NONLOCAL ELLIPTIC AND PARABOLIC PROBLEMS
BANACH CENTER PUBLICATIONS, VOLUME 66
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2004

ON THE EQUATION $u_t = \Delta u + M \exp u / \int \exp u \, dx$ IN PLANAR DOMAINS

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Abstract. The blow-up of solutions for a parabolic equation with nonlocal exponential nonlinearity is studied.

1. Introduction. Consider the equation

$$\frac{\partial u}{\partial t} = \Delta u + M \frac{Ve^u}{\int_D Ve^u dx}; \quad u(x,0) = u_0; \quad u|_\partial D = 0. \tag{1.1}$$

Here $D \subset R^2$ is a bounded domain $\alpha < V = V(x) < \beta$ is a continuous function on D where $0 < \alpha \leq \beta < \infty$. The constant $M > 0$ plays a significant rule in the global existence theory for this system, as we shall see below.

Equation (1.1) is a limit of some version of the Keller-Segel system [KS]

$$\varepsilon_1 \frac{\partial \rho}{\partial t} = \nabla \cdot (-\rho \nabla w + \nabla \rho), \tag{1.2}$$

$$\varepsilon_2 \frac{\partial u}{\partial t} = \Delta u + \rho, \tag{1.3}$$

where

1. $\rho = \rho(x,t)$ stands for the density of population of amoebae (or other living cells),
2. $w = w(x,t)$ stands for a chemical (sensitivity) attracting these cells,
3. $u(x,t)$ is the part of w which is produced by the cells themselves,
4. $w(x,t) = u(x,t) + \eta(x)$,where η is a *fixed* (in time) distribution of the chemical,
5. the no-flux boundary condition $(\rho \nabla w + \nabla \rho) \cdot \nu|_{\partial D} = 0$ where ν is the normal to ∂D, is assumed on (1.2), while the Dirichlet boundary condition $u|_{\partial D} = 0$ is assumed on (1.3).

2000 *Mathematics Subject Classification*: 35Q, 35K60, 35B40
Key words and phrases: nonlinear parabolic equation, blow-up of solutions.
The paper is in final form and no version of it will be published elsewhere.

The positive parameters ε_1 and ε_2 determine the rates of the cell and chemical dynamics, respectively.

The limit $\varepsilon_2 = 0$ is known in the literature and was studied by many authors, see [S], [BN] and references therein. In this case, equation (1.3) is reduced to a Poisson equation

$$\Delta u + \rho = 0 \quad \Longrightarrow \quad u(x,t) = \int_D G(x,y)\rho(y,t)dy. \tag{1.4}$$

where $G = \Delta^{-1}$ is the Green function associated with the Laplacian and Dirichlet b.c.

The second limit $\varepsilon_1 = 0$ is less familiar in the literature. In this case, equation (1.2) together with the no-flux boundary conditions yield

$$-\rho\nabla w + \nabla\rho \equiv 0 \Longrightarrow \rho(x,t) = M\frac{V(x)e^{u(x,t)}}{\int_D Ve^u},$$

where $V = e^\eta$ and M is the total (conserved) mass of the population

$$M = \int_D \rho(x,t)dx = \int_D \rho(x,0)dx.$$

Substituting the above in (1.3) (with $\varepsilon_2 = 1$) we obtain (1.1).

The system (1.2)–(1.3) can be presented as a generalized gradient system. Let the functional

$$\mathcal{F}(\rho,u) = \frac{1}{2}\int_D |\nabla u|^2 - \int_D \rho(\eta+u) + \int_D \rho\ln\rho$$

be defined on the domain $\Lambda_M \times \mathbb{H}_0^1(D)$, where

$$\Lambda_M = \left\{\rho \in \mathbb{L}_1(D),\ \rho \geq 0;\ \int_D \rho\ln\rho < \infty, \quad \int_D \rho = M\right\}.$$

Then, system (1.2)–(1.3) is rewritten as

$$\varepsilon_1\frac{\partial\rho}{\partial t} = \nabla\cdot\left[\rho\nabla\delta_\rho\mathcal{F}\right], \tag{1.5}$$

$$\varepsilon_2\frac{\partial u}{\partial t} = -\delta_u\mathcal{F}, \tag{1.6}$$

where δ_ρ (δ_u) stand for the standard first variational derivative with respect to ρ (u). The functional $\mathcal{F}$ is monotone nonincreasing along the solution. Indeed, using integration by parts and the boundary conditions for (1.2):

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(\rho(\cdot,t),u(\cdot,t)) &= \int_D \delta_\rho\mathcal{F}\frac{\partial\rho}{\partial t} + \int_D \delta_u\mathcal{F}\frac{\partial u}{\partial t} \\ &= -\varepsilon_2\int_D\left|\frac{\partial u}{\partial t}\right|^2 - \frac{1}{\varepsilon_1}\int_D \rho|\nabla\delta_\rho\mathcal{F}|^2 \leq 0.\end{aligned} \tag{1.7}$$

Let us revisit the limit $\varepsilon_2 = 0$. First, note that

$$\min_{u\in\mathbb{H}_0^1}\left[\frac{1}{2}\int_D|\nabla u|^2 - \int_D \rho u\right] = -\frac{1}{2}\int_D\int_D \rho(x)G(x,y)\rho(y)dxdy,$$

where G as defined in (1.4). Then define

$$E(\rho) = \inf_{u\in\mathbb{H}_0^1}\mathcal{F}(\rho,u) \equiv \int_D \rho\ln\rho - \frac{1}{2}\int_D\int_D \rho(x)G(x,y)\rho(y)dxdy - \int_D \rho\eta.$$

An immediate observation shows that the system (1.2) with (1.4) and $\varepsilon_2 = 0$, $\varepsilon_1 = 1$ is equivalent to

$$\frac{\partial \rho}{\partial t} = \nabla \cdot [\rho \nabla \delta_\rho E], \tag{1.8}$$

while

$$\frac{d}{dt} E(\rho(\cdot, t)) = -\int_D \rho \left|\nabla \delta_\rho E\right|^2 \leq 0,$$

namely, by replacing $\mathcal{F}$ in (1.5) and (1.7) by E.

A similar observation holds also in the case $\varepsilon_1 = 0$. A first look at (1.7) may suggests that it is a singular limit, since $1/\varepsilon_1$ appears on the RHS. However, we should expect that, for ε_1 small enough, the density ρ should be close to the minimum of $\mathcal{F}(\cdot, u)$, constrained by the conservation of mass $\int_D \rho = M$, which implies that $\delta_\rho \mathcal{F}$ is close to a constant $\lambda = \lambda(t)$ (which is, in fact, the Lagrange multiplier associated with the mass constraint). This leads us to define

$$H(u) = \inf_{\rho \in \Lambda_M} \mathcal{F}(\rho, u).$$

Next, we observe that the minimum above is obtained at

$$\rho(x,t) = M \frac{V(x) e^{u(x,t)}}{\int_D V e^u}, \qquad V(x) = e^\eta,$$

so

$$H(u) = \frac{1}{2} \int_D |\nabla u|^2 - M \ln \left(\int_D V e^u dx \right) \tag{1.9}$$

and the limit $\varepsilon_1 = 0$, $\varepsilon_2 = 1$ takes the form

$$\frac{\partial u}{\partial t} = -\delta_u H(u), \tag{1.10}$$

while

$$\frac{d}{dt} H(u) = -\int_D \left| \frac{\partial u}{\partial t} \right|^2 \tag{1.11}$$

that is, by replacing $\mathcal{F}$ in (1.6) and (1.7) by H.

2. Global existence and blow-up. It is known that, in the limit $\varepsilon_2 = 0$, a global strong solution exists under reasonable regularity conditions on the data $\rho_0 = \rho(x, 0)$, provided $\int_D \rho_0 = M < 8\pi$. In the case $M > 8\pi$ and D is starlike, there exist initial data for which the solution blows up in a finite time $T < \infty$. See [S], [BN].

However, almost nothing has been written on the second limit $\varepsilon_1 = 0$ of equation (1.10) or (1.1). It is easy to obtain local existence by standard theory of parabolic equations, cf. [LSU]. Global existence for $M < 8\pi$ is also not difficult due to the *Moser-Trudinger* inequality

$$\frac{1}{2} \int_D |\nabla u|^2 dx - 8\pi \int_D \ln \left(\frac{\int_D e^u}{|D|} \right) \geq 0 \quad \forall u \in \mathbb{H}_0^1(D).$$

For references on this inequality, see [B], [CSW] as well as [T], [ST] and references therein.

The question of blow-up for the case $M > 8\pi$ is much harder. A partial result in this direction was obtained in [W1]. I shall review this result below:

THEOREM 1. *If $M > 8\pi$ is not an integer multiple of 8π, then there exists a solution of (1.1) such that*

$$\lim_{t\to T}\int_D e^{u(x,t)}dx = \infty,$$

where $T \le \infty$ is the maximal time of existence of the local solution of (1.1).

Proof (sketch). It was proved in [W1] (Lemma 7), using results of [BM] and [L,S] that, for each bounded $D \subset R^2$, $V \in C^1(\overline{D})$ and $8k\pi \neq M > 8\pi$, $k \in \mathbb{N}$, there exists a constant $C = C(D, M)$ such that for any solution ϕ of the *stationary* problem

$$\Delta\phi + M\frac{Ve^\phi}{\int_D Ve^\phi} = 0\ ; \quad \phi|_{\partial D} = 0,$$

the inequality

$$H(\phi) > -C$$

is satisfied. Now, for $M > 8\pi$ the functional H is unbounded from below (sharpness of the Moser-Trudinger inequality). Let $u_0 \in \mathbb{H}^1_0$ for which $H(u_0) < -C$. By the monotonity of H (1.11) we have $H(u(\cdot,t)) \le H(u_0) < -C$ for any $t \in [0,T)$. On the other hand, if $\limsup_{t\to T}\int_D e^u < \infty$ then there is a uniform control over the $\mathbb{H}_1$ norm of $u(\cdot,t)$ for $t \in [0.T)$. By the local existence theorem (see [W1]), this is enough to guarantee the extension of the solution to time $T+\varepsilon$ for some $\varepsilon > 0$. This implies that $T = \infty$. Then

$$H(u(\cdot,T)) - H(u_0) = -\int_0^T\int_D |\delta_u H(u(\cdot,t))|^2\,dxdt$$

by (1.11), so

$$\int_0^\infty\int_D |\delta_u H(u(\cdot,t)|^2\,dxdt < \infty.$$

This, together with the assumed bound on the $\mathbb{H}_1$ norm of $u(,t)$, is enough to guarantee the existence of a sequence $u(,t_n)$, $t_n \to \infty$, which converges to a critical point ϕ of H which is a steady state. By lower semicontinuity of H we obtain $H(\phi) < -C$, a contradiction.

Finally, assume $\liminf_{t\to T}\int_D e^{u(x,t)}dx < \infty$. A similar argument, based on the bound from above of $u(.,t)$ for $t \in [0,T)$ and local existence theorem implies that $T = \infty$. In this case one can, again, isolate a subsequence $t_n \to \infty$ for which $u(\cdot,t_n)$ is uniformly bounded in $\mathbb{H}^1$ and converge weakly to a steady state ϕ. One can complete the argument as before.

Another result in [W1] shows a *conditional* blow-up.

THEOREM 2. *If the solution of (1.1) blows up in a finite time $T < \infty$, then there exists $x_0 \in D$ and $\gamma \ge 4\pi$ such that the measure*

$$\mu = \gamma\delta_{x_0} + \mu_0,$$

where μ_0 is nonatomic, is in the limit set $\lim_{t\to T} M\frac{Ve^{u(,t)}}{\int_D Ve^{u(,t)}}$.

The proof of Theorem 2 is based on a result in [W2], generalizing an elliptic estimate for the equation $\Delta u + f = 0$, $f \in \mathbb{L}_1$ of [BM], into a parabolic one:

$$\int_D e^{\beta u(x,t)} dx < \frac{C}{4\pi - \beta ||f(,t)||_1} \quad ; \quad t > 0$$

where $u(x,t)$ is a solution to the *linear* equation

$$u_t = \Delta u + f; (x,t) \in D \times \mathbb{R}^+; u(x,t)|_{\partial D} = 0; u(x,0) = 0;\ f \in \mathbb{L}_\infty \left(\mathbb{R}^+, \mathbb{L}_1(D)\right).$$

The main argument utilizes this estimate to show that, unless the limit set contains an atomic measure $\gamma\delta_{x_0}$ for some $x_0 \in D$, there is a uniform control on the $\mathbb{L}_p$ norm of $Ve^{u(,t)}/\int_D Ve^{u(x,t)}dx$ for some $p > 1$ as $t \to T$. This, in turn, implies $T = \infty$ due to local (in time) existence, as in Theorem 1.

References

[1] W. Beckner, *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, Ann. Math. 128 (1993), 213–242.

[2] H. Brezis and F. Merle, *Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions*, Comm. P.D.E 16 (1991), 1223–1253.

[3] P. Biler and T. Nadzieja, *Nonlocal parabolic problems in statistical mechanics*, Nonlinear Analysis Th., Meth. Appl. 30 (1997), 5343–5350.

[4] E. F. Keller and L. A. Segel, *Travelling bands for chemotactic bacteria, a theoretical analysis*, J. Theor. Biol., 80 (1971), 235–248.

[5] M. Chipot, I. Shafrir and G. Wolansky, *On the solutions of Liouville systems*, J. Diff. Eq. 140 (1997), 59-105.

[6] O. A. Ladyzenskaja, V. A. Solonnikov and N.N. Ural'ceva, *Linear and Quasi-Linear Equations of Parabolic Type*, Trans. of Math. Monographs, 23, AMS, 1968.

[7] Y. Y. Li and I. Shafrir, *Blow up analysis for solutions of $-\Delta u = Ve^u$ in dimension two*, Indiana Univ. Math. J. 43 (1994), 1255–1270.

[8] T. Suzuki and T. Senba, *Applied Analysis, Mathematical Methods in Applied Science*, World Scientific.

[9] M. Struwe and G. Tarantello, *On multivortex solutions in the Chern-Simons gauge theory*, Boll. UMI 8 1-B (1998), 109–121.

[10] G. Tarantello, *Multiple condensates solutions for the Chern-Simons-Higgs theory*, J. Math. Phys. 37, 8 (1996), 3769-3796.

[11] G. Wolansky, *A critical parabolic estimate and application to nonlocal equations arising in chemotaxis*, Applicable Analysis 66 (1997), 291–321.

[12] G. Wolansky, *A concentration theorem for the heat equation*, Monatsh. Math. 132 (2001), 255–261.