A BESSEL IDENTITY FOR THE THETA CORRESPONDENCE

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ABSTRACT. We establish a spectral identity between global Bessel distributions with respect to generic cuspidal representations of an odd orthogonal group and the metaplectic cover of a symplectic group which are related by the theta correspondence. We also provide analogous local identities for square-integrable representations.

CONTENTS

1. INTRODUCTION

In a seminal paper [Shi73], Shimura obtained a famous correspondence $f \mapsto F$ between modular forms of half-integral weight and those with integral weight. A well-known result of Waldspurger [Wal81] relates the Fourier coefficients of f to central values of the twisted L-functions of F. This relation has seen many applications and it was explicated and extended by several authors [Koh85, KZ81, Shi93, KS93, Koj00, Koj99, KM96] (to mention a few). The most general formula in this context is given in [BM07].

Nowadays, the Shimura correspondence is often viewed representation-theoretically in the framework of the *theta correspondence*. The basic idea, which is due to Howe [How79], is to relate two classical groups G, G' (or covers thereof) as a *dual reductive pair* inside a bigger metaplectic group and to use the Weil representation of the latter to obtain a correspondence between (a subset of) the representations of G and G'. (In Shimura's case the dual pair is $(\mathrm{SO}(2,1) = \mathrm{PGL}_2, \widetilde{\mathrm{SL}}_2)$.) Over the years, the theta correspondence became one of the most useful and well-developed tools in the study of representations of classical groups and their interrelations, both globally (automorphic representations) and locally.

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The properties of these correspondences are well studied in the literature. (The standard, albeit not very recent, source is [MVW87].)

The results of [BM07] use the *relative trace formula* which was introduced by Jacquet [Jac87a] to study Waldspurger's formula from a different point of view. A closely related approach is due to Iwaniec [Iwa87]. The trace formula approach can be formulated for other cases of dual reductive pairs. In this paper we focus on the dual pair (G, G') consisting of the split odd orthogonal group G and the metaplectic cover G' of the symplectic group, both of rank n. We consider this pair both in the p-adic and the number field case. The representation theory of the two groups are intimately related through the theta correspondence. Our main global result is a spectral identity between representations in the generic spectrum which are related by the theta correspondence. The precise statement is Theorem 2. Roughly speaking, it relates the square of a Fourier coefficient of forms on the metaplectic group to the central value of L-functions of their theta lifts.

The result was announced in [Jac05]. It is a typical spectral identity in the context of the relative trace formula and a direct generalization of the case $n = 1$ considered by Jacquet in [Jac87a]. In contrast, our approach here does not use the relative trace formula. Instead we use the theta correspondence. Our main tool is Furusawa's formulas [Fur95] relating the Fourier coefficients of theta lifts and Bessel coefficients. These formulas generalize the $n = 1$ case which was worked out by Waldspurger and they already play a major role in his work. The main new ingredient in deriving the spectral identity is to use the obvious, but crucial, fact that the adjoint of the theta lift is simply the theta lift in the converse direction.

We remark that the geometric comparison of the relative trace formula at hand was carried out in [MR04], once again using the theta correspondence. The results here can be viewed as the spectral counterpart of [ibid.], although they are formally independent of each other.

We also have an analogous result in the local case. Let F be a p-adic field and fix a non-trivial character ψ of the additive group of F. Let $G = SO(2n+1, F)$ (split) and $G' = \mathrm{Mp}(2n, F)$. Denote by $\Pi_2^{\psi}(G)$ (resp. $\Pi_2^{\psi}(G')$) the set of equivalence classes of irreducible square-integrable ψ-generic (genuine) representations of G (resp. G'). It will be seen below that the Howe duality provides a bijection (depending on ψ) between $\Pi_2^{\psi}(G)$ and $\Pi_2^{\psi}(G')$.[1] (Cf. [MS00] for an analogous result for *even* orthogonal groups.)

Let $\mathcal{S}(G)$, resp. $\mathcal{S}(G')$ denote the space of compactly supported locally constant (genuine) functions on G, (resp. G'). Denote by $\mathcal{S}^*(G)$ and $\mathcal{S}^*(G')$ the dual spaces of distributions on G and G' respectively. In §2.3 we define certain distributions $\mathbb{B}^{\pi} \in \mathcal{S}^*(G)$, (resp. $\mathbb{B}^{\pi'} \in \mathcal{S}^*(G')$) for $\pi \in \Pi_2^{\psi}(G)$ (resp. $\pi' \in \Pi_2^{\psi}(G')$).

We say that $f \in \mathcal{S}(G)$ and $f' \in \mathcal{S}(G')$ *match* if their respective orbital integrals are compatible. (See [MR04, Proposition 6.1] and §4.3 below.)

We prove the following identity of Bessel distributions.

[1] An analogous result for square-integrable representations was recently established by Ichino and Gan [GI] – see also [GS].

Theorem 1. *Suppose that $\pi \in \Pi_2^\psi(G)$ and $\pi' \in \Pi_2^\psi(G')$ correspond under the Howe ψ-duality. Then*

$$\mathbb{B}^\pi(f) = \mathbb{B}^{\pi'}(f'^\vee)$$

for all matching pairs $f \leftrightarrow f'$. Here $f'^\vee(g) = f'(g^{-1})$.

We remark that by adjusting the matching condition in [MR04], it is possible to restate the identity as $\mathbb{B}^\pi(f) = \mathbb{B}^{\pi'}(f')$ for any matching pairs f and f' under the revised matching condition.

Once again, the proof is based on the machinery of the Weil representation. We first derive an explicit Howe duality (in both directions) between π and π' realized in their corresponding Whittaker models. Indeed, this idea goes back to Waldspurger [Wal91] who used it in the case $n = 1$. It was used by Jiang-Soudry [JS03] in the supercuspidal case (for general n). As in the global case the spectral identity reduces to an adjointness relation between the explicit Howe duality maps in both directions. This approach is simpler than those used in [BM03] and [BM05], where a more detailed analysis of the Bessel distribution was required.

It should be possible to extend the local result to the general case (of generic representations). This will enable us to formulate and prove a precise formula for the square of Fourier coefficients of metaplectic cusp forms as in [BM07]. We hope to pursue this problem in the future.

Finally, we mention that there are additional closely related (and equally important) results of Waldspurger in the context of the theta correspondence ([Wal85]). They also admit a relative trace formula interpretation [Jac87b, Jac86], which in turn admit higher rank generalization. The geometric comparison of these trace formulas as well as others resulting from theta correspondence was carried out in [MR05, MR04, MR99b, MR99a]. It is likely that our approach can be applied to obtain the spectral identities underlying these comparisons.

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1.1. **Notation and Preliminaries.** Until §5 F is a local non-archimedean field of characteristic zero and $\mathcal{O}$ is its ring of integers. We fix a non-trivial character ψ of F.

For a vector space W, we use $\langle v_1, \ldots, v_m \rangle$ to denote the span of vectors $v_1, \ldots, v_m$ in W. Denote by $M_{a,b}$ the space of $a \times b$ matrices.

Let $V = M_{2n+1,1}(F)$, with the standard basis $e_1, \ldots, e_{2n+1}$, and the symmetric bilinear form $\langle \cdot, \cdot \rangle_V$ given by $\langle e_i, e_{2n+2-j} \rangle_V = \delta_{i,j}$. Then V has a splitting $V = V_+ \oplus V_- \oplus \langle e_{n+1} \rangle$, where $V_+ = \langle e_1, \ldots, e_n \rangle$ and $V_- = \langle e_{-1}, \ldots, e_{-n} \rangle$, (where we set $e_{-i} = e_{2n+2-i}$). We let $G = \mathrm{SO}(V)$ be the special orthogonal group of $(V, \langle \cdot, \cdot \rangle_V)$ (acting on the left). We denote by $\mathcal{S}(G)$ the space of locally constant compactly supported functions on G.

Let $V' = M_{1,2n}(F)$, with the standard basis $f_1, \ldots, f_{2n}$ and the anti-symmetric form $\langle \cdot, \cdot \rangle_{V'}$ given by $\langle f_i, f_{-j} \rangle_{V'} = \delta_{i,j}$ for $i, j = 1, \ldots, n$ where we set $f_{-j} = f_{2n+1-j}$. Then V'

has a splitting $V' = V'_+ \oplus V'_-$, where $V'_+ = \langle f_1, \dots, f_n\rangle$ and $V'_- = \langle f_{-1}, \dots, f_{-n}\rangle$. Denote by $\mathrm{Sp}(V')$ the symplectic group of $(V', \langle\cdot,\cdot\rangle_{V'})$ acting on the right.

Let $G' = \mathrm{Mp}(V')$ be the metaplectic cover of $\mathrm{Sp}(V')$. (Cf. [Kud, Chapter I] and [MVW87] for basic facts and conventions about G'.) An element in G' has the form $(g, \pm 1)$ with $g \in \mathrm{Sp}(V')$. Multiplication in G' is given by $(g_1, \varepsilon_1)(g_2, \varepsilon_2) = (g_1 g_2, \check{c}(g_1, g_2)\varepsilon_1\varepsilon_2)$ where $\check{c}$ is the cocycle defined in [RR93]. We write $\widetilde{g'} = (g', 1)$ for any $g' \in \mathrm{Sp}(V')$. (Of course, $g' \mapsto \widetilde{g'}$ is not a homomorphism.) We denote by $\mathcal{S}(G')$ the space of locally constant compactly supported functions on G' which are *genuine*, i.e., such that $\phi(g, -1) = -\phi(g, 1)$ for all $g \in \mathrm{Sp}(V')$.

Denote by e and e' the identity elements of G and G' respectively.

Let P (resp. P') be the Siegel parabolic subgroup of G (resp. $\mathrm{Sp}(V')$) and let U (resp. U') be its unipotent radical. We identify the Levi subgroups M and M' of P and P' with GL_n via

$$m(g) = \begin{pmatrix} g & & \\ & 1 & \\ & & g^* \end{pmatrix}, \quad m'(g) = \begin{pmatrix} g & \\ & g^* \end{pmatrix} \quad g \in \mathrm{GL}_n\,.$$

where we define

$$g^* = w_n\, {}^t g^{-1} w_n \quad g \in \mathrm{GL}_n$$

where w_n is the matrix with ones on the non-principal diagonal and zeros elsewhere.

Denote by N, N' and N'' the maximal unipotent subgroups of $\mathrm{SO}(V)$, $\mathrm{Sp}(V')$ and $G'' = \mathrm{GL}_n$ respectively, consisting of upper unitriangular matrices. Thus, $N = m(N'') \ltimes U$ and $N' = m'(N'') \ltimes U'$. Note that by the property of the cocycle, for all $n \in N'$ and $g' \in \mathrm{Sp}(V')$ we have $\widetilde{n'}\widetilde{g'} = \widetilde{n'g'}$ and $\widetilde{g'}\widetilde{n'} = \widetilde{g'n'}$. In particular $n' \mapsto \widetilde{n'}$ embeds N' in G'.

Denote by T'' the maximal torus of G'' consisting of diagonal matrices. Let $T = m(T'')$ and $T' = m'(T'')$ be the maximal tori of G and $\mathrm{Sp}(V')$ respectively. We fix good maximal compact subgroups K, K' and K'' of G, G' and $\mathrm{GL}_n(F)$ respectively, so that the Iwasawa decomposition $G = NTK$ holds for G (and similarly for G', G'').

When $t \in T$, we write $te_i = t_i e_i$ for $i = 1, \dots, n$ so that $t = \mathrm{diag}(t_1, \dots, t_n, 1, t_n^{-1}, \dots, t_1^{-1})$. It will sometimes be convenient to set $t_{n+1} = 1$ and $t_{-i} = t_i^{-1}$. Similarly, when $t' \in T'$, we write $f_i t' = t'_i f_i$, $i = 1, \dots, n$. We enumerate the simple roots of G by $\alpha_i(t) = t_i/t_{i+1}$, $i = 1, \dots, n$ (where $t_{n+1} = 1$). Similarly, we set $\alpha'_i(t') = t'_i/t'_{i+1}$, $i = 1, \dots, n$ (where $t'_{n+1} = 1$). (Note that for convenience we do not set α'_n to be a root for $\mathrm{Sp}(V')$.) We use $\delta(t)$ and $\delta'(t')$ to denote the modulus functions of the Borel subgroups of G and $\mathrm{Sp}(V')$ respectively. Thus $\delta(t)^{\frac{1}{2}} = \prod_{i=1}^n |t_i|^{n+\frac{1}{2}-i}$ and $\delta(t')^{\frac{1}{2}} = \prod_{i=1}^n |t'_i|^{n+1-i}$.

Let Z be a symplectic space over F with a polarization $Z = Z_+ \oplus Z_-$. We write a typical element of $\mathrm{Sp}(Z)$ (again, acting on the right on Z) as $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ where $A \in \mathrm{Hom}(Z_+, Z_+)$, $B \in \mathrm{Hom}(Z_+, Z_-)$, $C \in \mathrm{Hom}(Z_-, Z_+)$ and $D \in \mathrm{Hom}(Z_-, Z_-)$. Consider the Weil representation ω_ψ of the group $\mathrm{Mp}(Z)$ (with respect to the Rao cocycle defined by the splitting). It can be realized on $\mathcal{S}(Z_+)$ as follows. (Cf. [Kud, Chapter I].) For $g = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$, let

$$r(g)\phi(z) = \int_{Z_-/\ker C} \psi(\frac{1}{2}\langle zA, zB\rangle + \frac{1}{2}\langle z'C, z'D\rangle - \langle zB, z'C\rangle)\phi(zA + z'C)\; d_g z'$$

where the measure d_g on $Z_-/\ker C$ is uniquely determined by the property that $r(g)$ is unitary on $L^2(Z_+)$. Then we take $\omega_\psi(g)\phi(z) = \beta_\psi(g)[r(g)\phi](z)$ where $\beta_\psi(g)$ is a certain root of unity defined in [Kud, Theorem 4.5].[2]

In our case we take

$$Z = \mathrm{Hom}(V, V') \simeq V^* \otimes V' \simeq V \otimes V' \simeq M_{2n+1,2n} \tag{1}$$

where we identify V^* with V through $\langle \cdot, \cdot \rangle_V$. On the level of matrices, $G \times \mathrm{Sp}(V')$ acts on Z by

$$z(g, h) = g^{-1}zh \quad g \in G, h \in \mathrm{Sp}(V'), z \in Z.$$

We define the Weil representation ω_ψ using the polarization $Z_\pm = V \otimes V'_\pm$. Later on we will denote $Z_\pm$ by $Z^1_\pm$ (and ω_ψ by ω^1_ψ) to distinguish it from a different polarization that will be used.

We endow the additive group of F with the self-dual Haar measure with respect to ψ. We will use the following convention for the normalization of Haar measures on certain unipotent groups of G. Let α be a positive root of T in G. We have $\alpha(t) = t_i/t_j$ where either $j > i$ or $j < 0$ and $j \neq -i$. Let $U_\alpha = \{x_\alpha(t) : t \in F\}$ be the corresponding one-parameter subgroup of N where we normalize x_α by the relation $x_\alpha(s)e_j = e_j + se_i$. This identifies U_α with F and we take the corresponding Haar measure on U_α. If S is a T-stable subgroup of N then as an algebraic variety it is isomorphic (via multiplication, in any order) to the direct product of the U_α's contained in S [Hum75, §28.1]. We take the product measure as our choice of Haar measure on S. In a similar vein we fix a choice of Haar measures for subgroups of N'. We will also fix arbitrary Haar measures on G and G'.

2. Bessel distributions

For now let G be any totally disconnected group. Given an irreducible representation π of G, a functional ℓ on π and a functional $\ell^\vee$ on the contragredient $\pi^\vee$, we define the distribution

$$\mathcal{B}^\pi_{\ell,\ell^\vee}(f) = \ell^\vee(\ell \circ \pi(f)) \quad f \in \mathcal{S}(G).$$

In practice, it is convenient to realize $\ell^\vee$ as $\hat{\ell} \circ \iota$ where $\hat{\ell}$ is a functional on a representation $\hat{\pi}$ which is equivalent to $\pi^\vee$ via an intertwining operator $\iota : \pi^\vee \to \hat{\pi}$. In turn ι is defined in terms of a non-zero G-invariant pairing $(\cdot, \cdot)$ between π and $\hat{\pi}$, so that $(v, \iota(v^\vee)) = v^\vee(v)$. In this case we also write

$$\mathcal{B}^{(\pi,\hat{\pi},(\cdot,\cdot))}_{\ell,\hat{\ell}}(f) = \mathcal{B}^\pi_{\ell,\hat{\ell}\circ\iota}(f)$$

We refer to the triple $(\pi, \hat{\pi}, (\cdot, \cdot))$ as a pair of representations in duality.

In particular, suppose that (π, V) is unitary and $[\cdot, \cdot]$ is an invariant inner product on V. We can think of $(\pi, \bar{\pi}, [\cdot, \cdot])$ as a pair of representations in duality where $\bar{\pi}$ is realized on $\bar{V}$ (the vector space whose underlying set is V, but with conjugate $\mathbb{C}$-structure; as an operator $\bar{\pi}(g) = \pi(g)$). Fix a sequence K_n of compact open subgroups of K which form a basis for the topology. We say that an orthonormal basis $\mathfrak{B}$ of V is *admissible* if $\mathfrak{B} \cap V^{K_n}$

[2] The notation in [Kud] suggests that β depends on a choice of a standard basis as well. In fact, β depends only on the splitting.

spans V^{K_n} for any n. Let ℓ, ℓ' be linear functionals on V. Then $\overline{\ell'}$ is a linear functional on $\bar{V}$ where $\overline{\ell'}(v) = \overline{\ell'(v)}$. We then have

$$\mathcal{B}_{\ell,\overline{\ell'}}^{(\pi,\bar{\pi},[\cdot,\cdot])}(f) = \sum_{e_i \in \mathfrak{B}} \ell(\pi(f)e_i)\overline{\ell'}(e_i) = \sum_{e_i \in \mathfrak{B}} \ell(\pi(f)e_i)\overline{\ell'(e_i)}$$

for any admissible orthonormal basis $\mathfrak{B}$ of V (cf. [Jac10]).

Lemma 1. *Suppose that we have two pairs of representations in duality* $((\pi_i, V_i), (\hat{\pi}_i, \hat{V}_i), (\cdot,\cdot)_i)$ *of totally disconnected groups* G_i, $i = 1, 2$, *linear functionals* ℓ_i *of* V_i, $\hat{\ell}_i$ *of* $\hat{V}_i$, $i = 1, 2$, $f_i \in \mathcal{S}(G_i)$, $i = 1, 2$ *and linear transformations* $A : V_1 \to V_2$ *and* $\hat{A} : \hat{V}_2 \to \hat{V}_1$ *satisfying the following properties*

1. $(Av_1, \hat{v}_2)_2 = (v_1, \hat{A}\hat{v}_2)_1$ *for all* $v_1 \in V_1$, $\hat{v}_2 \in \hat{V}_2$,
2. $\pi_2(g) \circ A = A$ *for all* g *in a small open subgroup of* K,
3. $\ell_1 \circ \pi_1(f_1) = \ell_2 \circ A$,
4. $\hat{\ell}_2 \circ \hat{\pi}_2(f_2) = \hat{\ell}_1 \circ \hat{A}$.

Then for $f_2^\vee(g) = f_2(g^{-1})$ *we have*

$$\mathcal{B}_{\ell_1,\hat{\ell}_1}^{(\pi_1,\hat{\pi}_1,(\cdot,\cdot)_1)}(f_1) = \mathcal{B}_{\ell_2,\hat{\ell}_2}^{(\pi_2,\hat{\pi}_2,(\cdot,\cdot)_2)}(f_2^\vee)$$

Proof. Upon replacing ℓ_2 by $\frac{1}{\mathrm{vol}(K)} \int_K \ell_2 \circ \pi_2(k)\, dk$ we may assume that $\ell_2 \in V_2^\vee$. In this case $\iota_1(\ell_2 \circ A) = \hat{A}(\iota_2(\ell_2))$. Indeed, for all $v \in V_1$

$$(v, \iota_1(\ell_2 \circ A))_1 = \ell_2(Av) = (Av, \iota_2(\ell_2))_2 = (v, \hat{A}(\iota_2(\ell_2)))_1.$$

Therefore, we have

$$\mathcal{B}_{\ell_1,\hat{\ell}_1}^{(\pi_1,\hat{\pi}_1,(\cdot,\cdot)_1)}(f_1) = \hat{\ell}_1 \circ \iota_1(\ell_1 \circ \pi_1(f_1)) = \hat{\ell}_1 \circ \iota_1(\ell_2 \circ A) = \hat{\ell}_1 \circ \hat{A}(\iota_2(\ell_2)) =$$
$$\hat{\ell}_2 \circ \hat{\pi}_2(f_2)(\iota_2(\ell_2)) = \hat{\ell}_2 \circ \iota_2(\pi_2^\vee(f_2)\ell_2) = \hat{\ell}_2 \circ \iota_2(\ell_2 \circ \pi_2(f_2^\vee))) = \mathcal{B}_{\ell_2,\hat{\ell}_2}^{(\pi_2,\hat{\pi}_2,(\cdot,\cdot)_2)}(f_2^\vee).$$

□

2.1. **Whittaker models and invariant pairings.** From now on we are back to the case $G = \mathrm{SO}(V)$ and $G' = \mathrm{Mp}(V')$. Define non-degenerate characters ψ_N on N and $\psi_{N'}$ on N' by

$$\psi_N(u) = \psi(u_{1,2} + u_{2,3} + \ldots + u_{n,n+1}), \quad u \in N;$$
$$\psi_{N'}(u') = \psi(u'_{1,2} + u'_{2,3} + \ldots + u'_{n-1,n} + \frac{u'_{n,n+1}}{2}), \quad u' \in N'.$$

Let $C(N\backslash G, \psi_N)$ be the G-space of smooth functions $W : G \to \mathbb{C}$ such that $W(ug) = \psi_N(u)W(g)$ for all $u \in N$, $g \in G$, with G acting by right translation. Recall that for any $W \in C(N\backslash G, \psi_N)$ there exists a constant C such that $W(t) = 0$ if $t \in T$ and $|\alpha_i(t)| > C$ for some i. We say that an irreducible representation π of G is ψ-generic if it can be realized as a subspace $\mathcal{W}^\psi(\pi)$ of $C(N\backslash G, \psi_N)$. The space $\mathcal{W}^\psi(\pi)$ is uniquely determined by the equivalence class of π and is called the *Whittaker model* of π. We write $\Pi^\psi(G)$ for the set of equivalence classes of irreducible ψ-generic representations of G. (In fact, it is independent of ψ because T acts transitively on the ψ's.) If $\pi \in \Pi^{\psi}(G)$ then $\pi^{\vee} \in \Pi^{\psi^{-1}}(G)$ where $\pi^{\vee}$ denotes the contragredient of π. (In fact, $\pi^{\vee} \simeq \pi$.) We write λ^{ψ} for the linear functional on $\mathcal{W}^{\psi}(\pi)$ given by $\lambda^{\psi}(W) = W(e)$. Thus, λ^{ψ} is a Whittaker functional on $\mathcal{W}^{\psi}(\pi)$.

Similarly, an irreducible genuine representation π' of G' is called ψ-generic, (and write $\pi' \in \Pi^{\psi}(G')$) if it can be realized as a subspace $\widetilde{\mathcal{W}}^{\psi}(\pi')$ of the G'-space $C(N'\backslash G', \psi_{N'})$ consisting of smooth genuine functions $W' : G' \to \mathbb{C}$ such that $W'(\widetilde{u'}g') = \psi_{N'}(u')W'(g')$ for all $u' \in N'$, $g' \in G'$. (This time, $\Pi^{\psi}(G')$ depends on ψ.) Once again, the space $\widetilde{\mathcal{W}}^{\psi}(\pi')$ is uniquely determined by the equivalence class of π' [Szp07]. Also, if $\pi' \in \Pi^{\psi}(G')$ then $\pi'^{\vee} \in \Pi^{\psi^{-1}}(G')$-generic. (Note that $\pi'^{\vee}$ is not necessarily equivalent to π'.) Let $\widetilde{\lambda}^{\psi}$ be the Whittaker functional on $\widetilde{\mathcal{W}}^{\psi}(\pi')$ given by $\widetilde{\lambda}^{\psi}(W') = W'(e')$.

The following Lemma follows directly from [LM09, Theorem 3.1, Lemma 2.6, Corollary 3.4]; cf. also[CS80, §6].

Lemma 2. *Suppose that $\pi \in \Pi^{\psi}(G)$ is square-integrable. Then there exists $\lambda > 0$ and for any $W \in \mathcal{W}^{\psi}(\pi)$ there exists a non-negative $\phi \in \mathcal{S}(F^n)$ such that*

$$|W(t)| \leq \delta(t)^{\frac{1}{2}}\phi(\alpha_1(t), \dots, \alpha_n(t)) \prod_{i=1}^{n} |\alpha_i(t)|^{\lambda}, \ t \in T. \tag{2}$$

(Note that $\prod_{i=1}^{n} \alpha_i(t) = t_1$.) The bilinear form

$$(W, \hat{W})_{\mathcal{W}^{\psi}(\pi)} := \int_{N\backslash G} W(g)\hat{W}(g)\, dg \quad W \in \mathcal{W}^{\psi}(\pi), \hat{W} \in \mathcal{W}^{\psi^{-1}}(\pi^{\vee}) \tag{3}$$

is absolutely convergent and defines a G-invariant pairing between $\mathcal{W}^{\psi}(\pi)$ and $\mathcal{W}^{\psi^{-1}}(\pi^{\vee})$.

Similarly, suppose that $\pi' \in \Pi^{\psi}(G')$ is square-integrable. Then there exists $\mu > 0$ and for any $W' \in \widetilde{\mathcal{W}}^{\psi}(\pi')$ there exists a non-negative $\phi' \in \mathcal{S}(F^n)$ such that

$$|W'(\widetilde{t'})| \leq \delta'(t')\phi'(\alpha'_1(t'), \dots, \alpha'_n(t')) \prod_{i=1}^{n} |\alpha'_i(t')|^{\mu} \quad t' \in T'. \tag{4}$$

(Once again, $\prod_{i=1}^{n} \alpha'_i(t') = t'_1$.) The form

$$(W', \hat{W}')_{\widetilde{\mathcal{W}}^{\psi}(\pi')} = \int_{N'\backslash G'} W'(g')\hat{W}'(g')\, dg' \quad W' \in \widetilde{\mathcal{W}}^{\psi}(\pi'), \hat{W}' \in \widetilde{\mathcal{W}}^{\psi^{-1}}(\pi'^{\vee}) \tag{5}$$

defines a G'-invariant pairing between $\widetilde{\mathcal{W}}^{\psi}(\pi')$ and $\widetilde{\mathcal{W}}^{\psi^{-1}}(\pi'^{\vee})$.

Proof. The inner product formulas were established in [LM09]. The inequalities were not explicitly given, so we give a derivation here. We consider the inequality (2) of $W(t)$, the other inequality for $W'(\widetilde{t'})$ is established similarly.

Use the notations in [LM09]. Let B be the Borel subgroup of G containing N. By [LM09, Theorem 3.1], $W(t)$ is a finite combination of functions of the form $\delta_P^{\frac{1}{2}}(t)\phi_{P,\chi}(t)$ where P is a parabolic subgroup of G containing B, χ is a character of T so that the χ-generalized eigenspace of the Jacquet module $J_P(\pi)$ of π with respect to P contains a supercuspidal component, and $\phi_{P,\chi} \in \mathfrak{F}_{P,\chi}$. We only need to establish the bound of (2) for any such function $\delta_P^{\frac12}(t)\phi_{P,\chi}(t)$, as it is clear that the sum of two functions of t of the form (2) is again bounded by a function of the form (2).

By [LM09, (2.2) and Lemma 2.6(3)] $\delta_P^{\frac12}(t)\phi_{P,\chi}(t) = \delta_B^{\frac12}(t)\phi'_{B,\chi}(t)$ for a function $\phi'_{B,\chi} \in \mathfrak{F}_{B,\chi}$. By [LM09, Lemma 2.6(2)], $\phi'_{B,\chi}$ is bounded by $q^{-\langle \operatorname{Re}\chi, H(t)\rangle}\prod_{i=1}^n \phi'_i(\alpha_i(t))$ with ϕ'_i being Schwartz functions. From the square-integrability of π we have $q^{-\langle \operatorname{Re}\chi, H(t)\rangle} \le \prod_{i=1}^n |\alpha_i(t)|^{\lambda'}$ with $\lambda' > 0$ ([Cas]). Thus we get the bound (2) for $\delta_P^{\frac12}(t)\phi_{P,\chi}(t)$. $\square$

We write $\Pi_2^\psi(G)$ for the square-integrable representations in $\Pi^\psi(G)$. For any $\pi \in \Pi_2^\psi(G)$

$$\mathbb{W}^\psi(\pi) = (\mathcal{W}^\psi(\pi), \mathcal{W}^{\psi^{-1}}(\pi^\vee), (\cdot,\cdot)_{\mathcal{W}^\psi(\pi)})$$

is a pair of representations in duality. Similarly we define $\Pi_2^\psi(G')$ and

$$\widetilde{\mathbb{W}}^\psi(\pi') = (\widetilde{\mathcal{W}}^\psi(\pi'), \widetilde{\mathcal{W}}^{\psi^{-1}}(\pi'^\vee), (\cdot,\cdot)_{\widetilde{\mathcal{W}}^\psi(\pi')}).$$

Thus, if $\pi \in \Pi_2^\psi(G)$ then $\mathcal{W}^\psi(\pi)$ is an irreducible subspace of $L^2(N\backslash G, \psi_N)$ where the latter (by an abuse of notation) denotes the subspace of $C(N\backslash G, \psi_N)$ consisting of functions such that $\int_{N\backslash G} |f(g)|^2\, dg < \infty$. Conversely, any irreducible subspace of $L^2(N\backslash G, \psi_N)$ is (abstractly) a square-integrable (and ψ-generic) representation of G. This was proved recently independently by Delorme, Sakellaridis-Venkatesh and Tang [Del, SV, Tan]

2.2. **Local Bessel period.** Suppose again that $\pi \in \Pi_2^\psi(G)$. We define the *Bessel functional* of π on the space $\mathcal{W}^\psi(\pi)$. The Bessel subgroup is by definition

$$R = \{g \in G : ge_2 = e_2,\ ge_3 = e_3 \pmod{\langle e_2\rangle}, \dots, ge_{n+1} = e_{n+1} \pmod{\langle e_2, \dots, e_n\rangle}\}.$$

Note that $R = (R \cap N)H$ where H is the subgroup of G which fixes $e_2, \dots, e_{n+1}$ and fixes $e_{-2}, \dots, e_{-n}$ modulo $\langle e_{-1}\rangle$. Explicitly

$$H = \{m(A_{\xi,\eta}) : \xi \in F^{n-1}, \eta \in F^*\} \text{ where } A_{\xi,\eta} = \begin{pmatrix} \eta & & & \\ \xi_1 & 1 & & \\ \vdots & & \ddots & \\ \xi_{n-1} & & & 1 \end{pmatrix}$$

while

$$R \cap N = U^\sharp \rtimes \{m(\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}) : u \text{ upper unitriangular in } \mathrm{GL}_{n-1}\}$$

where $U^\sharp = \{u \in U : \langle ue_{n+1}, e_{-1}\rangle_V = 0\}$. For instance, for $n = 3$, R consists of the matrices in G of the form

$$\begin{pmatrix} * & 0 & 0 & 0 & * & * & * \\ * & 1 & * & * & * & * & * \\ * & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & * & * & * \end{pmatrix}$$

Define a character ψ_R on R by

(6) $$\psi_R(r) = \psi(r_{2,3} + \cdots + r_{n,n+1}), \ \ r \in R.$$

Set

(7) $$\mathcal{P}(W) = \int_{R \cap N \backslash R} W(r)\psi_R(r)^{-1} \, dr = \int_H W(h) \, dh$$

where dr is a Haar measure on $R \cap N\backslash R$ and dh is a suitable Haar measure on H.

Let also R_0, R_1 be the stabilizers of the vectors $e_2, \dots, e_{n+1}$ (resp., $e_1, \dots, e_{n+1}$). Thus, R_1 is the stabilizer of e_{n+1} in U, i.e. the derived group of U. Note that R_1 is T-stable and is endowed with a measure as in §1.1. We also remark that R_0, R_1 and R are unimodular, $R_0 = R_1 H$ and ψ_R is trivial on R_0. By (7) we get

(8) $$\mathcal{P}(W) = \int_{R_1 \backslash R_0} W(r) \, dr$$

for an appropriate Haar measure on $R_1\backslash R_0$.

Lemma 3. *The integral in* (7) *is absolutely convergent and defines an* (R, ψ_R)*-equivariant functional on* π.

Proof. The second part of the Lemma is clear. To show convergence we write the integral as

(9) $$\int_{F^*} \int_{F^{n-1}} W(m(A_{\xi,\eta})) \, d\xi \, |\eta|^{1-n} \, d^*\eta$$

Let $A_{\xi,\eta} = ntk$ be the Iwasawa decomposition with $n \in N''$, $k \in K''$ and $t = t_{\xi,\eta} = \mathrm{diag}(t_1, \dots, t_n)$. Then

(10) $$\prod_{j=k+1}^{n} |t_j| = \max(1, |\xi_k|, \dots, |\xi_{n-1}|) \ \ k = 1, \dots, n-1 \text{ and } \prod_{j=1}^{n} |t_j| = |\eta|.$$

Note that $\delta(t)^{\frac{1}{2}} = |t_1|^{n-\frac{1}{2}} \prod_{i=2}^{n} |\alpha_i(t)|^{m_i}$ with $m_i \geq 0$. On the other hand,

$$|t_1| = \frac{|\prod_{i=1}^n t_i|}{|\prod_{i=2}^n t_i|} = \frac{|\eta|}{\max(1, |\xi_1|, \dots, |\xi_{n-1}|)} \leq |\eta|.$$

Thus, by (2), the integrand in (9) is bounded by

$$\phi(\alpha_1(t), \dots, \alpha_n(t))|\eta|^{\frac{1}{2}}$$

for some Schwartz function $\phi \in \mathcal{S}(F^n)$. Indeed, the support condition of ϕ gives upper bounds on $|\alpha_i(t)|$, $i = 1, \dots, n$ and therefore upper bounds on $|t_i|$, $i = 1, \dots, n$ and hence on $|\xi_i|$, $i = 1, \dots, n-1$ and $|\eta|$ as well. Thus, we can ignore the non-negative powers of $|\alpha_i(t)|$ and $|\eta|$ occurring in (2). Thus the integral (9) is majorized by the convergent integral

$$\int_{|\xi_i| < C_i} \int_{|\eta| < C_n} C|\eta|^{\frac{1}{2}} \, d\xi \, d^*\eta$$

for some positive constants $C, C_1, \dots, C_n$. □

2.3. **Definition of local Bessel distribution.** Let $\pi \in \Pi_2^\psi(G)$. We define the local relative Bessel distribution of π by

$$\mathbb{B}^\pi(f) := \mathcal{B}^\pi_{\mathcal{P},\mathcal{W}}(f) = \mathcal{B}^{\mathbb{W}^\psi(\pi)}_{\mathcal{P},\lambda^{\psi^{-1}}}(f) = \sum_{W_i} \mathcal{P}(\pi(f)W_i)\overline{W_i(e)} \quad f \in \mathcal{S}(G)$$

where the sum is over an admissible orthonormal basis of $\mathcal{W}^\psi(\pi)$ with respect to the inner product

$$[W_1, W_2] = \int_{N\backslash G} W_1(g)\overline{W_2(g)}\, dg.$$

Note that $\mathcal{B}^\pi_{\mathcal{P},\mathcal{W}}$ does not depend on the choice of the Haar measure on G since rescaling it affects both $\pi(f)$ and the inner product formula (3) in the same way.

Similarly we define the local Bessel distribution of π' by

$$\mathbb{B}^{\pi'}(f') := \mathcal{B}^{\pi'}_{\widetilde{\mathcal{W}},\widetilde{\mathcal{W}}}(f') = \mathcal{B}^{\mathbb{W}^\psi(\pi')}_{\lambda^\psi,\lambda^{\psi^{-1}}}(f') = \sum_{W_i'} \pi'(f')W_i'(e')\overline{W_i'(e')} \quad f' \in \mathcal{S}(G')$$

where $\{W_i'\}$ is an admissible orthonormal basis of $\widetilde{\mathcal{W}}^\psi(\pi')$ with respect to

$$[W_1', W_2'] = \int_{N'\backslash G'} W_1'(g)\overline{W_2'(g)}\, dg.$$

As before, $\mathcal{B}^{\pi'}_{\widetilde{\mathcal{W}},\widetilde{\mathcal{W}}}(f')$ does not depend on the choice of Haar measure on G'. (It only depends on the measure on N' chosen above.)

3. Local theta correspondence

We now wish to realize the local Howe duality explicitly in terms of an integral transform on the corresponding Whittaker models. This was done in [JS03, §2] in case where π is supercuspidal, and is modeled on the global computations of Furusawa ([Fur95]). We need to extend this to the square-integrable case. The only issue is convergence.

Recall that G and G' comprise a dual pair inside $\mathcal{M} = \mathrm{Mp}(Z)$. Let ω_ψ be the Weil representation of $\mathcal{M}$.

3.1. **Explicit theta correspondence I.** Recall the realization $(\omega_\psi^1, \mathcal{S}(Z_+^1))$ of the Weil representation according to the G-invariant splitting $Z_\pm^1 = \mathrm{Hom}(V, V_\pm')$. We sometimes identify Z_+^1 with $M_{2n+1,n}$. The explicit action of ω_ψ^1 is described by formulas [MR04, (3.1)–(3.3)]. In particular:

$$\omega_\psi^1(g, \widetilde{m'(h)})\Phi(X) = |\det(h)|^{n+\frac{1}{2}} \frac{\gamma_\psi(1)}{\gamma_\psi((\det h)^{2n+1})}\Phi(g^{-1}Xh), \tag{11}$$

$$\omega_\psi^1(e, \widetilde{\left(\begin{smallmatrix} 1 & B \\ & 1 \end{smallmatrix}\right)})\Phi(X) = \Phi(X)\psi(\mathrm{Tr}({}^tXw_{2n+1}XBw_n)/2). \tag{12}$$

Here $\gamma_\psi(a)$ is a certain root of unity (Weil's constant).

Let $E_1 \in Z_+^1$ be given by $E_1e_i = E_1e_{-1} = 0$, and $E_1e_{-i-1} = f_i$, $i = 1, \ldots, n$. Thus, E_1 corresponds to $\sum_{i=1}^n e_{i+1} \otimes f_i$ under the isomorphism $Z \simeq V \otimes V'$ (cf. (1)).

Let $\pi \in \Pi_2^\psi(G)$. For $\Phi \in \mathcal{S}(Z_+^1)$ and $W \in \mathcal{W}^\psi(\pi)$ set

$$(\theta_\Phi^\psi W)(g') = \int_{R_1\backslash G} \omega_\psi^1(g, g')\Phi(E_1)W(g)\ dg. \tag{13}$$

The following is an extension of [JS03, Corollary 2.2] from the supercuspidal case to the square-integrable case.

Proposition 1. *The integral* (13) *converges absolutely and* $\theta_\Phi^\psi W \in L^2(N'\backslash G', \psi_{N'})$. *Moreover, there exists* Φ *and* W *such that* $\theta_\Phi^\psi W \not\equiv 0$.

Proof. For the convergence, we may assume, upon replacing Φ by $\omega_\psi^1(e, g')\Phi$ that $g' = e'$. Using Iwasawa decomposition, we have to show the convergence of

$$\int_{\mathrm{GL}_n} \int_{R_1\backslash U} |\omega_\psi^1(vm(h), e')\Phi_k(E_1)W(m(h)k)||\det h|^{-n}\ dv\ dh$$

for any $k \in K$ where $\Phi_k = \omega_\psi(k, e')\Phi$. Without loss of generality we can assume that $k = e$. By (11), the above integral can be written as

$$\int_{\mathrm{GL}_n} \int_{M_{n,1}(F)} |\Phi(h^{-1}e_2, \dots, h^{-1}e_n, e_{n+1} + h^{-1}v)W(m(h))||\det h|^{-n}\ dv\ dh.$$

By changing the variable we can write this as

$$\int_{\mathrm{GL}_n} \int_{M_{n,1}(F)} |\Phi(h^{-1}e_2, \dots, h^{-1}e_n, e_{n+1} + v)W(m(h))||\det h|^{1-n}\ dv\ dh.$$

Using the Iwasawa decomposition again, now for GL_n, we write $h = bk''$ where b is upper triangular and $k'' \in K''$. Let t be the diagonal part of b and write $b^{-1} = (x_{i,j})_{1\le i,j\le n}$. The Haar measure is given by

$$\prod_{i=1}^n |t_i|^{i-1}\ dt\ \ \otimes_{1\le i<j\le n} dx_{i,j}\ dk''.$$

Again, we may suppress k'', so it is enough to show the absolute convergence of the integral over b. By (2), the latter integral is majorized by

$$\iiint \phi(t_2^{-1}, \dots, t_n^{-1}, v, x_{i,j}, \alpha_1(t), \dots, \alpha_n(t)) \prod_{i=1}^n |t_i|^{\frac{1}{2}} \prod_{i=1}^n |\alpha_i(t)|^\lambda\ dv \otimes_{1\le i<j\le n} dx_{i,j}\ dt$$

where ϕ is a Schwartz function in $n - 1 + n + \binom{n}{2} + n$ variables and $\lambda > 0$. Note that the integrand is compactly supported in the variables $t_2, \dots, t_n \in F^*$. Therefore, the integral is majorized by the convergent integral

$$\int \phi'(t_1)|t_1|^{\lambda+\frac{1}{2}}\ d^*t_1$$

for an appropriate $\phi \in \mathcal{S}(F^*)$.

To show that θ_Φ^ψ is non-zero we can assume that $W(e) \neq 0$ and let Φ be a non-negative function supported in a small neighborhood Ξ of E_1 such that $\Phi(E_1) = 1$. Then the set

$\{g \in R_1\backslash G : \omega_\psi^1(g,e')\Phi(E_1) \neq 0\}$ consists of g with $g^{-1}E_1 \in \Xi$, which is an arbitrarily small neighborhood of R_1. Therefore, $\theta_\Phi^\psi W(e') \neq 0$.

To show that $\theta_\Phi^\psi W \in C(N'\backslash G', \psi_{N'})$, i.e., that

$$(\theta_\Phi^\psi W)(\widetilde{y'}g') = \psi_{N'}(y')(\theta_\Phi W)(g') \tag{14}$$

for all $y' \in N'$ we may separate to the case where $y' = m'(u')$, $u' \in N''$ and the case $y' \in U'$. In the first case, by (11)

$$(\theta_\Phi^\psi W)(\widetilde{m'(u')}g') = \int_{R_1\backslash G} \omega_\psi^1(g,g')\Phi(E_1u')W(g)\ dg = \int_{R_1\backslash G} \omega_\psi^1(x^{-1}g,g')\Phi(E_1)W(g)\ dg$$

where x is any element in N such that its $(1,j)$-th entries are 0 for $j>1$, and $(i+1,j+1)$-th entry is $u'_{i,j}$ for $1 \le i < j \le n$. Since $W(xg) = \psi_N(x)W(g) = \psi_{N'}(y')W(g)$, a change of variable $g \mapsto xg$ gives (14) in this case.

In the case $y' \in U'$, by (12) we have

$$\omega_\psi(g,\widetilde{y'}g')\Phi(E_1) = \psi(y'_{n,n+1}/2)\omega_\psi(g,g')\Phi(E_1) = \psi_{N'}(y')\omega_\psi(g,g')\Phi(E_1).$$

We again obtain (14).

To see that $\theta_\Phi W \in L^2(N'\backslash G', \psi_{N'})$ we write

$$\theta_\Phi^\psi W(g') = \int_{R_1\backslash G} \omega_\psi^1(g,g')\Phi(E_1)W(g)\ dg = \int_{N\backslash G}\int_{R_1\backslash N} \omega_\psi^1(ng,g')\Phi(E_1)W(ng)\ dg.$$

We will show the stronger statement

$$\int_{N\backslash G} |\int_{R_1\backslash N} \omega_\psi^1(ng,\cdot)\Phi(E_1)\psi_N(n)\ dn||W(g)|\ dg \in L^2(N'\backslash G'). \tag{15}$$

(By the previous argument we know that

$$\int_{R_1\backslash N} \omega_\psi^1(ng,\cdot)\Phi(E_1)\psi_N(n)\ dn \in C(N'\backslash G', \psi_{N'})$$

for all $g \in G$.) Using Iwasawa decomposition, this amounts to showing that

$$t' \mapsto \delta'(t')^{-\frac{1}{2}} \int_T |\int_{R_1\backslash N} \omega_\psi^1(nt,\widetilde{t'})\Phi(E_1)\psi_N(n)\ dn||W(t)|\delta^{-1}(t)\ dt \in L^2(T').$$

At this stage we will use the following Lemma which will be proved below.

Lemma 4. *There exists $\phi \in \mathcal{S}(F^{2n})$ such that*

$$(\delta'(t')\delta(t))^{-\frac{1}{2}}|\int_{R_1\backslash N} \omega_\psi^1(nt,\widetilde{t'})\Phi(E_1)\psi_N(n)\ dn| = \phi(y_1,\dots,y_n,z_1,\dots,z_n)\prod_{i=1}^n |t_i/t'_i|^{\frac{1}{2}}$$

where $y_i = \frac{t_i}{t'_i}$ and $z_i = \frac{t'_i}{t_{i+1}}$, $i=1,\dots,n$ (with $t_{n+1}=1$).

By the Lemma and the bounds (2) it suffices now to show that for any $\lambda > 0$ and $\phi \in \mathcal{S}(F^{2n})$ we have

$$t' \mapsto \int_T \phi(y_1, \dots, y_n, z_1, \dots, z_n) \prod_{i=1}^n |\alpha_i(t)|^\lambda |t_i/t'_i|^{\frac{1}{2}} \; dt \in L^2(T'),$$

or alternatively, by a change of variables $t_i \mapsto t_i t'_i$, that

$$t' \mapsto \int_T \phi(t_1, \dots, t_n, \alpha'_1(t')t_2^{-1}, \alpha_2(t')t_3^{-1}, \dots, \alpha'_n(t'))|t_1|^\lambda \prod_{i=1}^n |\alpha'_i(t')|^\lambda |t_i|^{\frac{1}{2}} \; dt \in L^2(T'). \tag{16}$$

Observe that for any $\varphi \in \mathcal{S}(F^2)$ and $\nu > 0$ there exists $\varphi' \in \mathcal{S}(F)$ such that

$$\int_{F^*} \varphi(t, x/t)|t|^\nu \; dt \le \varphi'(x).$$

It follows that the left-hand side of (16) is bounded by

$$\varphi'(\alpha'_1(t'), \dots, \alpha'_n(t')) \prod_{i=1}^n |\alpha'_i(t')|^\lambda$$

for some $\varphi' \in \mathcal{S}(F^n)$. Hence it belongs to $L^2(T')$ as required. □

Proof of Lemma 4. The integration over $R_1 \backslash N$ can be replaced by integration over the maximal unipotent of GL_{n+1}. We have

$$\omega^1_\psi(ut, \widetilde{t'})\Phi(E_1) = \gamma |\det(t')|^{n+\frac{1}{2}} \Phi(\sum_{i=1}^n t_i^{-1} u^{-1} e_{i+1} \otimes t'_i f_i)$$

where γ is a root of unity which depends on t, t' but not on u.

Thus, the argument of Φ is the matrix

$$\begin{pmatrix} t_1^{-1} x_{1,2} t'_1 & t_1^{-1} x_{2,3} t'_2 & \dots & t_1^{-1} x_{1,n+1} t'_n \\ t_2^{-1} t'_1 & t_2^{-1} x_{2,3} t'_2 & \dots & t_2^{-1} x_{2,n+1} t'_n \\ & t_3^{-1} t'_2 & \dots & t_3^{-1} x_{3,n+1} t'_n \\ & & \ddots & \vdots \\ & & & t'_n \end{pmatrix}$$

where $n^{-1} = (x_{i,j})$. Using the change of variables

$$x_{i,j} \mapsto \frac{t_i}{t'_{j-1}} x_{i,j} \quad 1 \le i < j \le n+1$$

the integral becomes $\prod_{i=1}^n |t_i|^{n+1-i} |t'_i|^{n+\frac{1}{2}-i}$ times

$$\int_{F^{\binom{n+1}{2}}} \Phi(\begin{pmatrix} x_{1,2} & x_{2,3} & \dots & x_{1,n+1} \\ t_2^{-1} t'_1 & x_{2,3} & \dots & x_{2,n+1} \\ & t_3^{-1} t'_2 & \dots & x_{3,n+1} \\ & & \ddots & \vdots \\ & & & t'_n \end{pmatrix}) \psi(\frac{t_1}{t'_1} x_{1,2} + \dots + \frac{t_n}{t'_n} x_{n,n+1}) \; dx_{i,j}.$$

The last integral is the value at $(y_1, \dots, y_n, z_1, \dots, z_n)$ of (the restriction to a certain subspace of) a partial Fourier transform of Φ. ☐

In general, for an irreducible representation π of G there exists a genuine representation $\Theta_\psi(\pi)$ of G' such that

$$\omega_\psi[\pi] := \omega_\psi / \cap_{\phi \in \operatorname{Hom}_G(\omega_\psi, \pi)} \ker \phi \simeq \pi \otimes \Theta_\psi(\pi)$$

as representations of $G \times G'$. It is known that $\Theta_\psi(\pi)$ is of finite-length. Let π' be an irreducible representation of G'. One says that π and π' *correspond under the Howe ψ-duality* if

$$\operatorname{Hom}_{G \times G'}(\omega_\psi, \pi \otimes \pi') \neq 0, \tag{17}$$

i.e., if π' is a quotient of $\Theta_\psi(\pi)$. By the *Howe duality conjecture* $\Theta_\psi(\pi)$, if non-zero, admits a unique irreducible quotient. In other words,

1. π' satisfying (17) is uniquely determined by π, and
2. $\dim \operatorname{Hom}_{G \times G'}(\omega_\psi, \pi \otimes \pi') = 1$.

The conjecture is known in many cases [Wal90, How89, LST11]. However, for our purposes we will only need the earlier weaker results of [Kud86].

Corollary 1. *Suppose that $\pi \in \Pi_2^\psi(G)$. Then the span V' of $\{\theta_\Phi^\psi W : \Phi \in \mathcal{S}(Z_+^1), W \in \mathcal{W}^\psi(\pi)\}$ is an irreducible subrepresentation π' of $L^2(N' \backslash G', \psi_{N'})$. The representations $\pi^\vee \simeq \pi$ and π' correspond under the Howe ψ-duality.*

Proof. The map $\Phi \mapsto (W \mapsto \theta_\Phi W)$ is clearly an element of

$$\operatorname{Hom}_G\left(\omega_\psi^1, \operatorname{Hom}(\mathcal{W}^\psi(\pi), L^2(N' \backslash G', \psi_{N'}))\right)$$

and hence can be viewed as an element of $\operatorname{Hom}_G\left(\omega_\psi^1, \mathcal{W}^\psi(\pi)^\vee \otimes L^2(N' \backslash G', \psi_{N'})\right)$ since $\mathcal{W}^\psi(\pi)^\vee \otimes L^2(N' \backslash G', \psi_{N'})$ is the G-smooth part of $\operatorname{Hom}\left(\mathcal{W}^\psi(\pi), L^2(N' \backslash G', \psi_{N'})\right)$ (since $\mathcal{W}^\psi(\pi)$ is admissible). In particular, V' is a quotient of $\Theta_\psi(\pi^\vee)$ and hence it is of finite length. Since V' is unitary, it is a direct sum of irreducible representations. On the other hand, since all irreducible quotients of $\Theta_\psi(\pi^\vee)$ have the same supercuspidal support [Kud86], it follows that V' is the unique irreducible generic representation with this supercuspidal support. ☐

Lemma 5. *We have*

$$(\theta_\Phi^\psi W)(e') = \mathcal{P}(\pi(f) W) \tag{18}$$

whenever f satisfies

$$\int_R f(rg) \psi_R(r) \, dr = \int_{R_0 \backslash R} \omega_\psi^1(rg, e') \Phi(E_1) \psi_R(r) \, dr. \tag{19}$$

Proof. By (8),

$$(\theta_\Phi^\psi W)(e') = \int_{R_0\backslash G} \omega_\psi^1(g,e')\Phi(E_1)\mathcal{P}(\pi(g)W)\ dg$$

$$= \int_{R\backslash G}\int_{R_0\backslash R} \omega_\psi^1(rg,e')\Phi(E_1)\psi_R(r)\mathcal{P}(\pi(g)W)\ dr\ dg$$

$$= \int_{R\backslash G}\int_{R} f(rg)\mathcal{P}(\pi(rg)W)\ dr\ dg = \int_{R\backslash G} f(g)\mathcal{P}(\pi(g)W)\ dg$$

as claimed. □

3.2. **Explicit theta correspondence II.** To go in the other direction we work with the mixed model realization of the Weil representation as in [MVW87]. Namely, decompose $Z_+^1 \simeq V\otimes V_+'$ as

$$\big((V_-\oplus\langle e_{n+1}\rangle)\otimes V_+'\big)\oplus\big(V_+\otimes V_+'\big)$$

and let $Z_+^2 = V_-\otimes V'\oplus e_{n+1}\otimes V_+'$. Define $T:\mathcal{S}(Z_+^1)\to\mathcal{S}(Z_+^2)$ to be the partial Fourier transform with respect to $V_+\otimes V_+'$ given by

$$T(\Phi)(X,Y) = \int_{V_+\otimes V_+'} \Phi(X,W)\psi(\langle Y,W\rangle)\ dW$$

where $X\in(V_-\oplus\langle e_{n+1}\rangle)\otimes V_+'$, $Y\in V_-\otimes V_-'$. Here we identify $V_+\otimes V_+'$ with F^{n^2} through the basis $e_i\otimes f_j$ and endow it with the product measure.

Define the realization ω_ψ^2 on $\mathcal{S}(Z_2^+)$ by

$$\omega_\psi^2(g)T(\Phi) = T(\omega_\psi^1(g)\Phi),\quad \Phi\in\mathcal{S}(Z_1^+).$$

Explicitly, identifying Z_+^2 with $M_{n,2n}\oplus M_{n,1}$ and letting $\Phi'=\Phi_1\otimes\Phi_2\in\mathcal{S}(Z_+^2)$ where $\Phi_1\in\mathcal{S}(M_{n,2n}(F))$ and $\Phi_2\in\mathcal{S}(M_{n,1}(F))$, we have ([MR04, (3.13)–(3.16)])

$$\omega_\psi^2(m(u),g')[\Phi_1\otimes\Phi_2](X,x) = [\Phi_1\otimes\omega_\psi(g')\Phi_2]((u^*)^{-1}Xg',x)\quad g'\in G', u\in N'', \tag{20}$$

$$\omega_\psi^2(u,e')[\Phi_1\otimes\Phi_2](E_2) = \psi_N^{-1}(u)[\Phi_1\otimes\Phi_2](E_2)\quad u\in U, \tag{21}$$

where $E_2 = e_{n+1}\otimes f_n - \sum_{i=1}^n e_{-i}\otimes f_{-i}$ (identified with $[(0_n,-1_n),f_n]$) and ω_ψ denotes the Weil representation on $\mathcal{S}(V_+')$. In particular,

$$\omega_\psi^2(g,\widetilde{u'}g')\Phi'(E_2) = \psi_{N'}(u')\omega_\psi^2(g,g')\Phi'(E_2),\ \ u'\in U'. \tag{22}$$

Let $\pi'\in\Pi^{\psi^{-1}}(G')$. For $\Phi'\in\mathcal{S}(Z_+^2)$ and $W'\in C(N'\backslash G',\psi_{N'}^{-1})$ define

$$(\theta'^{\psi}_{\Phi'}W')(g) = \int_{U'\backslash G'} \omega_\psi^2(g,g')\Phi'(E_2)W'(g')\ dg'. \tag{23}$$

Recall that the measure on U' is fixed in §1.1. By (22) the integrand is left U'-invariant.

Proposition 2. *For any $\pi'\in\Pi^{\psi^{-1}}(G')$ (not necessarily in $\Pi_2^{\psi^{-1}}(G')$) the integral in* (23) *is absolutely convergent and there exists Φ' and W' such that $\theta'^{\psi}_{\Phi'}W'\not\equiv 0$. Moreover, if $\pi'\in\Pi_2^{\psi^{-1}}(G')$ then $\theta'^{\psi}_{\Phi'}W'\in L^2(N\backslash G,\psi_N^{-1})$ for any $W'\in\widetilde{\mathcal{W}}^{\psi^{-1}}(\pi')$.*

Proof. The first part follows from [JS03, Corollary 2.1]. For completeness we provide a proof. By replacing Φ' with $\omega_\psi^2(g,e')\Phi'$, we can assume that $g=e$. By Iwasawa decomposition, to show absolute convergence, it suffices to show that the integrand is a Schwartz function in $m'\in M'$. (Of course, for this statement the extra modulus function $\delta_{P'}^{-1}$ in the integration formula is immaterial.) By (20)

$$\omega_\psi^2(e,\widetilde{m'})\Phi'(E_2)=\gamma|\det(x)|^{1/2}\Phi'((0_n,-x^*),*) \text{ for } m'=m'(x)\in M',$$

where γ is a root of unity and we don't specify the second argument of Φ'. It follows directly from (4) that $\omega_\psi^2(e,\widetilde{m'})\Phi'(E_2)\cdot W'(\widetilde{m'})$ is a Schwartz function in m'.

It is also clear that ${\theta'}^{\psi}_{\Phi'}W'$ can be made non-zero. Indeed, we can assume that $W'(e')\neq 0$ and let Φ' be a non-negative function supported in a small neighborhood of E_2 such that $\Phi'(E_2)=1$. Then it is easy to see that $\{g'\in U'\backslash G' : \omega_\psi^2(e,g')\Phi'(E_2)\neq 0\}$ is an arbitrarily small neighborhood of U'. Therefore, ${\theta'}^{\psi}_{\Phi'}W'(e)\neq 0$.

Next, we show that ${\theta'}^{\psi}_{\Phi'}\in C(N\backslash G,\psi_N^{-1})$, i.e. that

$$({\theta'}^{\psi}_{\Phi'}W')(ng)=\psi_N^{-1}(n)({\theta'}^{\psi}_{\Phi'}W')(g)$$

for all $n\in N$. It is enough to check this relation separately for $n=m(u)$, $u\in N''$ and $n\in U$. In the first case by (20) we have

$$\begin{aligned}
({\theta'}^{\psi}_{\Phi'}W')(ng) &= \int_{U'\backslash G'}\omega_\psi^2(m(u)g,g')\Phi'(E_2)W'(g')\,dg' \\
&= \int_{U'\backslash G'}\omega_\psi^2(g,\widetilde{m'(u^{-1})}g')\Phi'(E_2)W'(g')\,dg' \\
&= \int_{U'\backslash G'}\omega_\psi^2(g,g')\Phi'(E_2)W'(\widetilde{m'(u)}g')\,dg' \\
&= \psi_{N'}(m'(u))^{-1}\int_{U'\backslash G'}\omega_\psi^2(g,g')\Phi'(E_2)W'(g')\,dg' \\
&= \psi_N^{-1}(n)({\theta'}^{\psi}_{\Phi'}W')(g).
\end{aligned}$$

The second case follows from (21).

Finally, we show that ${\theta'}^{\psi}_{\Phi'}\in L^2(N\backslash G,\psi_N^{-1})$ if $\pi'\in\Pi_2^{\psi^{-1}}(G')$. We write

$$\begin{aligned}
{\theta'}^{\psi}_{\Phi'}W'(g) &= \int_{U'\backslash G'}\omega_\psi^2(g,g')\Phi'(E_2)W'(g')\,dg' \\
&= \int_{N'\backslash G'}\int_{U'\backslash N'}\omega_\psi^2(g,\widetilde{n'}g')\Phi'(E_2)\psi_{N'}^{-1}(n')\,dn'\;W'(g')\,dg'.
\end{aligned}$$

Thus, it suffices to show that

$$t\mapsto\delta(t)^{-\frac12}\int_{T'}\Big|\int_{U'\backslash N'}\omega_\psi^2(t,\widetilde{n'}\widetilde{t'})\Phi'(E_2)\psi_{N'}^{-1}(n')\,dn'\;W'(\widetilde{t'})\Big|\delta'(t')^{-1}\,dt'\in L^2(T).$$

As before, we use the following result.

Lemma 6. *There exists* $\phi \in \mathcal{S}(F^{2n})$ *such that*

$$(\delta(t)\delta'(t'))^{-\frac{1}{2}}\left|\int_{U'\backslash N'} \omega_\psi^2(t, \widetilde{n'}\widetilde{t'})\Phi'(E_2)\psi_{N'}^{-1}(n')\, dn'\right| = \phi(y_1, \dots, y_n, z_1, \dots, z_n)\prod_{i=1}^n |y_i|^{\frac{1}{2}}$$

where $y_i = t_i/t'_i$, $z_i = t'_i/t_{i+1}$.

The Lemma can be proved exactly as Lemma 4. Alternatively, it follows readily from the latter using Proposition 3 below. As before, using the Lemma and the bounds (4) we reduce to showing that for any $\mu > 0$ and $\phi \in \mathcal{S}(F^{2n})$ we have

$$t \mapsto \int_{T'} \phi(y_1, \dots, y_n, z_1, \dots, z_n)\prod_{i=1}^n |\alpha'_i(t')|^\mu |t_i/t'_i|^{\frac{1}{2}}\, dt' \in L^2(T).$$

(We may assume without loss of generality that $\mu < \frac{1}{2}$, since $\alpha'_i(t') = z_i y_{i+1}$ is bounded by the support of ϕ.) Alternatively, by a change of variables $t'_i \mapsto t_{i+1}t'_i$ (with the usual convention $t_{n+1} = 1$)

$$\int_{T'} \phi(\alpha_1(t)/t'_1, \alpha_2(t)/t'_2, \dots, \alpha_n(t)/t'_n, t'_1, \dots, t'_n)|t'_1 t_2|^\mu \prod_{i=1}^n |\alpha_i(t)/t'_i|^{\frac{1}{2}}\, dt' \in L^2(T). \tag{24}$$

Observe that for any $\varphi \in \mathcal{S}(F^2)$ and $\nu > 0$ there exists $\varphi' \in \mathcal{S}(F)$ such that

$$\int_{F^*} \varphi(t, x/t)|x/t|^\nu\, dt \le \varphi'(x)$$

for all $x \in F^*$. It follows that the left-hand side of (24) is bounded by

$$\varphi'(\alpha_1(t), \dots, \alpha_n(t))|\alpha_1(t)|^\mu |t_2|^\mu = \varphi'(\alpha_1(t), \dots, \alpha_n(t))\prod_{i=1}^n |\alpha_i(t)|^\mu$$

for some $\varphi' \in \mathcal{S}(F^n)$. Hence it belongs to $L^2(T)$ as required. ☐

As before, we obtain the following

Corollary 2. *The span* (π, V) *of* $\{\theta'^{\psi}_{\Phi'} W' : \Phi' \in \mathcal{S}(Z_+^2), W' \in \widetilde{\mathcal{W}}^{\psi^{-1}}(\pi')\}$ *is an irreducible subspace of* $L^2(N\backslash G, \psi_N^{-1})$*. The representations* π *and* $\pi'^{\vee}$ *correspond under the Howe* ψ*-duality.*

Lemma 7. *We have*

$$(\theta'^{\psi}_{\Phi'} W')(e) = (\pi'(f')W')(e') \tag{25}$$

whenever f' *satisfies*

$$\int_{N'} f'(\widetilde{n'}g')\psi_{N'}^{-1}(n')\, dn' = \int_{U'\backslash N'} \omega_\psi^2(e, \widetilde{n'}g')\Phi'(E_2)\psi_{N'}^{-1}(n')\, dn'. \tag{26}$$

Proof. The integral on the right-hand side of (26) makes sense by (22). We have

$$\begin{aligned}(\theta'^{\psi}_{\Phi'}W')(e) &= \int_{U'\backslash G'} \omega^2_\psi(e,g')\Phi'(E_2)W'(g')\ dg' \\ &= \int_{N'\backslash G'}\int_{U'\backslash N'} \omega^2_\psi(e,\widetilde{n'}g')\Phi'(E_2)\psi^{-1}_{N'}(n')W'(g')\ dn'\ dg' \\ &= \int_{N'\backslash G'}\int_{N'} f'(\widetilde{n'}g')W'(\widetilde{n'}g')\ dn'\ dg'\end{aligned}$$

as required. □

Corollary 3. *The Howe ψ-duality defines a bijection between $\Pi^\psi_2(G)$ and $\Pi^\psi_2(G')$.*

4. Proof of Theorem 1

4.1. **A relation between ω^1_ψ and ω^2_ψ.** We have chosen two models $(\omega^i_\psi, \mathcal{S}(Z^i_+))$, $i = 1,2$ for the Weil representation and an intertwining operator $\Phi \mapsto T(\Phi)$ between them.

Proposition 3. *When $\Phi' = T(\Phi)$,*

$$\int_{U'\backslash N'} \omega^2_\psi(g,\widetilde{n'}g')\Phi'(E_2)\psi^{-1}_{N'}(n')\ dn' = \int_{R_1\backslash N} \omega^1_\psi(ng,g')\Phi(E_1)\psi_N(n)\ dn. \tag{27}$$

Proof. The elements in Z^2_+ will be denoted by (A,B) where A is a $n\times 2n$ matrix and B is a vector in V'_+. The elements in Z^1_+ will be denoted by $(2n+1)\times n$ matrices.

Clearly we only need to establish the identity when g and g' are the identity elements. By (20), the left-hand side of (27) is

$$\int_{N''} T(\Phi)((0,-u^*),f_n)\psi^{-1}_{N''}(u)\ du$$

where

$$\psi_{N''}(u) = \psi(u_{1,2}+\ldots+u_{n-1,n}).$$

Writing $f_{-i}m'(u) = f_{-i} + \sum_{j=1}^{i-1} a_{i,j}f_{-j}$ then the above integral equals

$$\int_{F^{n(n-1)/2}} T(\Phi)(E_2 - \sum_{j=1}^n\sum_{i=j+1}^n a_{i,j}e_{-i}\otimes f_{-j})\,\psi(\sum_{i=1}^{n-1} a_{i+1,i})\ \otimes_{1\le j<i\le n} da_{i,j},$$

where as always $da_{i,j}$ is the self-dual Haar measure on F. Applying the Fourier inversion formula, this becomes

$$\int_{F^{n(n+1)/2}} \Phi(\sum_{i=1}^n e_{i+1}\otimes f_i + \sum_{i=1}^n\sum_{j=i}^n b_{i,j}e_i\otimes f_j)\,\psi(-\sum_{i=1}^n b_{i,i})\ \otimes_{1\le i\le j\le n} db_{i,j}.$$

Let $u\in N$ such that the $(i,j+1)-$th entry of u^{-1} is $b_{i,j}$ for $i=1,\ldots,n$ and $j=i,\ldots,n$, then clearly the above integrand is just $\Phi(u^{-1}E_1)\psi_N(u)$. We get the integral equals

$$\int_{R_1\backslash N} \omega^1_\psi(u,e')\Phi(E_1)\psi_N(u)\ du.$$

This proves the proposition. □

Remark 1. The proposition and its proof carry over verbatim to the archimedean case.

4.2. **Adjointness property.** We define a paring $(\cdot,\cdot)_1$ between $L^2(N\backslash G,\psi_N)$ and $L^2(N\backslash G,\psi_N^{-1})$ by

$$(W_1,W_2)_1=\int_{N\backslash G}W_1(g)W_2(g)\,dg,\ W_1\in L^2(N\backslash G,\psi_N),\ W_2\in L^2(N\backslash G,\psi_N^{-1}).$$

Similarly define paring $(\cdot,\cdot)_2$ between $L^2(N'\backslash G',\psi_{N'})$ and $L^2(N'\backslash G',\psi_{N'}^{-1})$.

Proposition 4. *Suppose that Φ and Φ' are related by (27). Let $\pi\simeq\pi^\vee\in\Pi_2^\psi(G)$ and $\pi'\in\Pi_2^\psi(G')$ correspond under Howe ψ-duality. Then $(\theta_\Phi^\psi W,W')_2=(W,\theta'^\psi_{\Phi'}W')_1$ for all $W\in\mathcal{W}^\psi(\pi)$ and $W'\in\widetilde{\mathcal{W}}^{\psi^{-1}}(\pi'^\vee)$.*

Proof. We have

$$\begin{aligned}(\theta_\Phi^\psi W,W')_2&=\int_{N'\backslash G'}\int_{R_1\backslash G}\omega_\psi^1(g,g')\Phi(E_1)W(g)\ dg\ W'(g')\ dg'\\&=\int_{N'\backslash G'}\int_{N\backslash G}\int_{R_1\backslash N}\omega_\psi^1(ng,g')\Phi(E_1)\psi_N(n)W(g)\ dg\ W'(g')\ dg'.\end{aligned}$$

On the other hand

$$(W,\theta'^\psi_{\Phi'}W')_1=\int_{N\backslash G}W(g)\int_{U'\backslash G'}\omega_\psi^2(g,g')\Phi'(E_2)W'(g')\ dg'\ dg,$$

which is

$$\int_{N\backslash G}W(g)\int_{N'\backslash G'}\int_{U'\backslash N'}\omega_\psi^2(g,\widetilde{n'}g')\Phi'(E_2)\psi_{N'}^{-1}(n')\ dn'\ W'(g')\ dg'\ dg.$$

Applying the relation (27), the integral becomes

$$\int_{N\backslash G}W(g)\int_{N'\backslash G'}\int_{R_1\backslash N}\omega_\psi^1(ng,g')\Phi(E_1)\psi_N(n)\ dn\ W'(g')\ dg'\ dg.$$

The proposition now follows from Fubini's Theorem, since we already know the convergence of

$$\int_{N'\backslash G'}\int_{N\backslash G}|\int_{R_1\backslash N}\omega_\psi^1(ng,g')\Phi(E_1)\psi_N(n)\ dn||W(g)W'(g')|\ dg\ dg'$$

by (15). □

4.3. **Bessel distribution identity.** We say f and Φ match if (19) is satisfied. Similarly we say f' and Φ' match if (26) is satisfied.

Corollary 4. *Assume $\pi\in\Pi_2^\psi(G)$ and $\pi'\in\Pi_2^\psi(G')$ correspond under Howe ψ-duality. Suppose that Φ and Φ' are related by (27), and f and f' match Φ and Φ' respectively. Then*

$$\mathbb{B}^\pi(f)=\mathbb{B}^{\pi'}(f'^\vee).$$

This follows from Lemma 1 applied to $(\pi_1, \hat{\pi}_1, (\cdot,\cdot)_1) = \mathbb{W}^\psi(\pi)$, $(\pi_2, \hat{\pi}_2, (\cdot,\cdot)_2) = \widetilde{\mathbb{W}^\psi}(\pi')$, $A = \theta^\psi_\Phi$, $\hat{A} = \theta'^\psi_{\Phi'}$, $(\ell_1, \hat{\ell}_1) = (\mathcal{P}, \lambda^{\psi^{-1}})$, $(\ell_2, \hat{\ell}_2) = (\widetilde{\lambda}^\psi, \widetilde{\lambda}^{\psi^{-1}})$ using Lemmas 5, 7 and Proposition 4.

We are now ready to prove Theorem 1. First, we recall the notion of matching functions and related concepts.

An $N'' \times N''$ orbit $\{n''_1 g'' n''_2\}$ of $g'' \in \mathrm{GL}_n$ is *relevant* if $n''_1 g'' n''_2 = g''$ implies $\psi_{N''}(n''_1) = \psi_{N''}(n''_2)$. Similarly an $R \times N$ orbit $\{rgn\}$ of $g \in G$ is relevant if $rgn = g$ implies $\psi_R(r) = \psi_N(n)^{-1}$; an $N' \times N'$ orbit $\{n'_1 g' n'_2\}$ of $g' \in \mathrm{Sp}(V')$ is relevant if $n'_1 g' n'_2 = g'$ implies $\psi_{N'}(n'_1) = \psi^{-1}_{N'}(n'_2)$.

Let S_l be a complete set of representatives of relevant orbits in GL_l. In [MR04, §4], two injective maps s and t are defined from $\cup_{0 \le l < n} S_l \times \{\pm 1\} \cup S_n$ to G and $\mathrm{Sp}(V')$ respectively, so that the images of s and t give complete sets of representatives of relevant orbits in G and $\mathrm{Sp}(V')$. For $x \in \cup_{0 \le l < n} S_l \times \{\pm 1\} \cup S_n$, define orbital integrals:

$$I_{s(x)}(f) = \int_R \int_{N \cap s(x)^{-1} R s(x) \backslash N} f(rs(x)n) \psi_R(r) \psi_N(n) \, dn \, dr,$$

$$J_{t(x)}(f') = \int_{N'} \int_{N' \cap t(x)^{-1} N' t(x) \backslash N'} f'(\widetilde{n'_1} \widetilde{t(x)} \widetilde{n'_2}) \psi^{-1}_{N'}(n'_1) \psi^{-1}_{N'}(n'_2) \, dn'_1 \, dn'_2.$$

Define transfer factors $\Delta(x)$ for $x \in \cup_{0 \le l < n} S_l \times \{\pm 1\} \cup S_n$ as in [MR04, §6]. We say f and f' match if $J_{t(x)}(f') = \Delta(x) I_{s(x)}(f)$ for all $x \in \cup_{0 \le l < n} S_l \times \{\pm 1\} \cup S_n$.

By [MR04, Proposition 6.1] to any $f \in \mathcal{S}(G)$ there exists a matching $\tilde{f} \in \mathcal{S}(G')$ More precisely, we can construct $\tilde{f}$ explicitly as follows. First by [ibid., Lemma 5.2] there exists Φ that relates to f through (19). Next, by [ibid., Lemma 5.6], (see [ibid., (5.5)]) there exists $\tilde{f} \in \mathcal{S}(G')$ such that the relation

$$\int_{N'} \tilde{f}(\widetilde{n'} g') \psi^{-1}_{N'}(n') \, dn' = \int_{R_1 \backslash N} \omega^1_\psi(n, g') \Phi(E_1) \psi_N(n) \, dn$$

holds for all $g' \in G'$. From this and (27) we see that $\tilde{f}$ and $T(\Phi)$ match in the sense of (26). By Corollary 4 we infer that

$$\mathbb{B}^\pi(f) = \mathbb{B}^{\pi'}(\tilde{f}^\vee).$$

On the other hand $\mathbb{B}^{\pi'}(f'^\vee)$ is a bi-$(N', \psi_{N'})$-equivariant distribution on $\mathcal{S}(G')$ and therefore, it depends only on the orbital integrals of f' by [GK75]. Theorem 1 follows.

Remark 2. In [MR04, Theorem 7.1], it is established that at almost all places (the odd places where ψ is unramified), when f is an element in the Hecke algebra of G, f' the corresponding element in Hecke algebra of G', the functions f and f' match.

Remark 3. If we adjust the definition of orbital integral in [MR04] to:

$$J'_{t(x)}(f') = \int_{N'} \int_{N' \cap t(x) N' t(x)^{-1} \backslash N'} f'(\widetilde{n'_1} (\widetilde{t(x)})^{-1} \widetilde{n'_2}) \psi_{N'}(n'_1) \psi_{N'}(n'_2) \, dn'_1 \, dn'_2,$$

(and change the definition of relative trace formula in [MR04] accordingly), then clearly $J'_{t(x)}(f'^{\vee}) = J_{t(x)}(f')$. If we redefine matching of functions by the condition $J'_{t(x)}(f') = \Delta(x)I_{s(x)}(f)$, then Theorem 1 can be restated as $\mathbb{B}^{\pi}(f) = \mathbb{B}^{\pi'}(\tilde{f})$ whenever f and $\tilde{f}$ match under the revised matching condition.

5. Global Bessel distribution identity

In this section, we consider the global counterpart of Theorem 1. Let F be a number field and $\mathbb{A}$ its ring of adeles. We retain the notation V, V', Z^1_+, Z^2_+ from §1.1 and §3; and denote by $G'_{\mathbb{A}}$ the metaplectic cover of $\mathrm{Sp}(V'_{\mathbb{A}})$. (We refer to [JS07] for the precise definition and standard facts about the metaplectic group over the adeles.) Let ω^i_ψ be the Weil representation of the metaplectic cover of $\mathrm{Sp}(V_{\mathbb{A}} \otimes V'_{\mathbb{A}})$ on $\mathcal{S}(Z^i_{+,\mathbb{A}})$ defined with respect to the splitting $V \otimes V' = Z^i_+ \oplus Z^i_-$, $i = 1, 2$. Define $\mathcal{S}(G_{\mathbb{A}})$ to be the space of Schwartz-Bruhat functions on $G_{\mathbb{A}}$; similarly for $\mathcal{S}(G'_{\mathbb{A}})$. We define theta functions $\theta^{i,\phi}_\psi(g, g')$ on $G_{\mathbb{A}} \times G'_{\mathbb{A}}$ by:

$$\theta^{i,\phi}_\psi(g, g') = \sum_{z \in Z^i_{+,F}} \omega^i_\psi(g, g')\phi(z), \ \phi \in \mathcal{S}(Z^i_{+,\mathbb{A}}).$$

Let T be as in §3.2. By Poisson summation formula we have

$$\theta^{1,\phi}_\psi = \theta^{2,T(\phi)}_\psi \tag{28}$$

for any $\phi \in \mathcal{S}(Z^1_{+,\mathbb{A}})$. (Cf. the proof of [MR05, Proposition 3.1].)

Let $\pi = \otimes_v \pi_v$ be an irreducible cuspidal representation of $G_{\mathbb{A}}$ realized in $L^2(G_F \backslash G_{\mathbb{A}})$. We denote by $\theta_\psi(\pi)$ the (possibly zero) representation of $G'_{\mathbb{A}}$ generated by the functions

$$\Theta^\phi_\psi[\varphi](g') = \int_{G_F \backslash G_{\mathbb{A}}} \theta^{1,\phi}_\psi(g, g')\varphi(g)\, dg, \ \varphi \in \pi, \ \phi \in \mathcal{S}(Z^1_{+,\mathbb{A}}).$$

Similarly for an irreducible genuine cuspidal representation $\widetilde{\pi}$ of $G'_{\mathbb{A}}$ realized in $L^2(G'_F \backslash G'_{\mathbb{A}})$, we denote by $\theta'_\psi(\widetilde{\pi})$ the representation of $G_{\mathbb{A}}$ generated by functions of the form:

$$\widetilde{\Theta}^\phi_\psi[\widetilde{\varphi}](g) = \int_{G'_F \backslash G'_{\mathbb{A}}} \theta^{2,\phi}_\psi(g, g')\widetilde{\varphi}(g')\, dg', \ \widetilde{\varphi} \in \widetilde{\pi}, \ \phi \in \mathcal{S}(Z^2_{+,\mathbb{A}}).$$

For cusp forms φ, $\hat{\varphi}$ of $G_{\mathbb{A}}$ let $(\varphi, \hat{\varphi})_{G_F \backslash G_{\mathbb{A}}} = \int_{G_F \backslash G_{\mathbb{A}}} \varphi(g)\hat{\varphi}(g)\, dg$. Similarly for G'_A. By (28) we have the adjointness relation

$$(\Theta^\phi_\psi[\varphi], \varphi')_{G'_F \backslash G'_{\mathbb{A}}} = (\varphi, \widetilde{\Theta}^{T(\phi)}_\psi[\varphi'])_{G_F \backslash G_{\mathbb{A}}}. \tag{29}$$

For any cuspidal automorphic form φ on $G_{\mathbb{A}}$ define the Whittaker coefficient by

$$\mathcal{W}^\psi(\varphi)(g) = \int_{N_F \backslash N_{\mathbb{A}}} \varphi(ng)\psi_N(n)^{-1}\, dn$$

and the Bessel period $\mathcal{P}(\varphi) = \mathcal{P}^\psi(\varphi)$ by

$$\mathcal{P}(\varphi)(g) = \int_{R_F \backslash R_{\mathbb{A}}} \varphi(rg)\psi_R(r)^{-1}\, dr.$$

We recall that by unfolding the global zeta integral for G for $\operatorname{Re} s \gg 0$, if $\mathcal{P}(\varphi) \not\equiv 0$ then $\mathcal{W}^{\psi}(\varphi) \not\equiv 0$ [Gin90].

Similarly, the Whittaker coefficient $\widetilde{\mathcal{W}}^{\psi}(\varphi')$ of an automorphic form φ' on $G'_{\mathbb{A}}$ is given by

$$\widetilde{\mathcal{W}}^{\psi}(\varphi')(g') = \int_{N'_F \backslash N'_{\mathbb{A}}} \varphi'(\widetilde{n'} g') \psi_{N'}^{-1}(n') \, dn'.$$

We write $\mathcal{W}^{\psi}(\varphi)$ for $\mathcal{W}^{\psi}(\varphi)(e)$ and similarly for $\mathcal{P}(\varphi)$ and $\widetilde{\mathcal{W}}^{\psi}(\varphi')$.

Furusawa, extending results of Waldspurger for $n = 1$, obtained the following formulas which are the global analogues of the results in section 3.

Proposition 5. *(*[Fur95]*)*

(1) *Let $\varphi \in \pi$, $\phi = \otimes \phi_v \in \mathcal{S}(Z^1_{+,\mathbb{A}})$, $\varphi' = \Theta^{\phi}_{\psi}[\varphi]$. Then $\widetilde{\mathcal{W}}^{\psi}(\varphi') = \mathcal{P}(\pi(f)\varphi)$ where $f = \otimes f_v$ such that f_v and ϕ_v are related through* (19)*. (We recall that such f always exists.)*

(2) *Let $\varphi' \in \pi'$, $\phi' = \otimes \phi'_v \in \mathcal{S}(Z^2_{+,\mathbb{A}})$ and $\varphi = \widetilde{\Theta}^{\phi'}_{\psi}[\varphi']$. Then $\mathcal{W}^{\psi^{-1}}(\varphi) = \widetilde{\mathcal{W}}^{\psi^{-1}}(\pi'(f')\varphi')$ where $f' = \otimes f'_v$ such that f'_v and ϕ'_v are related through* (26)*. (Once again, such f' exists.)*

In the terminology of [PS79] a *ψ-hypercuspidal automorphic form* is one for which $\mathcal{W}^{\psi}(\varphi) \equiv 0$ and a cusp form is *ψ-generic* if it is orthogonal (in $L^2(G_F \backslash G_A)$) to all ψ-hypercuspidal automorphic forms. In other words, φ is ψ-generic if $(\varphi, \hat{\varphi})_{G_F \backslash G_{\mathbb{A}}} = 0$ for all ψ^{-1}-hypercuspidal $\hat{\varphi}$. We denote by $\mathcal{A}^{\psi}(G)$ the space of ψ-generic cusp forms. (In fact, this space does not depend on ψ.) Define $\mathcal{A}^{\psi}(G')$ similarly. (This time, the dependence on ψ' is important.) It is a formal consequence of local uniqueness of Whittaker models that $\mathcal{A}^{\psi}(G)$ and $\mathcal{A}^{\psi}(G')$ are multiplicity-free. (Cf. [PS79].) Let Ξ^{ψ} be the set of irreducible cuspidal representations π of $G_{\mathbb{A}}$ in $\mathcal{A}^{\psi}(G)$ for which $\theta_{\psi}(\pi)$ is non-zero. Similarly let Ξ'^{ψ} be the set of irreducible genuine cuspidal representations π' of $G'_{\mathbb{A}}$ in $\mathcal{A}^{\psi}(G')$ such that $\theta_{\psi^{-1}}(\pi') \neq 0$. From Proposition 5 and the relation (29) we deduce

Corollary 5. (1) *Suppose that $\pi \in \Xi^{\psi}$. Then $\theta_{\psi}\pi \in \Xi'^{\psi}$.*
(2) *Suppose that $\pi' \in \Xi'^{\psi}$. Then $\theta'_{\psi^{-1}}\pi' \in \Xi^{\psi}$.*

Combining Corollary 5 and the results of Jiang-Soudry ([JS07]) we conclude

Proposition 6. *θ_{ψ} defines a bijection between Ξ^{ψ} and Ξ'^{ψ} whose inverse map is $\theta'_{\psi^{-1}}$. Moreover, Ξ^{ψ} is the set of irreducible representations π in $\mathcal{A}^{\psi}(G)$ such that $L(\frac{1}{2}, \pi) \neq 0$, or equivalently, $\mathcal{P} \not\equiv 0$ on π, while Ξ'^{ψ} consists of all irreducible representations in $\mathcal{A}^{\psi}(G')$.*

Let $\pi \in \Xi^{\psi}$ with its automorphic realization V in $\mathcal{A}^{\psi}(G)$. Then $\pi^{\vee} \in \Xi^{\psi^{-1}}$. Let $(\hat{\pi}, \hat{V})$ be the automorphic realization of $\pi^{\vee}$ in $\mathcal{A}^{\psi^{-1}}(G)$. (In fact, $\hat{\pi} = \pi$ and $\hat{V} = V$.) Similarly, let (π', V') be the automorphic realization of $\pi' \in \Xi'^{\psi}$ in $\mathcal{A}^{\psi}(G')$ and let $(\hat{\pi}', \hat{V}')$ be the automorphic realization of $\pi'^{\vee}$ in $\mathcal{A}^{\psi^{-1}}(G')$. Thus, $\hat{V}' = \{\overline{\varphi} : \varphi \in V'\}$ where $\overline{\varphi}(g) = \overline{\varphi(g)}$.

Suppose that $\pi \in \Xi^{\psi}$ and $\pi' = \theta_{\psi}(\pi)$. Then by (29) we have $\theta'_{\psi}(\hat{\pi}') = \hat{\pi}$. Let $\iota_{\pi} : \pi^{\vee} \mapsto \hat{\pi}$ and $\iota_{\pi'} : \widetilde{\pi}^{\vee} \mapsto \hat{\pi}'$ be the intertwining maps defined by pairings $(\cdot, \cdot)_{G_F \backslash G_{\mathbb{A}}}$ and $(\cdot, \cdot)_{G'_F \backslash G'_{\mathbb{A}}}$

respectively. Define distributions on $G_{\mathbb{A}}$ and $G'_{\mathbb{A}}$ with respect to π and π' as follows:

$$\mathbb{B}^{\pi}(f) = \mathcal{B}_{\mathcal{P},\mathcal{W}^{\psi^{-1}}}^{(\pi,\hat{\pi},(\cdot,\cdot)_{G_F\backslash G_{\mathbb{A}}})}(f) = \mathcal{W}^{\psi^{-1}} \circ \iota_{\pi}(\mathcal{P} \circ \pi(f)) = \sum_i \mathcal{P}(\pi(f)\varphi_i)\overline{\mathcal{W}^{\psi}(\varphi_i)},$$

$$\mathbb{B}^{\pi'}(f') = \mathcal{B}_{\widetilde{\mathcal{W}}^{\psi},\widetilde{\mathcal{W}}^{\psi^{-1}}}^{(\pi',\hat{\pi}',(\cdot,\cdot)_{G'_F\backslash G'_{\mathbb{A}}})}(f') = \widetilde{\mathcal{W}}^{\psi^{-1}} \circ \iota_{\pi'}(\widetilde{\mathcal{W}}^{\psi} \circ \pi'(f')) = \sum_i \widetilde{\mathcal{W}}^{\psi}(\pi'(f')\varphi'_i)\overline{\widetilde{\mathcal{W}}^{\psi}(\varphi'_i)}$$

where φ_i is an admissible orthonormal basis of π, that is, the restricted tensor product of admissible bases of π_v, and similarly for φ'_i.

Theorem 2. *Given $f = \otimes f_v \in \mathcal{S}(G_{\mathbb{A}})$ there exists a matching $f' = \otimes f'_v \in \mathcal{S}(G'_{\mathbb{A}})$ (that is, f'_v matches f_v for all v) such that for any $\pi \in \Xi^{\psi}$ we have*

$$\mathbb{B}^{\pi}(f) = \mathbb{B}^{\pi'}(f'^{\vee}) \tag{30}$$

where $\pi' = \theta_{\psi}(\pi)$.

Proof. We argue as in the proof of Theorem 1. Given $f = \otimes f_v \in \mathcal{S}(G_{\mathbb{A}})$ we construct $f' = \otimes f'_v$ using [MR04] by first constructing ϕ_v related to f_v and then choosing f'_v related to $T(\phi_v)$. (In [ibid.] f_v is restricted to be compactly supported, but in the archimedean case the proof works in fact for any $f_v \in \mathcal{S}(G(F_v))$.) Applying the global analogue of Lemma 1 we get (30). □

Remark 4. Assuming the validity of the Gelfand-Kazhdan localization principle in the archimedean case, every bi-(N', ψ')-equivariant distribution on $G'_{\mathbb{A}}$ would be determined by its orbital integrals. Therefore, we would be able to rephrase Theorem 2 by simply saying that the relation (30) holds for any matching f, f'.

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