_i$ with $c_i \geq 0$ for all i. To show that $u \in C^*(S \cup \Delta^*)$ it is enough to show that $< u_i, x > \geq 0$ for all $x \in S \cup \Delta^*$.

(i) Assume that $v_i \in S$, hence $u_i = v_i^S$.
If $x = v_j \in S$ then

$$< u_i, x >=< v_i^S, v_j >= \delta_{i,j} \geq 0.$$

If $x = v_j^* \in \Delta^*$ and $v_j \notin S$ then

$$< u_i, x >=< v_i^S, v_j^* >= 0$$

since v_i^S is in the span of S. If $x = v_j^* \in \Delta^*$ and $v_j \in S$ then we write $u_i = v_i^S = \sum_{v_t \in S} d_t v_t$. Since S is a set of linearly independent obtuse vectors it follows from Remark 3.2 that $d_i \geq 0$. Hence

$$< u_i, x >=< v_i^S, v_j^* >= \sum_{v_t \in S} d_t < v_t, v_j^* >= d_j < v_j, v_j^* > \geq 0$$

(ii) Assume $u_i = v_i^{S \bigcup \{v_i\}}$ where $v_i \notin S$. Similar arguments as above will show that $< u_i, x > \geq 0$ for all $x \in S \cup \Delta^*$ hence we are done. □

4. Method of Proof

The main method of proof in this paper is to use the Bruhat decomposition for a cell by cell analysis of the functions that we are interested in. It is important to understand how the Bruhat decomposition compares with the Iwasawa decomposition.

We present an explicit method of obtaining such information which follows a simple pattern. The idea is to analyze the Bruhat cells inductively going from the closed cell up to the open cell. The induction is on the height of the respective Weyl element. Another induction takes place inside an individual cell where we "peel" the root groups one by one. For this process we shall appeal repeatedly to Lemma 2.6 which allows us to obtain information on a larger cell from a smaller cell.

The main results in this paper are proved using this method. In this section we illustrate the method by proving a result that we will need later. This result is probably known to experts. For the case of $GL_3(F)$ see ([4], Section 3).

4.1. **Iwasawa decomposition.** Let $K = GL_n(O)$. It is well known that

$$G = NAK$$

For every $|\alpha| \in |X|$ we extend $|\alpha|$ (see [10]) to G by defining

$$|\alpha|(g) = |\alpha|(a) \tag{4.1}$$

where $g = nak$, $n \in N, a \in A, k \in K$, is an Iwasawa decomposition of g. It is easy to see that $|\alpha|$ is independent of the choice of decomposition.

Recall that $\Delta^* = \{\lambda_1, ..., \lambda_n\}$ is the set of fundamental weights where $\lambda_i = |\alpha|_{n-i,...,n-i,i,...,i}$ (See (2.3).) We view λ_i as a function on G as above. The main theorem of this section is the following:

Theorem 4.1. *Let $\lambda \in \Delta^*$ and $w \in \mathbb{W}$. Then*

$$\lambda(wn) \leq 1$$

for every $n \in N_w^-$.

Remark 4.2. Since every $n \in N$ can be written in the form $n = n_+n_-$ for $n_+ \in N_w^+$ and $n_- \in N_w^-$ it follows that

$$wn = wn_+n_- = n_+^w wn_-$$

where $n_+^w = w^{-1}n_+w \in N$. Hence $\lambda(wn) = \lambda(wn_-)$. It follows that the statement in the above Theorem is equivalent to the statement $\lambda(wn) \leq 1$ for all $n \in N$ and $\lambda \in \Delta^*$.

Proof. We will proceed with two inductions. The first induction is on $l(w)$.

$l(w) = 0$:

In this case $w = e$, $N_w^- = \{e\}$. Hence, $wn = e$. Since $\lambda(e) = 1$ we are done.

Now let $w \in \mathbb{W}$ be such that $l(w) > 0$ and assume that the Theorem is true for all $w_1 \in \mathbb{W}$ such that $l(w_1) < l(w)$.

We order the roots in

$$S^-(w) = \{\alpha \in \Phi^+ \,:\, w(\alpha) < 0\} = \{\alpha_1, ..., \alpha_l\}$$

as in Lemma 2.6 so that height$(\alpha_i) \geq$ height(α_{i+1}) for $i = 1, ..., l-1$. If $n \in N_w^-$ then we can write

$$n = x_{\alpha_1}(r_1)x_{\alpha_2}(r_2)\cdots x_{\alpha_t}(r_t) \tag{4.2}$$

with $r_1, ..., r_t \in F$ and $t \leq l$. Notice that we can always take $t = l$ at the cost of having the last r_is being zeroes. However, we are interested in having t as small as possible. We would like to prove that $\lambda_i(wn) \leq 1$. We shall do so by induction on t.

Since w is a permutation matrix whose entries are 0 or 1 it follows that $w \in K$. Hence if $t = 0$ then $wn = w$ and $\lambda_i(w) = 1$. So assume that n is of the form (4.2) with $t > 0$ and assume that the Theorem is true for $t-1$. We divide into two cases.

If $|r_t| \leq 1$ then $x_{\alpha_t}(r_t) \in K$ hence

$$\lambda(wn) = \lambda(wx_{\alpha_1}(r_1)\cdots x_{\alpha_{t-1}}(r_{t-1})x_{\alpha_t}(r_t)) = \lambda(wx_{\alpha_1}(r_1)\cdots x_{\alpha_{t-1}}(r_{t-1})).$$

Hence we can use our second induction assumption to conclude that $\lambda(wn) \leq 1$.

If $|r_t| \geq 1$ then $x_{-\alpha_t}(-r_t^{-1}) \in K$. Hence we have

$$\lambda(wn) = \lambda(wnx_{-\alpha_t}(-r_t^{-1})) = \lambda(wx_{\alpha_1}(r_1)\cdots x_{\alpha_t}(r_t)x_{-\alpha_t}(-r_t^{-1})).$$

By Lemma 2.6 we have that $wnx_{-\alpha_t}(-r_t^{-1}) = n_1a_1w_1n_2$ with $w_1 < w_2$, $n_1, n_2 \in N$ and $a_1 = h_{w(\alpha_t)}(r_t)$. Hence

$$\lambda(wn) = \lambda(h_{w(\alpha_t)}(r_t))\lambda(w_1n_2)$$

By Remark 2.1, we have $\lambda(h_{w(\alpha_t)}(r_t)) \leq 1$. By our first induction assumption we have $\lambda(w_1n_2) \leq 1$. Hence we get the result. □

5. Spaces of Whittaker functions

In this section we define a subspace of the space of Whittaker functions on G and prove some properties of this subspace. In particular we prove Theorem 5.7 which asserts that certain functions on unipotent subgroups are compactly supported. This is one of our main theorems in this paper.

5.1. **Whittaker functions.** Let ψ_F be a character of F and assume that ψ_F is identically one on O and nontrivial on P^{-1}. For a unipotent matrix $y \in N$ we set

$$\psi(y) = \psi_F(y_{1,2} + y_{2,3} + \dots + y_{n-1,n}). \tag{5.1}$$

Where $y_{i,j}$ are the entries of y. We let $\mathcal{W} = \mathcal{W}(G, \psi)$ be the set of functions $W : G \to \mathcal{C}$ such that W is smooth on the right and

$$W(ng) = \psi(n)W(g), \quad n \in N, g \in G.$$

Examples of such functions are Whittaker functions associated with generic representations of G. Other examples are given by projecting compactly supported and locally constant functions to this space as follows.

$$W_f(g) = \int_N f(ng)\psi^{-1}(n)dn, \quad f \in C_c^\infty(G).$$

we shall study the space of such functions $\{W_f : f \in C_c^\infty(G)\}$ in Section 9.

For every $|\alpha| \in |X|$ we extend $|\alpha|$ to G as in (4.1)

For $w \in \mathbf{W}$ we let $S^0(w)$ be the set of simple roots defined in (2.2). That is,

$$S^0(w) = S(ww_0)$$

where $S(ww_0)$ is defined in (2.18).

Definition 5.1. *Let $\mathcal{W} = \mathcal{W}(G, \psi)$ be the space of Whittaker functions defined above. We define $\mathcal{W}^0 = \mathcal{W}^0(G, \psi) \subset \mathcal{W}$ to be the set of functions $W \in \mathcal{W}$ such that for every $w \in \mathbb{W}$ and every $\alpha \in S^0(w)$ there exist positive constants $D_\alpha < E_\alpha$ such that if $g \in BwB$ then*

$$W(g) \neq 0 \implies D_\alpha < |\alpha|(g) < E_\alpha, \ \ \alpha \in S^0(w). \tag{5.2}$$

In other words, $W \in \mathcal{W}^0$ if for each $w \in \mathbb{W}$ and each $\alpha \in S^0(w)$ the support of W in BwB has bounded image under α.

Remark 5.2. The condition $|\alpha|(g) < E_\alpha$, $\alpha \in S^0(w)$ that appears in (5.2) is redundant since by [10] the support of W is contained in the set $\{g : |\alpha|(g) < C, \alpha \in \Delta\}$ for some positive number C. Moreover, if W is a Whittaker function in the Whittaker model of a supercuspidal representation of G then it follows from [10] that W is compactly supported mod NZ. Hence it follows that W satisfies condition (5.2) for every $\alpha \in \Delta$ and every $g \in G$ and in particular $W \in \mathcal{W}^0$.

Definition 5.3. *Let $\alpha \in \Delta$. We define the sets*

$$X_{C_1,C_2}(\alpha) = \{g \in G | C_1 < |\alpha|(g) < C_2\}, \quad A_{C_1,C_2}(\alpha) = \{a \in A | C_1 < |\alpha|(a) < C_2\}$$

and the sets

$$X_{C_1,C_2} = \bigcap_{\alpha \in \Delta} X_{C_1,C_2}(\alpha), \quad A_{C_1,C_2} = \bigcap_{\alpha \in \Delta} A_{C_1,C_2}(\alpha).$$

Lemma 5.4. *Let $\alpha \in \Delta$, $C_1 < C_2$ positive numbers and R a compact set in G. Then*
(a) There exist constants $C_1' < C_2'$ such that

$$X_{C_1,C_2}(\alpha) R \subset X_{C_1',C_2'}(\alpha)$$

(b) Let Y be a subset of G and assume that for every $y \in Y$ there exists $r \in R$ such that $yr \in X_{C_1,C_2}(\alpha)$. Then there exist positive constants $C_1' < C_2'$ such that $Y \subset X_{C_1',C_2'}(\alpha)$.

Proof. (a) We can write $X_{C_1,C_2}(\alpha) = NA_{C_1,C_2}(\alpha)K$. It is clear that $|\alpha|(X_{C_1,C_2}(\alpha)R) = |\alpha|(A_{C_1,C_2}(\alpha))|\alpha|(KR)$. Since KR is a a compact set in G and $|\alpha|$ is continuous the result follows.
(b) Let $y \in Y$ and let $y = n_0 a_0 k_0$ be an Iwasawa decomposition for y. If $r \in R$ then $|\alpha|(yr) = |\alpha|(y)||\alpha|(k_0 r)$. Since KR is a compact set, there exist positive constants $D_1 < D_2$ such that $D_1 < |\alpha(k_0 r)| < D_2$ for all $k_0 \in K$ and $r \in R$. By our assumption, there exists $r_0 \in R$ such that $C_1 < |\alpha|(yr_0) < C_2$. Hence $C_1/D_2 < |\alpha|(y) < C_2/D_1$ and we can choose $C_1' = C_1/D_2$ and $C_2' = C_2/D_1$. □

Corollary 5.5.
(a) The set $\mathcal{W}^0$ is invariant under right translations by B, i.e, if $W \in \mathcal{W}^0$ then for every $b \in B$, $W_b \in \mathcal{W}^0$ where $W_b(g) = W(gb)$.
(b) W^0 is invariant under right integration by compact open subset of closed subgroups of B, i.e, if H is a closed subgroup of B and $X \subset H$ is open and compact in H then for every $W \in \mathcal{W}^0$, $W_X \in \mathcal{W}^0$ where $W_X(g) = \int_X W(gh)dh$.

Proof. (a) We take R to be the Singleton, $R = \{b^{-1}\}$, where $b \in B$. By Lemma 5.4, $X_{C_1,C_2}(\alpha)b^{-1} \subset X_{C_1',C_2'}(\alpha)$. Thus if W restricted to the set

BwB is supported on $X_{C_1,C_2}(\alpha) \cap BwB$ then W_b restricted to BwB will be supported on the set $X_{C_1',C_2'}(\alpha) \cap BwB$.
(b) Since W is smooth on the right, W_X is a finite linear combination of W_{b_i} for some $b_i \in B$, hence (b) follows from (a) $\square$

For each $w \in W$ we let $\Delta(S^0(w))$ be the basis of $|V|$ defined in (3.3) and let $\Delta(S^0(w))^*$ be the dual basis (up to scalars) that we fixed in Lemma 3.4 (b). Let M be a positive constant. We define the cone $A^M(w) \subset A$ to be

$$A^M(w) = \{a \in A : |\beta|(a) < M, \quad \text{for all } \beta \in \Delta(S^0(w))^*\}$$

Lemma 5.6. *Let $w_1, w \in \mathbb{W}$ and $M > 0$. If $w_1 < w$ then there exists a constant $M_1 > 0$ such that*

$$A^M(w) \subset A^{M_1}(w_1).$$

Proof. By (2.19) and Lemma 3.6

$$C(\Delta(S^0(w_1))^*) \subseteq C(\Delta(S^0(w))^*).$$

Hence every $\lambda \in \Delta(S^0(w_1))^*$ can be written as a non-negative linear combination of elements in $\Delta(S^0(w))^*$. Thus there exists a constant $M_1 > 0$ such that $|\lambda|(a) < M_1$ for all $a \in A^M(w)$ and $\lambda \in \Delta(S^0(w_1))^*$. $\square$

Our first main theorem of this paper is the following:

Theorem 5.7. *Let $W \in \mathcal{W}^0$ and M a positive constant. Then the function*

$$(a, n) \mapsto W(awn)$$

defined on the set $A^M(w) \times N_w^-$ is compactly supported in N_w^-. That is, if $W(awn) \neq 0$ and $a \in A^M(w), n \in N_w^-$ then n is in some compact set independent of a.

Note that if $w = w_0$ then $S^0(w) = \emptyset$ hence $\Delta(S^0(w)) = \Delta^*$ and $\Delta(S^0(w))^* = \Delta$. It follows that $A^M(w) = A^M$ as defined in (1.4). Since $N_w^- = N$ in that case, Theorem 1.1 follows from the above Theorem.

Proof. Our proof will use a double induction argument as in the proof of Theorem 4.1. We begin by induction on $l(w)$.

$l(w) = 0$**:** That is, $w = e$.

In this case, $N_w^- = \{e\}$ and there is nothing to prove. Now let $w \in \mathbb{W}$ and assume the Theorem is true for all $w_1 \in \mathbb{W}$ such that $l(w_1) < l(w)$.

We order the roots in

$$S^-(w) = \{\alpha_1, ..., \alpha_l\}$$

as in Lemma 2.6 so that height$(\alpha_i) \geq$ height(α_{i+1}) for $i = 1, ..., l-1$. If $n \in N_w^-$ then we can write

$$n = x_{\alpha_1}(r_1) x_{\alpha_2}(r_2) \cdots x_{\alpha_t}(r_t) \tag{5.3}$$

with $r_1, ..., r_t \in F$ and $t \leq l$. Here we will use induction on t. The precise induction statement is the following: Fix $W \in \mathcal{W}^0$ and M a positive constant. Let $n \in N_w^-$ be written as in (5.3) and $a \in A^M(w)$. If

$$W(awn) \neq 0$$

then there exists a constant $C = C(W, w, M) > 0$ independent of a such that $|r_i| \leq C$ for $i = 1, ..., t$.

Assume $t = 1$. Then we can write $n = x_{\alpha_1}(r_1)$. We assume $W(awn) \neq 0$ with $a \in A^M(w)$. Since W is smooth on the right, there exists $D > 0$ so that

$$W(gx_{-\alpha_1}(-r_1^{-1})) = W(g)$$

for every $r_1 \in F$ such that $|r_1| > D$ and every $g \in G$. Hence if $|r_1| > D$ then

$$W(awn) = W(awx_{\alpha_1}(r_1)x_{-\alpha_1}(-r_1^{-1})) \neq 0$$

By Lemma 2.6 we have that $awx_{\alpha_1}(r_1)x_{-\alpha_1}(-r_1^{-1})) = n_1a_1w_1n_2$ with $w_1 = ww_{\alpha_1}$, $a_1 = ah_{w(\alpha_1)}(r_1)$ and $n_1, n_2 \in N$. More precisely, it is easy to see that in this case

$$awx_{\alpha_1}(r_1)x_{-\alpha_1}(-r_1^{-1})) = n_3ah_{w(\alpha_1)}(r_1)w_1$$

for some $n_3 \in N$. Set

$$\alpha_0 = w(\alpha_1)$$

We get that $W(awn) \neq 0$ implies that

$$W(ah_{\alpha_0}(r_1)w_1) \neq 0.$$

Set $S_1 = S^0(w_1)$. Since $w_1 \in K$ and since $W \in \mathcal{W}^0$ we have that for every $\beta \in C(S_1)$ there exists a positive constant D_β such that

$$\beta(ah_{w(\alpha_1)}(r_1)) \geq D_\beta \tag{5.4}$$

Since $a \in A^M(w)$ and since $w_1 < w$ it follows from Lemma 5.6 that there exists $M_1 > 0$ such that $a \in A^{M_1}(w_1)$. Hence for every $\gamma \in C(\Delta(S_1)^*) = C^*(\Delta(S_1))$ there exists $E_\gamma > 0$ depending only on M_1 and γ such that

$$\gamma(a) \leq E_\gamma \tag{5.5}$$

By Corollary 2.5 we have that $\alpha_0 \in \Phi(S_1)$. Since α_0 is negative it follows that

$$\alpha_0 = \sum_{\alpha \in S_1} c_\alpha \alpha \tag{5.6}$$

with $c_\alpha \leq 0$ for all $\alpha \in S_1$. If $\gamma \in C^*(\Delta(S_1))$ then it follows from the definition of $\Delta(S_1)$ that

$$< \gamma, \alpha > \geq 0, \quad \text{for all } \alpha \in S_1$$

Hence it follows from (5.6) that for all $\gamma \in C^*(\Delta(S_1))$

$$< \gamma, \alpha_0 > \leq 0.$$

Since $C^*(\Delta(S_1)) = C(\Delta(S_1)^*)$ contains the basis $\Delta(S_1)^*$ and since $\alpha_0 \neq 0$ it follows that there exists $\gamma_0 \in C(\Delta(S_1)^*)$ such that

$$< \gamma_0, \alpha_0 > < 0. \tag{5.7}$$

Let $\Delta(S_1)^* = \{\gamma_1, ..., \gamma_{n-1}\}$ where γ_i is defined by (3.2). Then we can write

$$\gamma_0 = \sum d_i \gamma_i$$

with $d_i \geq 0$. Since $\alpha_0 \in \Phi(S_1)$ it follows that $< \alpha_0, \gamma_i > = 0$ for all i such that $\alpha_i \notin S_1$. Hence we can (and will) assume that $d_i = 0$ for i such that $\alpha_i \notin S_1$. (That is, we are replacing γ_0 with $\tilde{\gamma}_0 = \sum_{\alpha_i \in S} d_i \gamma_i$. It is clear that $\tilde{\gamma}_0 \in C(\Delta(S_1)^*)$ and that $< \tilde{\gamma}_0, \alpha_0 > < 0$)

Since S_1 is an obtuse set we get that $\gamma_i = \alpha_i^{S_1}$ is in $C(S_1)$ for all i such that $\alpha_i \in S_1$. Hence $\gamma_0 \in C(S_1)$. Hence by (5.4)

$$\gamma_0(ah_{\alpha_0)}(r_1)) \geq D_{\gamma_0}$$

Since

$$\gamma_0(ah_{\alpha_0}(r_1)) = \gamma_0(a)\gamma_0(h_{\alpha_0}(r_1))$$

and since $\gamma_0(a) \leq E_{\gamma_0}$ by (5.5) we get that

$$\gamma_0(h_{\alpha_0}(r_1)) \geq \frac{D_{\gamma_0}}{E_{\gamma_0}}$$

Write $\gamma_0 = |\alpha|_{\lambda_1,...,\lambda_n}$ (see (2.3). and $\alpha_0 = |\alpha|_{i,j}$ (see (2.13)). Then by (5.7)

$$< \gamma_0, \alpha_0 > = < |\alpha|_{\lambda_1,...,\lambda_n}, |\alpha|_{i,j} > = \lambda_i - \lambda_j < 0$$

On the other hand

$$\gamma_0(h_{\alpha_0}(r_1)) = \gamma_0(h_{i,j}(r_1)) = |r_1|^{\lambda_i - \lambda_j} \geq \frac{D_{\gamma_0}}{E_{\gamma_0}}$$

Hence there exists $C > 0$ depending on W,w,α_1 and M but not on $a \in A^M(w)$ such that

$$W(awx_{\alpha_1}(r_1)) \neq 0 \;\Rightarrow\; |r_1| \leq C$$

We now prove the general case. Let $t > 1$. Assume that our second induction hypothesis holds for $t-1$. Let $S^-(w) = \{\alpha_1, ..., \alpha_l\}$ with height$(\alpha_i) \geq$ height(α_{i+1}) and let $n \in N_w^-$ be of the form

$$n = x_{\alpha_1}(r_1)x_{\alpha_2}(r_2) \cdots x_{\alpha_t}(r_t)$$

Let $a \in A^M(w)$ and assume that

$$W(awn) \neq 0$$

Let $D > 0$ be such that if $|r_t| \geq D$ then

$$W(gx_{-\alpha_t}(-r_t^{-1})) = W(g)$$

for all $g \in G$. Assume $|r_t| \geq D$. Then

$$W(awn) = W(awnx_{-\alpha_t}(-r_t^{-1})) \neq 0.$$

Let $g_1 = awnx_{-\alpha_1}(-r_t^{-1})$. and $\alpha_0 = w(\alpha_t)$. Then by Lemma 2.6

$$g_1 = n_1 a h_{\alpha_0}(r_t) w_1 n_2$$

with $n_1 \in N$, $w_1 = ww_{\alpha_t}$ and $n_2 \in N_{w_1}^-$. Since $W(g_1) \neq 0$ we get that

$$W(ah_{\alpha_0}(r_t)w_1 n_2) \neq 0$$

with $l(w_1) < l(w)$. We wish to invoke our first induction assumption for w_1. Notice that we have assumed that $a \in A^M(w)$ and that $|r_t| \geq D$. To use the induction assumption we need to show that $ah_{\alpha_t}(r_t) \in A^{M_2}(w_1)$ for some constant $M_2 > 0$ which depends only on D and M. By Lemma 5.6, $A^M(w) \subset A^{M_1}(w_1)$ for some constant $M_1 > 0$.

Let $S_1 = S^0(w_1)$. Let $\gamma \in \Delta(S_1)^*$. By the same arguments as in the $t = 1$ case we have that

$$< \gamma, \alpha_0 > \leq 0.$$

Hence, (see $t = 1$ case) $\gamma(h_{\alpha_0}(r_t)) = |r_t|^p$ with $p \leq 0$. It follows that

$$\gamma(h_{\alpha_0}(r_t)) \leq D^p$$

Now since

$$\gamma(ah_{\alpha_0}(r_t)) = \gamma(a)\gamma(h_{\alpha_0}(r_t)) \leq M_1 D^p$$

it follows that there exists $M_2 > 0$ such that $ah_{\alpha_0}(r_t) \in A^{M_2}(w_1)$ for all $a \in A^M(w)$ and $|r_t| \geq D$.

Now it follows from our induction hypothesis that n_2 is inside a compact set in N_w^- independent of a and r_t.

Since $W \in \mathcal{W}^0$ it follows that for every $\alpha \in S_1$ there exist positive constants $D_\alpha < E_\alpha$ such that

$$D_\alpha \leq \alpha(ah_{\alpha_0}(r_t)w_1 n_2) \leq E_\alpha$$

Since n_2 is in a fixed compact set it follows from Lemma 5.4 (a) that there exist positive constants $D'_\alpha \leq E'_\alpha$ such that

$$D'_\alpha \leq \alpha(ah_{\alpha_0}(r_t)) \leq E'_\alpha$$

Hence if $\beta \in C(S_1)$ there exists a positive constant D'_β such that

$$D'_\beta \leq \beta(ah_{\alpha_0}(r_t))$$

The proof now follows word for word the $t = 1$ case. That is, we find $\gamma_0 \in C^*(\Delta(S_1)) \cap C(S_1)$ such that $< \gamma_0, \alpha_0 > < 0$. Since $a \in A^{M_1}(w_1)$ there exists $M_{\gamma_0} > 0$ such that

$$\gamma_0(a) \leq M_{\gamma_0}$$

By (5.4)

$$\gamma_0(ah_{\alpha_0}(r_t)) = \gamma_0(a)\gamma_0(h_{\alpha_0}(r_t)) \geq D'_{\gamma_0}$$

hence

$$\gamma_0(h_{\alpha_0}(r_t)) \geq \frac{D'_{\gamma_0}}{M_{\gamma_0}}$$

Since $<\gamma_0,\alpha_0> < 0$ we have that $\gamma_0(h_{\alpha_0}(r_t)) = |r_t|^p$ with $p<0$ hence $|r_t|$ is bounded.

To summarize, we have just proved that if

$$W(awn) \neq 0$$

with $a \in A^M(w)$ and n written as in (5.3) then there exists $C_t > 0$ independent of a and $r_1, ..., r_{t-1}$ so that $|r_t| \leq C_t$.

It remains to prove that $r_1, ..., r_{t-1}$ are also bounded. Consider the space $\{\rho(x_{\alpha_t}(r_t))W \ : \ |r_t| \leq C_t\}$ where

$$\rho(x_{\alpha_t}(r_t))W)(g) = W(gx_{\alpha_t}(r_t)).$$

Since W is smooth on the right it follows that this space is spanned by a finite number of functions $W_1, ..., W_p$. By Corollary 5.5 (b), each such function is in $\mathcal{W}^0$. Let $a \in A^M(w)$ and $n' = x_{\alpha_1}(r_1) \cdots x_{\alpha_{t-1}}(r_{t-1})$. It follows from our induction assumption on t that for each such function W_i there exists a constant A_i such that

$$W_i(awn') \neq 0 \implies |r_j| < A_i, \ j = 1, ..., t-1 \tag{5.8}$$

Let $A = \max\{A_1, ...A_p\}$. Then it is clear that (5.8) holds with A replacing A_i. We now write

$$\begin{aligned} W(awn) &= W(awx_{\alpha_1}(r_1) \cdot \cdot x_{\alpha_{t-1}}(r_{t-1})x_{\alpha_t}(r_t)) \\ &= c_1(r_t)W_1(awx_{\alpha_1}(r_1) \cdot \cdot x_{\alpha_{t-1}}(r_{t-1})) + ... + c_p(r_t)W_l(awx_{\alpha_1}(r_1) \cdot \cdot x_{\alpha_{t-1}}(r_{t-1})) \end{aligned}$$

for every r_t such that $|r_t| \leq C_t$. If $W(awn) \neq 0$ then at least one of the summands does not vanish and we can conclude that $|r_i| \leq A$ for $i = 1, .., t-1$. Now taking $C = \max\{A, C_t\}$ we get our result □

6. Bessel functions for supercuspidal representations

In this section we attach Bessel functions to irreducible supercuspidal representations of $G = GL_n(F)$. This section is not needed in the sequel since we will later attach Bessel function to every irreducible generic representation of $GL_n(F)$. The reason we include this section is that the situation for supercuspidal representations is nicer than the general situation and both the formulas and proofs are simpler.

Lemma 6.1. *Let (π, V) be a supercuspidal representation of G and let $\mathcal{W}(\pi,\psi)$ be the Whittaker model of π (see (1.2)). Then $\mathcal{W}(\pi,\psi) \subset \mathcal{W}^0(G,\psi)$.*

Proof. Let $W \in \mathcal{W}(\pi,\psi)$. Then by [10], W is compactly supported mod NZ. It follows that for every $\alpha \in \Delta$ and every $w \in \mathbb{W}$ the support of W in BwB has bounded image under α. (Since the support of W in G already has bounded image under α.) Hence $W \in \mathcal{W}^0$. □

The main result that allows the definition of the Bessel functions is the following: (For the proof see Corollary 1.2).

Corollary 6.2. *Let* (π, V) *be a supercuspidal representation of* G *and let* $W \in \mathcal{W}(\pi, \psi)$. *Let* $w \in \mathbb{W}$ *and fix* $g \in BwB$. *Then the function*

$$n \to W(gn)$$

from N_w^- *to* $\mathbf{C}$ *is compactly supported in* N_w^-.

Let $w \in \mathbb{W}$. We define the subtorus A_w to be

$$A_w = \{a \in A : \psi(n) = \psi(n^{aw}), \text{ for all } n \in N_w^+ .\} \tag{6.1}$$

Here $n^g = gng^{-1}$. It is easy to see that $A_e = Z(G)$ and that $A_{w_0} = A$.

Definition 6.3. *We say that* w *is a relevant Weyl element if* $A_w \neq \emptyset$.

It is well known (see [11]) that w is relevant if and only if $w = w_0 w_S$ where $S \subset \Delta$ and w_S is the longest Weyl element in the standard parabolic subgroup given by S. The set of relevant Weyl elements is the set of Weyl element of the form

$$\begin{pmatrix} & & & & I_{m_1} \\ & & & I_{m_2} & \\ & & \cdot & & \\ & \cdot & & & \\ I_{m_l} & & & & \end{pmatrix}$$

where I_m is identity matrix of order m and $m_1 + m_2 + \ldots + m_l = n$.

Fix a relevant Weyl element w and fix $g \in NA_wwN = NA_wwN_w^-$. Let (π, V) be an irreducible supercuspidal representation of G. Let $W \in \mathcal{W}(\pi, \psi)$. Define

$$L_g(W) = \int_{N_w^-} W(gn)\psi^{-1}(n)dn$$

By Corollary 6.2 this integral is absolutely convergent. Let G act on $\mathcal{W}(\pi, \psi)$ by right translations as in (1.3).

Lemma 6.4.

$$L_g(\rho(n)W) = \psi(n)L_g(W). \quad n \in N.$$

Proof. This is obvious if $n \in N_w^-$. Assume $n_1 \in N_w^+$. Then for $n \in N_w^-$ and $g = n_2awn_3$ with $n_2 \in N$, $a \in A_w$ and $n_3 \in N_w^-$ we have

$$\begin{aligned} W(gnn_1) &= W(n_2awn_3nn_1) \\ &= \psi(n_2)W(awn_1^{n_3n}n_3n_1) \\ &= \psi(n_2)\psi(n_1^{n_3n})W(awn_3n_1) \\ &= \psi(n_1)W(n_2awn_3n) \\ &= \psi(n_1)W(gn) \end{aligned}$$

Here we have used that N_w^- normalizes N_w^+ and that $\psi(n^{-1}n^+n) = \psi(n^+)$ for every $n \in N_w^-$ and $n^+ \in N_w^+$. Writing $(\rho(n_1)W)(gn) = W(gnn_1)$ and computing the integral defining $L_g(\rho(n_1)W)$ we get our result for $n \in N_w^+$. Since every $n \in N$ can be written in the form $n = n^+n_-$ for some $n^+ \in N_w^+$ and $n_- \in N_w^-$ we get our result for a general $n \in N$. □

It follows that L_g is a Whittaker functional on π. Hence by the uniqueness of the Whittaker functional we get that there exists a scalar $j_{\pi,\psi,w}(g) \in \mathbf{C}$ such that

$$L_g(W) = j_{\pi,\psi,w}(g)W(e), \quad W \in \mathcal{W}(\pi,\psi). \tag{6.1}$$

$j_{\pi,\psi,w}$ is a function on NA_wwN which we call the Bessel function associated to π and w. We shall show in Section 8 that $j_{\pi,\psi,w}$ is locally constant on NA_wwN. When $w = w_0$ we set $j_\pi = j_{\pi,\psi} = j_{\pi,\psi,w_0}$. To get a formula for $j_{\pi,\psi,w}$ we notice that since π is supercuspidal there exists a function $W \in \mathcal{W}(\pi,\psi)$ such that $W(e) = 1$. (This follows from the existence of a nontrivial Whittaker functional on π.) Hence we get from (6.1) that

Corollary 6.5. *Let π be a supercuspidal representation of G and let w be a relevant Weyl element. Then there exists $W \in \mathcal{W}(\pi,\psi)$ such that*

$$j_{\pi,\psi,w}(g) = \int_{N_w^-} W(gn)\psi^{-1}(n)dn, \quad g \in NA_wwN.$$

7. Projection into $\mathcal{W}^0(G,\psi)$

In this section we shall show that every $W \in \mathcal{W}(G,\psi)$ can be projected into $\mathcal{W}^0(G,\psi)$ by integrating it on a compact unipotent group versus a character of that group. We start with some preliminary results about Howe vectors. The proofs can be found in ([4], Section 5).

7.1. **Howe vectors.** For a positive integer m we denote by K_m the congruence subgroup of K given by $K_m = e + M_n(P^m)$. We let $A_m = A \cap K_m$. Let

$$d = \begin{pmatrix} 1 & & & & & \\ & \varpi^2 & & & & \\ & & \varpi^4 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \varpi^{2n-2} \end{pmatrix}$$

Let $J_m = d^mK_md^{-m}$.Notice that J_m is expanding above the main diagonal and shrinking on and below the main diagonal. Let

$$N_m = N \cap J_m. \tag{7.1}$$

Let $\bar{N}_m = \bar{N} \cap J_m$ and $\bar{B}_m = \bar{B} \bigcap J_m$. Using similar properties of K_m, it is easy to see that

$$J_m = \bar{N}_m A_m N_m = \bar{B}_m N_m.$$

Moreover, for $\alpha \in \Phi^+$ let

$$J_\alpha = N_\alpha \cap J_m = \{x_\alpha(r) : \; |r| \leq q^{(2\text{height}(\alpha)-1)m}\} \tag{7.2}$$

and for $\alpha \in \Phi^-$ let

$$J_\alpha = N_\alpha \cap J_m = \{x_\alpha(r) : \; |r| \leq q^{-(2\text{height}(\alpha)+1)m}\}. \tag{7.3}$$

Then

$$N_m = \prod_{\alpha \in \Phi^+} J_\alpha, \quad \bar{N}_m = \prod_{\alpha \in \Phi^-} J_\alpha \tag{7.4}$$

We fix a character ψ_F on F as in Section 3. In particular $\psi_F = 1$ idenitically on the ring of integers O and $\psi_F(P^{-1}) \neq 1$. Let ψ be a character of N obtained from ψ as in (5.1). For $m \geq 1$ we define a character ψ_m on J_m by

$$\psi_m(j) = \psi(n_j)$$

where $j = \bar{b}_j n_j$, $\bar{b}_j \in \bar{B}_m$, $n_j \in N_m$ is the unique decomposition of j. It is easy to see that ψ_m is a character on J_m. For each $W \in \mathcal{W}(G, \psi)$ we define $W_m = W_{N_m,\psi}$ by

$$W_m(g) = W_{N_m,\psi}(g) = \int_{N_m} W(gn)\psi^{-1}(n)dn. \tag{7.5}$$

Since $N_{m+1} \supset N_m$ it is a simple application of Fubini to show that if $m \geq k$ then

$$W_m(g) = \text{vol}(N_k)^{-1} \int_{N_m} W_k(gn)\psi^{-1}(n)dn. \tag{7.6}$$

For $g_1 \in G$ we let $(\rho(g_1)W)(g) = W(gg_1)$. The proof of the following Lemma is the same as the proof of Lemma 5.1 in [4].

Lemma 7.1. *Let M be such that $\rho(K_M)W = W$ and let m be an integer such that $m > 3M$. Then*

$$\rho(j)W_m = \psi_m(j)W_m, \;\; j \in J_m. \tag{7.7}$$

Formulating Lemma 7.1 for functions we get that for $m > 3M$

$$W_m(gj) = \psi_m(j)W_m(g) \;\text{ for all }\; g \in G, j \in J_m. \tag{7.8}$$

We call a vector W in a representation space of G satisfying (7.7) (or (7.8)) a Howe vector. The above Lemma shows that if the representation space affords a nontrivial Whittaker functional then non-zero Howe vectors exist. This property and some uniqueness properties of Howe vectors for irreducible admissible representations of $GL_n(F)$ were established in [7]. We now continue to study the behavior of Whittaker functions satisfying (7.8).

Lemma 7.2. *Let $w \in \mathbb{W}, a \in A$ and $\alpha \in S^0(w)$. Assume $W \in \mathcal{W}$ satisfies (7.8) for some $m \geq 1$. Then*

$$W(aw) \neq 0 \implies \alpha(a) \in 1 + P^m$$

Proof. We divide into two cases. First assume that w is not relevant (see Definition 6.3) , that is, w is not of the form $w = w_S w_0$ for some subset S of simple roots. (Notice that $\{w_S w_0 : S \subset \Delta\} = \{w_0 w_S : S \subset \Delta\}$.) Then by [13] Lemma 89, there exists a simple root β such that $\alpha = w(\beta) > 0$ but $w(\beta)$ is not a simple root. Let $r \in P^{-m}$. Then

$$\psi_F(r)W(aw) = W(awx_\beta(r)) = W(x_\alpha(\alpha(a)r)aw) = W(aw) \tag{7.9}$$

By our assumptions on the conductor of ψ there exists $r \in P^{-m}$, such that $\psi_F(r) \neq 0$. Hence $W(aw) = 0$ and our statement is trivially true.

Assume $w = w_S w_0$. Then $S = S^0(w)$. (See [9], Section 1.8, ex.2). Let $\alpha \in S$ and let $\beta = w_0 w_S(\alpha)$. β is a positive simple root. Arguing as in (7.9) we get

$$\psi_F(r)W(aw) = \psi_F(\alpha(a)r)W(aw)$$

for all $|r| \leq q^{-m}$. Hence $W(aw) \neq 0$ implies that $\alpha(a) - 1 \in P^m$ which is the required conclusion. □

Our main theorem of this section is the following. It implies (and in fact is equivalent to) Theorem 1.3 in the introduction.

Theorem 7.3. *Let $W \in \mathcal{W}(G, \psi)$. Then there exists a positive integer M such that $W_m = W_{N_m,\psi} \in \mathcal{W}^0(G, \psi)$ for every $m \geq M$.*

Proof. We need to show that there exists M such that for every fixed $m \geq M$ and every $w \in \mathbb{W}$, the support of W_m in BwB has bounded image under every $|\alpha| \in S^0(w)$. In other words, the statement of the theorem is equivalent to the following statement:

(A) Fix $w \in \mathbb{W}$ and $\alpha \in S^0(w)$. Then there exists an integer $M > 0$ and constants $C < D$ (depending on m) such that if $g \in BwB$ and $W_m(g) \neq 0$ then $C < |\alpha(g)| < D$.

We shall prove statement (A) by induction on $l(w)$.

$l(w) = 0$**:** That is, $w = e$.

In this case $BwB = B = NA$. By Lemma 7.1 there exists a positive integer M such that W_m satisfies (7.8) for every $m \geq M$. Let $m \geq M$ and assume that $g = na$ is in the support of W_m. Then $W_m(g) = W_m(na) = \psi(n)W_m(a) \neq 0$. Hence $W_m(a) \neq 0$ and by Lemma 7.2, $\alpha(a) \in 1 + P^m$ for every $\alpha \in S^0(e) = \Delta$. Since $\alpha(g) = \alpha(a)$ we get statement (A) for $w = e$.

For the general case, fix $w \in \mathbb{W}$, $w \neq e$. Let

$$S^-(w) = \{\alpha_1, ..., \alpha_l\}$$

and assume that $\text{height}(\alpha_i) \geq \text{height}(\alpha_{i+1})$ for $i = 1, ..., l-1$. We can write every $g \in BwB$ uniquely in the form

$$g = nawn_1 = nawx_{\alpha_1}(r_1)x_{\alpha_2}(r_2)\cdots x_{\alpha_l}(r_l) \tag{7.10}$$

with $n \in N$, $a \in A$, $r_1, ..., r_l \in F$. Here

$$n_1 = x_{\alpha_1}(r_1)x_{\alpha_2}(r_2)\cdots x_{\alpha_t}(r_t) \tag{7.11}$$

First case: Fix $\alpha \in S^0(w)$. Assume $W_m(g) \neq 0$ and assume that $n_1 \in N_m$. Then $W_m(g) = \psi(n_1)W_m(gn_1^{-1})$ hence if we let $g_1 = gn_1^{-1}$ we get that $W_m(g_1) \neq 0$. Now $W_m(g_1) = W_m(naw) = \psi(n)W_m(aw)$. Hence $W_m(aw) \neq 0$ and by Lemma 7.2, $\alpha(a)$ is in a compact set. Since $w \in K$ we have that $\alpha(g_1) = \alpha(a)$. Hence we proved that if g is of the form (7.10) with $n_1 \in N_m$ and $W(g) \neq 0$ then there exists $r_g \in R = N_m$ such that $\alpha(gr_g)$ is in a fixed compact set. By Lemma 5.4 (b), $\alpha(g)$ is in a fixed compact set.

We shall now consider the second case where $n_1 \notin N_m$. If $n_1 \notin N_m$ then there exists i such that $x_{\alpha_i}(r_i) \notin N_m$. This is equivalent to $r_i = r_i(g)$ satisfying $|r_i(g)| > q^{(2j+1)m}$ where $j = \text{height}(\alpha_i)$. (See (7.2)).

Lemma 7.4. *Let m be a positive integer and assume that $g \in BwB$ is of the form*

$$g = nawn_1 = nawx_{\alpha_1}(r_1)x_{\alpha_2}(r_2)\cdots x_{\alpha_i}(r_i)$$

with $i \leq l$ and $x_{\alpha_i}(r_i) \notin N_m$. Let $n_2 \in N_m$ and let $g_1 = gn_2$. Then in the decomposition of g_1 into (7.10) we have $|r_i(g_1)| = |r_i(g)| = |r_i|$.

Proof. We can write

$$n_2 = n_3x_{\alpha_i}(b_i)x_{\alpha_{i+1}}(b_{i+1})\cdots x_{\alpha_l}(r_l)$$

with n_3 a product over the positive root subgroups that are different than $\alpha_i, ..., \alpha_t$. This decomposition of n_2 is unique and $x_{\alpha_j}(b_j) \in N_m$ for $j = i, ..., l$. It follows from (2.8) that $r_j(g_1) = b_j$ for $j = i+1, ..., l$ and that $r_i(g_1) = r_i + b_i$. Since $|r_i| > q^{(2\text{height}(\alpha_i)+1)m}$ and since $|b_i| \leq q^{(2\text{height}(\alpha_i)+1)m}$ we get that $|r_i(g_1)| = |r_i(g)|$. □

Fix $\alpha \in S^0(w)$. To finish the proof we need to show that there exists a positive integer M such that if $m \geq M$ and if $W_m(g) \neq 0$ for g of the form (7.10) with $n_1 \notin N_m$ then $\alpha(g)$ is in a fixed compact set. Since $n_1 \notin N_m$ there exists a maximal i in the decomposition of n_1 in (7.11) such that $x_{\alpha_i}(r_i) \notin N_m$. We shall prove our Theorem by downward induction on this maximal i. That is, our second induction statement is the following:

(B) Fix i, $1 \leq i \leq l$. There exists an integer $M > 0$ such that if $m \geq M$ and if $W_m(g) \neq 0$ with g of the form (7.10) with $x_{\alpha_i}(r_i) \notin N_m$ and i is the maximal such index than $\alpha(g)$ is in a fixed bounded set (depending on m, w and W but not on such g).

We consider the case $i = l$. Let M_1 be such that for $m \geq M_1$, W_m satisfies (7.8) and such that W_m satisfies the induction assumption (A) for every $w_1 \in \mathbb{W}$ such that $l(w_1) < l(w)$ and for every $\alpha_1 \in S^0(w_1)$. That is, we assume that if $g \in Bw_1B$ and if $W_m(g) \neq 0$ then $\alpha_1(g)$ is in a fixed compact set. We can enlarge this fixed compact set to be good for every such w_1 and every such α_1. Let $M = 3M_1$. Assume $m \geq M$ and assume that $W_m(g) \neq 0$ where g is of the form (7.10) with $x_{\alpha_l}(r_l) \notin N_m$. By our assumption that $\text{height}(\alpha_{j+1}) \leq \text{height}(\alpha_j)$ we have that α_l is a simple root and $|r_l| > q^m$. By (7.6) we have

$$W_m(g) = \frac{1}{\text{vol}(N_{M_1})} \int_{N_m} W_{M_1}(gn)\psi^{-1}(n)dn$$

Since $W_m(g) \neq 0$ there exists $n_2 \in N_m$ such that $W_{M_1}(gn_2) \neq 0$. Let $g_1 = gn_2$. By Lemma 7.4 we have that

$$|r_l(g_1)| = |r_l(g)| > q^m \geq q^{3M_1}$$

It follows that $x_{-\alpha_l}(-1/r_l(g_1)) \in J_{M_1}$, hence by (7.8)

$$W_{M_1}(g_1 x_{-\alpha_l}(-1/r_l(g_1))) = W_{M_1}(g_1) \neq 0$$

Let $g_2 = g_1 x_{-\alpha_l}(-1/r_l(g_1))$. By Lemma 2.6, $g_2 \in Bw_1B$ with $l(w_1) < l(w)$. Moreover, by (2.19), $\alpha \in S^0(w_1)$. Hence by our assumptions on M_1 above, $W_{M_1}(g_2) \neq 0$ implies that $\alpha(g_2)$ is in a fixed compact set. Hence we proved that for every g satisfying the conditions above such that $W_m(g) \neq 0$ there exists $r_g \in R = N_m J_{M_1}$ such that $\alpha(gr)$ is in a fixed compact set. By Lemma 5.4 (b), $\alpha(g)$ is in a fixed compact set.

We now prove the general case. Fix $1 \geq i < l$. Let M_1 be as in the case $i = l$. By our induction assumption (B) we can also assume (by enlarging M_1) that if $m \geq M_1$ and if $W_m(g) \neq 0$ and if g is of the form (7.10) with $x_{\alpha_j}(r_j) \notin N_m$ for some $j > i$ then $\alpha(g)$ is in a fixed compact set.

Let $M = 3M_1$ and let $m \geq M$. Assume that $W_m(g) \neq 0$ where g is in the form (7.10) with $x_{\alpha_i}(r_i) \notin N_m$ and $x_{\alpha_j}(r_j) \in N_m$ for $j > i$. Then

$$W_m(g) = \psi_F(r_{i+1}) \cdots \psi_F(r_l) W_m(nawx_{\alpha_1}(r_1)) \cdots x_{\alpha_i}(r_i)) \neq 0$$

Let $n_2 = x_{\alpha_l}(-r_1)) \cdots x_{\alpha_{i+1}}(-r_i))$ Then $n_2 \in N_m$ and the above equation implies that for $g_1 = gn_2 = nawx_{\alpha_1}(r_1)) \cdots x_{\alpha_i}(r_i)$ we have $W_m(g_1) = W_m(gn_2) \neq 0$. We also have

$$W_m(g_1) = \frac{1}{\text{vol}(N_{M_1})} \int_{N_m} W_{M_1}(g_1 n)\psi^{-1}(n)dn$$

Since $W_m(g_1) \neq 0$ it follows that there exists $n_3 \in N_m$ such that $W_{M_1}(g_1 n_3) \neq 0$. Let $g_2 = g_1 n_3$ By Lemma 7.4 we have that $|r_i(g_2)| = |r_i(g)| = |r_i|$. We divide into two cases. First assume that there exists $j > i$ such that $x_{\alpha_j}(r_j(g_2)) \notin N_{M_1}$. Then it follows by our assumptions on M_1 that $\alpha(g_2)$ is in a fixed compact set. Since $g_2 = gr_g$ for $r \in R = N_m$ we get that $\alpha(g)$ is in a fixed compact set.

Next assume that $x_{\alpha_j}(r_j(g_2)) \in N_{M_1}$ for every $j > i$. Using (7.8) as above we get that $W_{M_1}(g_3) \neq 0$ where

$$g_3 = nawx_{\alpha_1}(r_1(g_2)) \cdots x_{\alpha_i}(r_i(g_2))$$

and $g_3 = g_2 n_4$ with $n_4 \in N_{M_1}$ Since

$$|r_i(g_2)| = |r_i(g)| > q^{(2\text{height}(\alpha_i)+1)m} \geq q^{(2\text{height}(\alpha_i)+1)3M_1}$$

it follows from (7.3) that $x_{-\alpha_l}(-1/r_i(g_2)) \in J_{M_1}$ hence by the same arguments as in the case $i = l$ we get that $\alpha(g)$ is in a fixed compact set. □

8. Bessel functions

In this section we attach Bessel functions to irreducible generic representation of $G = GL_n(F)$. The definition of these functions depend on Theorem 5.7 and Theorem 7.3 and is identical to the definition of the Bessel functions in [4]. Since the proofs are the same as in ([4], Section 6) we shall omit them. Given an irreducible generic representation of G we will attach a Bessel function for each relevant Weyl element w. This Bessel function will be defined on a subset of BwB and will be locally constant there. If the representation is supercuspidal then our definition here will coincide with the definition in Section 6 making Section 6 redundant. We are primarily interested in the Bessel function which is attached to the longest Weyl element w_0 which we call the main (or principal) Bessel function. We shall provide full proofs in this case for the sake of completeness.

Let $w \in \mathbb{W}$ be a relevant Weyl element. That is, there exists $S \subset \Delta$ such that $w = w_S w_0$. Let N_w^+ and N_w^- be the subgroups of N as defined in (2.21). We define the subtorus A_w as in Section 6 to be

$$A_w = \{a \in A : \psi(n) = \psi(n^{aw}), \text{ for all } n \in N_w^+ .\}$$

Here $n^g = gng^{-1}$. Let (π, V) be an irreducible generic representation of G and let $W \in \mathcal{W}(\pi, \psi)$. By Theorem 7.3 there exists a positive integer M such that if $m \geq M$ then $W_m \in \mathcal{W}^0$ (See (7.5) for the definition of W_m.) Fix $m \geq M$ and let $g \in NA_w wN_w^-$. We define

$$L_{g,w}(W) = \frac{1}{\text{vol}(N_m)} \int_{N_w^-} W_m(gn)\psi^{-1}(n)dn \tag{8.1}$$

By writing $g = n_1 awn_2$ and using Theorem 5.7 it follows that the integral above converges. (See also Corollary 6.2). The main result of this section is the following:

Proposition 8.1.
(a) $L_{g,w}(W)$ *is independent of* $m \geq M$.
(b) $L_{g,w}$ *is a Whittaker functional on* $\mathcal{W}(\pi,\psi)$*, that is, for every* $n \in N$, $L_{g,w}(\pi(n)W) = \psi(n)L_{g,w}(W)$.
(c) If $W \in \mathcal{W}^0$ *then*

$$J_{g,w}(W) = \int_{N_w^-} W(gn)\psi^{-1}(n)dn.$$

The proof is the same as in ([4] Proposition 6.1.) We will prove the Proposition for the case $w = w_0$. (see also the introduction for the case $w = w_0$.)

Proof. In that case $N_w^- = N$, $N_w^+ = \{e\}$ and $A_w = A$. Using of the Fubini theorem, it is easy to see that if $m_1 \geq m \geq M$ then

$$\frac{1}{\mathrm{vol}(N_m)}\int_{N_{m_1}} W_m(gn)\psi^{-1}(n)dn = \int_{N_{m_1}} W(gn)\psi^{-1}(n)dn.$$

Since N_{m_1} cover N when $m_1 \to \infty$ it follows that

$$\begin{aligned} L_{g,w_0}(W) &= \frac{1}{\mathrm{vol}(N_m)}\int_N W_m(gn)\psi^{-1}(n)dn \\ &= \lim_{m_1\to\infty}\frac{1}{\mathrm{vol}(N_m)}\int_{N_{m_1}} W_m(gn)\psi^{-1}(n)dn \\ &= \lim_{m_1\to\infty}\int_{N_{m_1}} W(gn)\psi^{-1}(n)dn \end{aligned}$$

Now (a) and (c) follow from the last line of the above equation. For part (b) we fix $n_1 \in N$ and consider the above limit for $\rho(n_1)W$. Since N_m cover N we have that there exists M_1 such that $n_1 \in N_m$ for all $m \geq M_1$. Now a simple change of variable in the integral above will give the result. □

By (b), $L_{g,w}$ is a Whittaker functional, hence by the uniqueness of Whittaker functionals it follows that there exists a scalar $j_{\pi,\psi,w}(g)$ such that

(8.2) $$L_{g,w}(W) = j_{\pi,\psi,w}(g)W(e) \quad g \in NA_wwN, W \in \mathcal{W}(\pi,\psi).$$

We call $j_{\pi,\psi,w}(g)$ the Bessel function attached to w and denote by $j_{\pi,\psi} = j_{\pi,\psi,w_0}(g)$ the Bessel function attached to π. It is easy to see that

(8.3) $$j_{\pi,\psi,w}(n_1gn_2) = \psi(n_1)\psi(n_2)j_{\pi,\psi,w}(g), \quad g \in NA_wwN, n_1, n_2 \in N.$$

Lemma 8.2. *There exists* $W \in \mathcal{W}^0(\pi,\psi)$ *such that*

$$j_{\pi,\psi,w}(g) = \int_{N_w^-} W(gn)\psi^{-1}(n)dn, \quad g \in NA_wwN.$$

Proof. It follows from Theorem 7.3 that there exists $W \in \mathcal{W}^0(\pi,\psi) = \mathcal{W}(\pi,\psi)\cap\mathcal{W}^0(G,\psi)$ such that $W(e) = 1$. The result now follows from from Proposition 8.1 (c). □

Corollary 8.3. *$j_{\pi,\psi,w}(g)$ is locally constant on NA_wwN.*

Proof. By (8.3) it is enough to prove that $j_{\pi,\psi,w}(g)$ is locally constant on A_ww. Let W be as in Lemma 8.2. By Theorem 5.7 $n \mapsto W(awn)$ is compactly supported on the set $(A^M(w) \cap A_w) \times N_w^-$. It follows from Lemma 8.2 that $j_{\pi,\psi,w}$ is locally constant on $A^M(w) \cap A_w)$. Since $A^M(w)$ cover $A_w)$ when $M \to \infty$ we get our result. □

We end this section by describing the Bessel functions attached to the contragredient representation.

Lemma 8.4. *Let π be a generic representation of G and $\tilde{\pi}$ the representation contragredient to π. Then*

$$j_{\tilde{\pi},\psi^{-1},w}(g) = j_{\pi,\psi,w_0ww_0}(g^{-1}) \quad g \in NA_wwN.$$

Proof. Let τ be the involution of G defined by $\tau(g) = w_0^t g^{-1} w_0$ For each $W \in \mathcal{W}(\pi,\psi)$ we define $W^\tau(g) = W(\tau(g))$. By [10] the mapping $W \mapsto W^\tau$ is a bijection between $\mathcal{W}(\pi,\psi)$ and $\mathcal{W}(\tilde{\pi},\psi^{-1})$. If $g \in G$ is written in the Iwasawa decomposition in the form $g = nak$ where n is upper triangular a is diagonal and $k \in GL_n(O)$ then

$$\tau(g) = (w_0^t n^{-1} w_0)(w_0 a^{-1} w_0)(w_0^t k^{-1} w_0)$$

is an Iwasawa decomposition for $\tau(g)$. Hence, if $W \in \mathcal{W}^0(\pi,\psi)$ then $W^\tau \in \mathcal{W}^0(\tilde{\pi},\psi^{-1})$. Using Lemma 8.2 we get that

$$j_{\tilde{\pi},\psi^{-1},w}(g) = j_{\pi,\psi,\tau(w)}(\tau(g)) \quad g \in NA_wwN.$$

Now $\tau(w) = w_0ww_0$ and we claim that $j_{\pi,\psi,w_0ww_0}(\tau(g)) = j_{\pi,\psi,w_0ww_0}(g^{-1})$ for all $g \in NA_wwN$.. Since both functions satisfy (8.3) with ψ^{-1} replacing ψ it is enough to show that they coincide on the set A_ww. Since $\tau(g) = g^{-1}$ for all $g \in A_ww$ we get our result. □

Corollary 8.5. *Let $j_{\pi,\psi} = j_{\pi,\psi,w_0}$ be the Bessel function attached to π. Then*

$$j_{\tilde{\pi},\psi^{-1}}(g) = j_{\pi,\psi}(g^{-1}), \quad g \in Bw_0B.$$

9. Orbital integrals

In this section we show that the Bessel functions for the longest Weyl element (the main Bessel function) defined in Section 8 are given locally by orbital integrals. These integrals were studied in [11]. We will do this in two steps. We will show that the Bessel function restricted to a compact set in G is given by an integral of a Whittaker function which is compactly supported mod N. That is, if we restrict ourselves to this small neighborhood, we can replace a Whittaker function in the representation space with a different Whittaker function (not necessarily in the representation space) which is compactly supported mod NZ. Then we use the fact that each Whittaker function which is compactly supported mod NZ comes from an integral of a function in $C_c^\infty(G)$. We will start from the second part. Let ω be a character of Z and let $\mathcal{W}_\omega(G,\psi) \subseteq \mathcal{W}(G,\psi)$ be the subspace of functions $W \in \mathcal{W}(G,\psi)$ satisfying

$$W(gz) = \omega(z)W(g) \quad g \in G, z \in Z. \tag{9.1}$$

Let $C_c^\infty(G)$ be the space of locally constant functions on G with compact support. For each $f \in C_c^\infty(G)$ we let

$$W_f(g) = W_f^\psi(g) = \int_N f(ng)\psi^{-1}(n)dn.$$

It is clear that $W_f \in \mathcal{W}(G,\psi)$. We also define

$$W_{f,\omega}(g) = \int_Z \int_N f(nzg)\psi^{-1}(n)\omega^{-1}(z)dndz, \quad f \in C_c^\infty(G). \tag{9.2}$$

It is clear that $W_{f,\omega} \in \mathcal{W}_\omega(G,\psi)$. The image of these maps is well known. (See for example [4], Lemma 7.1). It is given in the following Lemmas:

Lemma 9.1. *Let $f \in C_c^\infty(G)$. Then W_f is compactly supported mod N and the map $f \mapsto W_f$ is a linear map onto the space of compactly supported functions mod N in $\mathcal{W}(G,\psi)$.*

Lemma 9.2. *Let $f \in S(G)$. Then $W_{f,\omega}$ is compactly supported mod NZ and the map $f \mapsto W_{f,\omega}$ is a linear map onto the space of compactly supported functions mod N in $\mathcal{W}_\omega(G,\psi)$.*

Let $|V|$ be the subspace of $|X|$ given by $|V| = \{|\alpha|_{r_1,r_2,...,r_n} : r_1 + r_2 + ... + r_n = 0\}$. (see (2.9)). Let $Q = \{\beta_1, ..., \beta_{n-1}\}$ be a basis for $|V|$. Let $C_1 < C_2$ be positive constants and define

$$A_Q(C_1,C_2) = \{a \in A : \ C_1 < \beta_i(a) < C_2, \ i = 1, ..., n-1\}.$$

Lemma 9.3. *A function W on G is compactly supported mod NZ if and only if there exist constants C_1, C_2 such that W is supported on $NA_Q(C_1,C_2)K$.*

Proof. We can write $A_Q(C_1,C_2) = ZA'$ where $A' = \{d(a_1,a_2,...,a_{n-1},1) \in A_Q(C_1,C_2)\}$. Since Q is a basis, it is clear that A' is compact. Hence if W is supported on $NA_Q(C_1,C_2)K$ then it is compactly supported mod NZ. Now assume W is compactly supported mod NZ. Then W is supported on a set of the form NZR for some compact set R. Since the sets $NA_Q(C_1,C_2)K$ for different choices of C_1 and C_2 are open sets that cover G we get that the sets of the form $NA_Q(C_1,C_2)K$ cover R. Since R is compact there exist constants C_1', C_2' so that $R \subset NA_Q(C_1',C_2')K$. Hence $NZR \subset NA_Q(C_1',C_2')K$. □

For each $w \in \mathbb{W}$ we define the set $M(w) \subset \Delta^*$ as follows.

$$M(w) = \{\alpha^* | \alpha \in \Delta, \ \alpha \notin S^0(w)\}$$

Remark 9.4. If $w_1 < w$ then by (2.19), $S^0(w_1) \supset S^0(w)$, hence $M(w_1) \subset M(w)$.

Let E be a positive constant. We let

$$A_w(E) = \{a \in A : \ |\lambda|(a) > E \ \text{ for every } \lambda \in M(w).\}$$

Theorem 9.5. *Let $W \in \mathcal{W}^0$, $w \in \mathbb{W}$ and $E > 0$. There exists a function $W_1 \in \mathcal{W}(G, \psi)$ compactly supported mod NZ such that*

$$W_1(n_1 awn_2) = W(n_1 awn_2) \tag{9.3}$$

for all $a \in A_w(E)$ and $n_1, n_2 \in N$.

Remark 9.6. Let $C_1 < C_2$ be positive constants and let $A_{C_1,C_2} = A_\Delta(C_1, C_2)$. By Lemma 9.3 we have that W_1 being compactly supported mod NZ is equivalent to W_1 being supported on a set of the form $NA_{C_1,C_2}K$ for some C_1, C_2. Hence we can find $W_1 \in \mathcal{W}$ compactly supported mod NZ such that (9.3) holds if and only if we can find constants C_1, C_2 such that the function

$$W_1(g) = \begin{cases} W(g), & \text{if } g \in NA_{C_1,C_2}K; \\ 0, & \text{otherwise.} \end{cases} \tag{9.4}$$

satisfies (9.3). Hence, we shall use (9.4) to define the desired W_1. Notice that if we define W_1 by (9.4) then $W(g) = 0 \Rightarrow W_1(g) = 0$ hence we only need to prove (9.3) for $g = n_1 awn_2$ such that $a \in A_w(E)$ and $W(g) \neq 0$.

Proof. We shall prove this theorem by an induction on $l(w)$ as in the proof of Theorem 4.1, Theorem 5.7 and Theorem 7.3.

$l(w) = 0$

In this case $w = e$, $S^0(w) = \Delta$, $M(w) = \emptyset$ and $A_w(E) = A$. We need to show the existence of a Whittaker function W_1, compactly supported mod NZ such that $W_1 = W$ on B. Since $W \in \mathcal{W}^0$ it follows that the support of W on B is contained in a set of the form NA_{C_1,C_2}. Define W_1 as in (9.4). Then W_1 satisfies the requirements of the Theorem.

We turn to the general case: $l(w) \geq 1$. Fix $w \in \mathbb{W}$ such that $l(w) \geq 1$.

Remark 9.7. By the induction assumption, and by Remark 9.6, if we are given a set of positive constants $\{E_{w_1} : l(w_1) < l(w)\}$ then there exists a Whittaker function W_1 compactly supported mod NZ such that (9.3) holds for every w_1 such that $l(w_1) < l(w)$ and every $a \in A^{w_1}(E_{w_1})$.

Fix $E > 0$ and let $a \in A_w(E)$. We need to show the existence of a function W_1 as above such that

$$W_1(awn) = W(awn) \tag{9.5}$$

for all $a \in A_w(E)$ and all $n \in N_w^-$. Let $S_w^- = \{\alpha_1, ..., \alpha_l\}$. We can assume that $\text{height}(\alpha_i) \geq \text{height}(\alpha_{i+1})$, $i = 1, ..., l-1$. Every $n \in N_w^-$ can be written (not uniquely) in the form

$$n = x_{\alpha_1}(r_1) \cdots x_{\alpha_j}(r_j) \tag{9.6}$$

for $0 \leq j \leq l$. We shall prove by an induction on j that there exists a Whittaker function W_1 as above such that (9.5) holds for every $a \in A_w(E)$ and every n of the form (9.6).

$j = 0$.

For $j = 0$ we need to show the existence of W_1 as above such that

$$W_1(aw) = W(aw)$$

for all $a \in A_w(E)$. By the remark above it is enough to consider the case where $a \in A_w(E)$ and $W(aw) \neq 0$. Since every $\beta \in \Delta^*$ is a positive linear combination of positive simple roots (see Remark 3.2), it follows from Remark 5.2 that for every $\beta \in M(w) \subset \Delta^*$ there exists a constant D_β such that

$$|\beta(a)| < D_\beta \tag{9.7}$$

Since $a \in A_w(E)$ it follows that for every $\beta \in M(w)$ we have

$$E < |\beta(a)|. \tag{9.8}$$

It is possible that the set of such a is empty in which case we take $W_1 = 0$ (or W_1 given by (9.4) for any constants $C_1 < C_2$.) By Lemma 7.2, we have that for every $\alpha \in S^0(W)$ there exist constants $C_\alpha < D_\alpha$ such that

$$C_\alpha < |\alpha(a)| < D_\alpha \tag{9.9}$$

Putting together (9.7), (9.8), (9.9) and using that $M(w) \cup S^0(w)$ is a basis for $|V|$ (see Lemma 3.4 (a)) we get by Lemma 9.3 that a satisfying the conditions above is in a set of the form A_{C_1,C_2} for some constants $C_1 < C_2$. Hence we can use (9.4) to define W_1.

The general case: Assume $j \geq 1$ and let $n \in N_w^+$ be in the form (9.6). Since W is smooth on the right, there exists a positive constant D such that if $|r| \geq D$ then

$$W(gx_{-\alpha_j}(-r^{-1})) = W(g), \quad g \in G.$$

Assume that n is of the form (9.6) with $a \in A_w(E)$ and $|r_j| \geq D$. We have

$$W(awn) = W(awnx_{-\alpha_j}(-r_j^{-1})). \tag{9.10}$$

By Lemma 2.6, $g = awnx_{-\alpha_j}(-r_j^{-1}) \in Bw_1B$ with $w_1 < w$. Moreover, if we write $g = n_1a_1w_1n_2$ for $n_1 \in N$, $a_1 \in A$ and $n_2 \in N_{w_1}^-$ then we have $a_1 = ah_{w(\alpha_j)}(r_j)$. Let $\beta \in M(w_1)$. Since $|r_j| \geq D$ it follows from Remark 2.1 that there exists $C_{\beta,D} > 0$ such that $|\beta(h_{w(\alpha_j)}(r_j))| > C_{\beta,D}$. By Remark 9.4 we have that $\beta \in M(w)$ hence $|\beta(a)| > E$. Hence we get that for every $|r_j| \geq D$ and every $a \in A_w(E)$

$$|\beta(a_1)| = |\beta(ah_{w(\alpha_j)}(r_j))| > EC_{\beta,D}.$$

It follows that if we take $E_1 = \min\{EC_{\beta,D} : \beta \in M(w)\}$ then $a_1 \in A_{w_1}(E_1)$.

Remark 9.8. If $M(w) = \emptyset$ then $M(w_1) = \emptyset$ and $A_{w_1}(E_1) = A$ for every $E_1 > 0$. Hence, in that case, we can take any E_1 that we like and a_1 will be in $A_{w_1}(E_1)$.

It follows from our first induction assumption that there exists a function W_1 given by (9.4) so that

$$W_1(n_1a_1w_1n_2) = W(n_1a_1w_1n_2) \tag{9.11}$$

for every $n_1 \in N$, $a_1 \in A_{w_1}(E_1)$ and $n_2 \in N_{w_1}^-$. Since W_1 is also smooth on the right it follows that there exists a constant $D_1 \geq D$ such that

$$W_1(awnx_{-\alpha_j}(-r_j^{-1})) = W_1(awn) \tag{9.12}$$

when $|r_j| \geq D_1$. Combining (9.10), (9.11) and (9.12) we get that

$$W_1(awn) = W(awn)$$

for every $a \in A_w(E)$ and every n of the form (9.6) with $|r_j| \geq D_1$. We now consider the case $|r_j| < D_1$. Fix r such that $|r| < D_1$ and let $W' \in \mathcal{W}^0$ be defined by

$$W' = \rho((x_{\alpha_j}(r))W \tag{9.13}$$

By our induction hypothesis on j, there exists a Whittaker function W_1 compactly supported mod NZ such that

$$W_1(awn) = W'(awn) \tag{9.14}$$

for all $a \in A_w(E)$ and all n of the form

$$n = x_{\alpha_1}(r_1) \cdots x_{\alpha_{j-1}}(r_{j-1}) \tag{9.15}$$

Let $W_2 = \rho((x_{\alpha_j}(-r))W_1$. Then W_2 is compactly supported mod NZ and from (9.13) and (9.14) we have

$$W_2(awnx_{\alpha_j}(r)) = W(awnx_{\alpha_j}(r))$$

for every $a \in A_w(E)$ and every n of the form (9.15). Since both W and W_2 are smooth on the right it follows that the above equality holds in a neighborhood of r. Since the set $\{r : |r| < D_1\}$ is compact it follows that we can cover it by a finite number of such neighborhoods. Since we can assume that each $W_2 = W_2(r)$ is given by (9.4) it follows that we can take the constants in (9.4) such that the equality will hold for all r_j such that $|r_j| < D_1$. □

Remark 9.9. Let ω be a quasi-character of Z. If $W \in \mathcal{W}^0$ satisfies $W(zg) = \omega(z)W(g)$ for every $z \in Z$ and $g \in G$ then W_1 given by (9.4) satisfies $W_1(zg) = \omega(z)W_1(g)$ for every $z \in Z$ and $g \in G$.

For $W \in \mathcal{W}^0$ and $g \in Bw_0B$ we let

$$J(W, g) = J_{g,w_0}(W) = \int_N W(gn)\psi^{-1}(n)dn$$

Corollary 9.10. *Let $W \in \mathcal{W}^0$. Let U be a compact set in G. Then there exists a function $W_1 \in \mathcal{W}$ compactly supported mod NZ such that*

$$J(W_1, g) = J(W, g)$$

for every $g \in U \cap Bw_0B$.

Proof. Let $\Delta^* = \{\lambda_1, ..., \lambda_{n-1}\}$. Since U is compact it follows that λ_i is bounded on U for $i = 1, ..., n-1$. Hence there exists a constant E such that $\lambda(g) > E$ for every $\lambda \in \Delta^*$ and every $g \in U$. Let $g \in U \bigcap Bw_0B$. Then $g = n_1 a w_0 n_2$ for some $n_1, n_2 \in N$ and $a \in A$. Let $\lambda \in \Delta^*$. By Theorem 4.1 we have

$$|\lambda(g)| = |\lambda(n_1 a w_0 n_2)| = |\lambda(a)||\lambda(w_0 n_2)| \leq |\lambda(a)|.$$

Since $\lambda(g) > E$ it follows that $\lambda(a) > E$, hence $a \in A_w^E$. By Theorem 9.5 we can find W_1 in $\mathcal{W}$ compactly supported mod NZ such that

$$W_1(a w_0 n) = W(a w_0 n)$$

for every $n \in N$ and $a \in A_w(E)$. In particular we have $W_1(gn) = W(gn)$ for every $g \in U \cap Bw_0B$ and $n \in N$. Hence $J(W_1, g) = J(W, g)$ for such g which is what we wanted to prove. □

Let $\phi \in C_c^\infty(G)$ and ω a quasi-character of Z. Let $g \in Bw_0B$. We define

$$J_{\psi,\omega}(\phi, g) = \int_N \int_N \int_Z f(n_1 z g n_2) \psi^{-1}(n_1) \psi^{-1}(n_2) \omega^{-1}(z) dn_1 dn_2 dz$$

It follows from [11] that this integral converges absolutely. It is easy to see that

$$J_{\psi,\omega}(\phi, g) = \int_N W_{\phi,\omega}(gn) \psi^{-1}(n) dn = J(W_{\phi,\omega}, g)$$

where $W_{\phi,\omega}$ is defined by (9.2).

The following Corollary implies Theorem 1.7 in the introduction.

Corollary 9.11. *Let π be an irreducible admissible generic representation of G with central character ω_π. Let $j_{\pi,\psi}$ be the Bessel function of π as defined in Section 8. Let U be a compact open set in G. Then there exists a function $\phi \in C_c^\infty(G)$ such that*

$$J_{\psi,\omega_\pi}(\phi, g) = j_{\pi,\psi}(g) \tag{9.16}$$

for every $g \in U \cap Bw_0B$.

Proof. Recall that $j_{\pi,\psi}(g) = j_{\pi,\psi,w_0}(g)$ is the Bessel function associated to the longest Weyl element w_0. By (8.2) there exists $W \in \mathcal{W}^0(\pi, \psi)$ such that $j_{\pi,\psi}(g) = J(W, g)$ for every $g \in Bw_0B$. Since the central character of π is ω_π it follows that $W \in \mathcal{W}^0_{\omega_\pi}(G, \psi) = \mathcal{W}_{\omega_\pi}(G, \psi) \cap \mathcal{W}^0(G, \psi)$. By Corollary 9.10 and by remark 9.6 and Remark 9.9 there exists $W_1 \in \mathcal{W}_\omega(G, \psi)$ which is compactly supported mod NZ such that $J(W_1, g) = J(W, g)$ for every

$g \in U \cap Bw_0B$. By Lemma 9.2 there exists $\phi \in C_c^\infty(G)$ such that $W_{\phi,\omega} = W_1$. Hence

$$J_{\psi,\omega}(\phi, g) = J(W_{\phi,\omega}, g) = J(W_1, g) = J(W, g) = j_{\pi,\psi}(g)$$

for every $g \in U \cap Bw_0B$. □

Remark 9.12. If π is supercuspidal then there exits $\phi \in C_c^\infty(G)$ such that (9.16) holds for every $g \in Bw_0B$. That is, in this case the Bessel function is globally given by an orbital integral. This follows from Remark 5.2, from Lemma 9.2 and (8.2).

10. Concluding remarks

10.1. **Simply laced groups.** Our results in this paper remain valid for split, simply laced algebraic groups over a local non-archimedean field. The main facts that we need is that Lemma 2.6 remains valid in that setting and that the root system of such groups is obtuse (see Section 3). The proofs will be the same as in the $GL(n)$ case. Since Lemma 2.6 is not valid in the setting of a split reductive group we can not yet generalize the results for that case. However, we believe that the results are still true in the general setting of a quasi-split group over a non-archimedean local field.

10.2. **Distributions.** In [3], Bessel functions were defined using a distribution approach for every generic representation of a quasi-split group over a local field. In [2],[4] it is proved that in the case of $GL_2(F)$ and $GL_3(F)$ these are the same as the functions we defined here. Moreover, these functions are locally integrable and give the Bessel distribution everywhere. We believe that this should be true for the Bessel functions defined here. In Corollary 1.9 we have reduced the question of local integrability of the Bessel functions to the Local integrability of the orbital integrals. By [11], these orbital integrals are given asymptotically by certain germs. It remains to find good bounds for these germs.

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