THE THIN FILM EQUATION WITH "BACKWARDS" FORCING

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Dedicated to Roberta Dal Passo (1956-2007)

Abstract. In this paper, we focus on the thin film equation with lower order "backwards" diffusion which can describe, for example, structure formation in biofilms and the evolution of thin viscous films in the presence of gravity and thermo-capillary effects. We treat in detail the equation

$$u_t + \{u^n(u_{xxx} + \nu u^{m-n}u_x - Au^{M-n}u_x)\}_x = 0,$$

where $\nu = \pm 1$, $0 < n$, $m < M$, and $0 \leq A$. Global existence of weak non-negative solutions is proven when $-2 < m - n$, and $A > 0$ or $\nu = -1$, and when $-2 < m - n < 2$ if $A = 0$, $\nu = 1$. From the weak solutions, we get strong entropy solutions under the additional the constraint that $m > n - 3/2$ if $\nu = 1$. A local energy estimate is obtained when $2 \leq n < 3$ under some additional restrictions. Finite speed of propagation is proven for the case of "strong slippage," $0 < n < 2$, when $m > n/2$ and $\nu = 1$, based on local entropy estimates, and for the case of "weak slippage," $2 \leq n < 3$, when $m < n/2$, based on local entropy and energy estimates.

Keywords: thin film equation, backwards diffusion, higher order parabolic equations, degenerate parabolic equations, finite speed of propagation

AMS Subject Classifications: 35K65, 35K30, 35K35, 35G25, 76A20, 76D08

1. Introduction

The thin film equation [20]

$$u_t + \{u^n(u_{xxx})\}_x = 0, \quad n > 0, \tag{1.1}$$

often needs to be augmented in modeling specific physical systems in order to take into account the presence of additional physical effects. See [18] for a survey and review. The systems which we wish to accommodate in the present paper include equations for

(i) structure formation and the dynamics of biofilms [14]

$$u_t + \{u(1-u)(u_{xxx} + h'(u)u_x)\}_x = 0, \tag{1.2}$$

where $h(u) = a(u^2 - bu^3)$, and a, b are positive constants,

(ii) the evolution of thin viscous films in the presence of gravity and thermo-capillary effects

$$u_t + \{u^n(u_{xxx} + u^{m-n}u_x - Au^{M-n}u_x)\}_x = 0, \tag{1.3}$$

where m, n, M, A are constants such that $0 \leq A$, $0 < n, m < M$, and the perhaps more accurate variant of (1.3) given by

$$u_t + \{u^n(u_{xxx} + h'(u)u_x)\}_x = 0, \tag{1.4}$$

where $h'(u) = u(1+Bu)^{-2}$ and B is a positive constant [18, 24, 19], as well as the simpler equation

$$u_t + \{u^n(u_{xxx} + u^{m-n}u_x)\}_x = 0, \tag{1.5}$$

which models the evolution of thin viscous films in the presence of thermo-capillary effects but without gravity.

(iii) An additional example in the spirit of the present paper is

$$u_t + \{u^n(u_{xxx} + h'(u)u_x)\}_x = 0, \tag{1.6}$$

which describes (i) the evolution of a thin viscous film in the presence of attractive polar forces if $h(u) = -ae^{-u/b}$ and a, b are positive constants, or (ii) the evolution of a thin viscous film in the presence of attractive van der Waals forces if $h(u) = Au^{-\alpha}$, where $A < 0$ is a (negative) Hamaker constant and α is a positive constant. See [16, 17, 18].

To approach these different model equations, we shall focus on the equation

$$u_t + \{u^n(u_{xxx} + \nu u^{m-n}u_x - Au^{M-n}u_x)\}_x = 0, \tag{1.7}$$

where $\nu = \pm 1$, and n, m, M, A are constants such that $0 \leq A$, $0 < n$, $m < M$, with some further restrictions to be imposed in the sequel. Typical boundary conditions are

$$u_x = u^n u_{xxx} = 0, \quad x = \pm a, \tag{1.8}$$

and these boundary conditions are adopted here. All of the examples which have been listed above may be written in this form directly, except for (1.4), (1.6 i) in which

$$(a)\ \ h'(u) = u(1+Bu)^{-2} \ \text{ and } \ (b)\ \ h'(u) = \frac{a}{b}e^{-u/b},$$

respectively. Note that

$$0 < \frac{u}{(1+Bu)^2} < \frac{1}{B^2 u}, \tag{1.9}$$

$$0 < \frac{a}{b} e^{-u/b} < \frac{a}{b}, \tag{1.10}$$

for $u \geq 0$, and hence (a), (b) have upper bounds of the form $h'(u) = \nu u^{m-n} - A u^{M-n}$ with $\nu = 1$, $A = 0$, and $m - n = -1$ and $m - n = 0$ respectively, which facilitate the analysis of (1.4), (1.6 i). Our treatment of (1.7) can be generalized to encompass (1.4) and (1.6 i) as well; comments in this direction appear as remarks which follow the statement of our main results.

The term "$\{u^n(\nu u^{m-n} u_x)\}_x$," with $\nu = +1$ in (1.7), is often referred to as a "backwards diffusion" term, since if one considers dynamics dominated by this term alone

$$u_t + \{u^n(u^{m-n} u_x)\}_x = 0,$$

and one linearizes about uniform positive state, then the resultant dynamics is given by the backwards (ill-posed) diffusion equation. Similarly, if $\nu = -1$, the term $(u^n(\nu u^{m-n} u_x))_x$ in (1.7) is often referred to as "forward diffusion," for obvious reasons. We shall often refer to equation (1.7) with $\nu = +1$ as the "unstable case" and to equation (1.7) with $\nu = -1$ as the "stable case," since in the context of thin films, (1.7) with $\nu = +1$ models limiting attractive (or destabilizing) forces and (1.7) with $\nu = -1$ models limiting repulsive (or stabilizing forces).

While (1.7) has often been treated in the presence of forward or stabilizing diffusion (see [8] and references therein)

$$u_t + \{u^n(u_{xxx} - u^{m-n} u_x)\}_x = 0,$$

the backwards or unstable variant

$$u_t + \{u^n(u_{xxx} + u^{m-n} u_x)\}_x = 0, \tag{1.11}$$

has yet to be analyzed in depth. For example, existence and the finite speed of propagation property were proven for (1.11) in [7], but subject to the constraint that $m \geq 0$. Similarly, destabilizing lower order terms were included in the proof of existence and finite speed of propagation given in [12], but the analysis there required the inclusion of stabilizing lower order terms as well. Certain properties of the solutions of the thin film equation in the presence of lower order destabilizing terms have been studied. For example, in Beretta [5], source type solutions with compact support are shown to exist for $0 < n < 3$, $m = n + 2$, and $\nu = \pm 1$; these source type solutions are $\mathcal{C}^1$ solutions such that $u^n u'''$ and $u^{n+2} u'$ are differentiable. There is by now a very rich literature on compactly supported self-similar solutions, steady state solutions and their stability, and blow up, see e.g. [15],[23], [21] and references therein. Notably [7, 23], for blow up to occur within the framework of (1.7), it is necessary to require that $A = 0$, $\nu = -1$, and $m \geq n + 2$, with $m = 3 = n + 2$ constituting a critical case when $n = 1$.

The focus of the present paper, however, is not on blow up, but rather on conditions that guarantee existence, regularity, and finite speed of propagation. As a first step in this direction, the existence of weak nonnegative solutions (see Definition 1) is demonstrated in §2. This is accomplished by means of the following energy estimate

$$\sup_{0 \leq t \leq T} \int_\Omega u_x^2 + \int_0^T \int_\Omega u^n (u_{xxx} + \nu u^{m-n} u_x - A u^{M-n} u_x)^2 \, dx \, dt \leq C, \tag{1.12}$$

where C is time independent and depends only on the problem parameters and the initial conditions. The estimate (1.12) is demonstrated to hold for $0 < T < \infty$, when $\nu = \pm 1$, $0 < n$, $-2 < m - n$, $m < M$ if $A > 0$, and $m - n < 2$ if $\nu = 1$ and $A = 0$. Additionally, an entropy-like estimate is obtained based on a Gronwall inequality for regularized solutions. It is the use of this Gronwall inequality, which is explicitly depending on the regularization parameter, which allows us to control lower order forcing terms which are more singular than those treated up to now. These estimates, together with mass conservation,

$$\int_{\Omega} u(x,\, t)\, dx = \int_{\Omega} u(x,\, 0)\, dx, \tag{1.13}$$

implies global bounds from which existence of weak nonnegative solutions can then be concluded using arguments which are now fairly standard, see Bernis & Friedman [2], Giacomelli [10].

To obtain the existence of a strong ($\mathcal{C}^1(\Omega)$ for a.e. $t > 0$) solution, a local entropy estimate is derived in §3. For the case $\nu = 1$, the additional constraint $m - n > -\frac{3}{2}$ is imposed and the local entropy estimate obtained can be written as

$$\begin{aligned}
&\frac{1}{\alpha(\alpha+1)} \int_{\Omega} \zeta^4 u^{1+\alpha}(x,\, T)\, dx + A \int_{Q_T} \zeta^4 u^{\alpha+M-1} u_x^2 + \\
&\qquad c_1 \left[\int_P \zeta^4 u^{\alpha+n-2\gamma+1} (u^{\gamma})_{xx}^2 + \int_{Q_T} \zeta^4 u^{\alpha+n-3} u_x^4 \right] \le \\
&\qquad c_2 \int_{Q^T} (|\zeta_x|^4 + |\zeta\zeta_{xx}|^2) u^{n+\alpha+1} + c_3 \int_{Q_T} |(\zeta^3 \zeta_x)_x| u^{\alpha+m+1} + \\
&\qquad\qquad c_4 \int_{Q_T} \zeta^4 u^{\alpha+2m-n+1} + \frac{1}{\alpha(\alpha+1)} \int_{\Omega} \zeta^4 {u_0}^{\alpha+1}(x)\, dx,
\end{aligned} \tag{1.14}$$

and holds for certain $\alpha \in (\max\{-2m + n - 1,\, -m - 1\},\, 2 - n) \setminus \{0,\, -1\}$ and for γ satisfying (3.8). For the case $\nu = -1$, a similar estimate is obtained with no additional restrictions. For both the cases, $\nu = \pm 1$, the local entropy estimate implies the global entropy estimate

$$c_5 \int_{\Omega_T} u^{\alpha+n-3} u_x^4 + c_6 \int_{\Omega_T} u^{\alpha+n-1} u_{xx}^2 \le c_7 T, \tag{1.15}$$

for $0 < T < \infty$. Positivity and strong solutions are implied by (1.15) following the arguments of Beretta, Bertsch & Dal Passo [1]. We present a careful new refinement of Theorem 3.1 from [1] (see Lemma 3.1 in §3), which clarifies the set of β for which $C^1([-a,\, a])$ regularity for almost every $t > 0$ is implied for $u^{1/\beta}(\cdot,\, t)$ by the local entropy estimates. A local energy estimate is proven for $\nu = \pm 1$, $2 \le n < 3$ under the additional constraint that $m > (2n - 2)/3$ if $2 \le n \le 5/2$ and $m > n - 3/2$ if $5/2 < n < 3$.

In §4 and §5, we investigate sharp conditions for FSP, the finite speed propagation property, for equation (1.1). For the "standard" thin film equation ($\nu = 0$, $A = 0$), this property was proven by F. Bernis for $0 < n < 2$ in [3], and then for $2 \le n < 3$ in [4]. For equation (1.1) with "forward" (normal) diffusion ($\nu = -1$, $A = 0$), conditions for FSP as well as sharp estimates for the speed of propagation were obtained in [8] for the "strong slippage" case ($0 < n < 2$) only. Here we study the much more delicate case of backward diffusion, where the lower order diffusion term "encourages" the destruction of the FSP property for all values of m. Our analysis makes use of some ideas from [11]. The proof given in §4 is for the "strong slippage case" in which $0 < n < 2$, and requires that $m > n/2$ if $\nu = 1$. It is based on the local entropy estimate from §3 for α positive and the Stampacchia Lemma for systems. We demonstrate that if $\operatorname{supp} u_0 \subset \{x \leq 0\}$, then there exists a continuous function, $s(t)$ satisfying $s(0) = 0$, and a positive time T_0 such that $\operatorname{supp} u(\cdot, t) \subset [-a, s(t)]$, $s(t) < a \quad \forall t < T_0$, and $s(T_0) = a$. The proof in §5 is for the "weak slippage case" in which $2 < n < 3$, and requires that $m > n/2$. It is based on combining local entropy estimates for $-1 < \alpha < 0$ with the local energy estimates from §3, and again makes use of the Stampacchia Lemma for systems. It is our conjecture that the restriction that $m > n/2$ when $\nu = 1$ is sharp, and we hope to investigate this point further in a later publication.

The outline of the paper is as follows. The existence of weak non-negative solutions is proven in §2. Existence of strong energy-entropy solutions is demonstrated in §3. Finite speed of propagation is proven in §4 for the case of weak slippage and in §5 for the case of strong slippage.

2. Weak solutions

In this section, we follow Bernis & Friedman [2], relying on local parabolic regularity theory [9, 10] to attain global existence.

Notation. Let $\Omega = (-a, a)$ where $a \in (0, \infty)$ is arbitrary, and $Q_t = \Omega \times (0, t)$, $0 < t < \infty$, and set $P_t = \overline{Q}_t \setminus \{u = 0 \text{ or } t = 0\}$, $Q = Q_\infty$, $P = P_\infty$.
We shall assume that the initial conditions $u_0(x)$ satisfy

$$u_0 \in H^1(\Omega), \quad u_0 \geq 0, \quad u_0 \not\equiv 0. \tag{2.1}$$

Let us now consider the problem

$$(\mathbb{P}) \begin{cases} u_t + (u^n(u_{xxx} + \nu u^{m-n} u_x - A u^{M-n} u_x))_x = 0, \quad (x, t) \in Q_T, \\ u_x(\pm a, t) = 0, \quad 0 \in (0, T), \\ u_{xxx}(\pm a, t) = 0 \text{ when } u(\pm a, t) \neq 0, \quad t \in (0, T), \\ u(x, 0) = u_0(x), \quad x \in \bar{\Omega}, \end{cases}$$

where $\nu = \pm 1$.

We remark that while we shall be looking for solutions on a finite interval, we can always consider a parallel Cauchy problem obtained by extending the initial conditions via periodicity and reflection. This will allow us, for example, to directly implement generalized Bernis inequalities for nonnegative periodic functions, [11, Lemma B.1].

Definition 1. *A function $u \in \mathcal{C}^{0, 1/2, 1/8}(\bar{\Omega} \times [0, \infty)) \cap L^\infty([0, \infty); H^1(\Omega))$ is said to be a weak solution of $(\mathbb{P})$ if:*
(a) $u \in C^{4,1}(P), \quad u \geq 0$,
(b) $u_x(x,t) = u_{xxx}(x,t) = 0$ *when* $u(x,t) \neq 0$, *for* $(x,t) \in \partial\Omega \times (0,T)$,
(c) $J \equiv u^n(u_{xxx} + \nu u^{m-n} u_x - A u^{M-n} u_x) \in L^2(P)$,
(d) for all $\phi \in \operatorname{Lip}(\bar{\Omega} \times (0, \infty))$ *with compact support,* u *satisfies:*

$$\int_Q u\, \phi_t \, dxdt + \int_P u^n \left(u_{xxx} + \nu u^{m-n} u_x - A u^{M-n} u_x\right) \phi_x \, dxdt = 0, \tag{2.2}$$

(e) $u(x, 0) = u_0(x)$ *for* $x \in \bar{\Omega}$.

Remark 2.1. *The regularity on P, the positivity set, guarantees that the boundary conditions (1.8) hold on $\{-a,\, a\} \cap P$.*

Given this definition, we may formulate the following

Theorem 1. *If $0 < n$, $\nu = \pm 1$, $A \geq 0$, $-2 < m - n$, $m < M$ if $A > 0$, and $m - n < 2$ if $\nu = 1$ and $A = 0$, then there exists a solution to $(\mathbb{P})$ in the sense of Definition 1 for arbitrary initial datum, $u_0(x)$, satisfying (2.1).*

The approach here is to find a weak solution to $(\mathbb{P})$ as the limit of a subsequence of smooth positive solutions of a regularized problem, $(\mathbb{P}_\epsilon)$, satisfying regularized initial conditions, $u_{0\epsilon}$. We shall require that for $\lambda \in (0,\, 1)$, $\theta \in (0,\, 2/5]$, $u_{0\epsilon}$ satisfies

$$\begin{aligned} &u_{0\epsilon} \in \mathcal{C}^{4,\,\lambda}(\bar{\Omega}), \quad u'_{\epsilon 0}(\pm a) = u'''_{\epsilon_0}(\pm a) = 0, \\ &u_0 + \epsilon^\theta \leq u_{0\epsilon} \leq u_0 + 1, \quad u_{\epsilon 0} \to u_0 \text{ in } H^1((-a,\, a)) \text{ as } \epsilon \to 0. \end{aligned} \tag{2.3}$$

We note that it is possible at this point to pass to a periodic variant of this problem by extending the initial conditions by reflection then imposing periodicity.

We define as in [2], $f_\epsilon(s) = \frac{|s|^{n+4}}{\epsilon |s|^n + s^4}$. We shall adopt the convention, here and in the section which follows, that c_i, d_i denote positive constants that are independent of ϵ, and $C_i(t)$ denotes a positive increasing function defined on $(0,\, \infty)$ that is independent of ϵ; c_i, d_i, $C_i(t)$ may depend on Ω, u_0, and the problem parameters, and their value may change from line to line.

Proof. We consider the approximating Cauchy problem $(\mathbb{P}_\epsilon)$

$$(\mathbb{P}_\epsilon) \begin{cases} u_t + \{f_\epsilon(u)(u_{xxx} + \nu u^{m-n} u_x - A u^{M-n} u_x)\}_x = 0, & (x,\, t) \in Q_T, \\ u_x(\pm a,\, t) = u_{xxx}(\pm a,\, t) = 0, & t \in (0,\, T), \\ u(x,\, 0) = u_{0\epsilon}(x), & x \in \bar{\Omega}. \end{cases}$$

Problem $(\mathbb{P}_\epsilon)$ possess a unique maximal positive solution, u_ϵ, such that $u_\epsilon \in \mathcal{C}^{4,\,\lambda,\,\lambda/4}(\bar{\Omega} \times [0,\, \tau_\epsilon))$, $\tau_\epsilon > 0$, see [9, Theorem 6.3, p 302] as well as the remark following the proof given there. That the periodic variant would maintain the periodicity and reflection properties of $u_{0\epsilon}$ can be seen by translating and reflecting the solution, and invoking uniqueness of solutions to the Cauchy problem $(\mathbb{P}_\epsilon)$.

We now obtain a global energy estimate. By testing $(\mathbb{P}_\epsilon)$ with $\phi \equiv 1$ and recalling (2.1), (2.3), it follows that

$$0 < \overline{u_\epsilon(t)} = \overline{u_{0\epsilon}} \leq \overline{u}_0 + 1, \tag{2.4}$$

where $\overline{v} := |\Omega|^{-1} \int_\Omega v$. Note that (2.4) holds also for similarly defined approximating solutions for the exceptional cases, (1.4), (1.6 i).

Let $\nu = \pm 1$, and set

$$h(s) = \begin{cases} \frac{\nu s^{m-n+1}}{m-n+1} - A \frac{s^{M-n+1}}{M-n+1}, & m, M \neq \{n-1\}, \\ \nu \ln s - A \frac{s^{M-n+1}}{M-n+1}, & m = n-1, \\ \frac{\nu s^{m-n+1}}{m-n+1} - A \ln s, & M = n-1, \end{cases}$$

and

$$H(s) = \begin{cases} \frac{\nu s^{m-n+2}}{(m-n+2)(m-n+1)} - \frac{A s^{M-n+2}}{(M-n+2)(M-n+1)}, & m, M \neq \{n-1\}, \\ \nu(s \ln s - s) - \frac{A s^{M-n+2}}{(M-n+2)(M-n+1)}, & m = n-1, \\ \frac{\nu s^{m-n+2}}{(m-n+2)(m-n+1)} - A(s \ln s - s), & M = n-1. \end{cases}$$

Testing $(\mathbb{P}_\epsilon)$ with $-u_{\epsilon xx} - h(u_\epsilon)$, we obtain

$$\int_\Omega \Big[\frac{1}{2} {u_{\epsilon x}}^2 - H(u_\epsilon)\Big] + \int_{Q_t} f_\epsilon(u_\epsilon)(u_{\epsilon xxx} + h'(u_\epsilon) u_{\epsilon x})^2 = \int_\Omega \Big[\frac{1}{2} {u_{0\epsilon x}}^2 - H(u_{0\epsilon})\Big]. \quad (2.5)$$

If $\nu = 1$, $A = 0$, and $-1 \leq m - n < 2$, we may use the Gagliardo-Nirenberg inequality

$$||u||_{L^p(\Omega)} \leq c_1 ||u_x||^{1/2}_{L^2(\Omega)} ||u||^{1/2}_{L^1(\Omega)} + c_2 ||u||_{L^1(\Omega)}, \quad 1 < p < 4,$$

in conjunction with positivity of solutions and (2.4), to conclude that

$$\int_\Omega H(u_\epsilon)(t) \leq \frac{1}{4} \int_\Omega {u_{\epsilon x}}^2 + c_3 \Big(\int_\Omega u_\epsilon\Big)^{c_4} = \frac{1}{4} \int_\Omega {u_{\epsilon x}}^2 + c_5. \quad (2.6)$$

If the restrictions on the parameters stated in Theorem 1 hold, with the additional restriction that $-2 < m - n < -1$ if $\nu = 1$ and $A = 0$, then by Young's inequality and (2.4),

$$\int_\Omega H(u_\epsilon) \leq c_6(\overline{u}_0 + 1) = c_7. \quad (2.7)$$

From (2.5), (2.6), (2.7), we may conclude that under the conditions on the parameters stated in Theorem 1

$$\frac{1}{4} \int_\Omega {u_{\epsilon x}}^2(t) + \int_{Q_t} f_\epsilon(u_\epsilon)(u_{\epsilon xxx} + \nu {u_\epsilon}^{m-n} u_{\epsilon x} - A {u_\epsilon}^{M-n} u_{\epsilon x})^2 \leq c_8. \quad (2.8)$$

(The case $m - n = 2$, $A = 0$ with certain additional side constraints is also possible to treat, see e.g. [7].)

From (2.4), (2.8), we obtain that

$$||u_\epsilon||_{L^\infty(0,\,t;\,H^1(\Omega))} \leq c_9. \quad (2.9)$$

Noting that $u_t = -J_x$, and since by (2.8)

$$||f_\epsilon^{1/2}(u_{\epsilon xx} + h(u_\epsilon))_x\,||_{L^2(Q_t)} \leq c_8,$$

it follows by (2.9) that

$$||u_t||_{L^2(0,\,t;\,H^{-1}(\Omega))},\ ||J||_{L^2(Q_t)} \leq c_{10}.$$

As in [2], we obtain the uniform Hölder estimate

$$||u_\epsilon||_{\mathcal{C}^{0,\,1/2,\,1/8}(\overline{Q_t})} \leq c_{11},$$

where c_{11} is time independent.

Remark 2.2. *It is also possible to implement the above discussion when u^n is replaced by $f(u)$ in ($\mathbb{P}$), for "suitable" $f(u) \in \mathcal{C}(0, \infty) \to \mathbb{R}^+$, and to work with* [6]

$$f_\epsilon(u) = \frac{f(u)u^4}{\epsilon f(u) + u^4},$$

where "suitable" is defined more or less stringently depending on whether $A = 0$ or $A > 0$.

We now demonstrate roughly as in [2] that $u_\epsilon \geq 4\sigma > 0$ in $\bar{\Omega} \times [0, \tau_\epsilon]$ for some $\sigma > 0$. On $[0, \tau_\epsilon)$, we know that $u_\epsilon(x, t) > 0$. Hence we may multiply the equation in ($\mathbb{P}_\epsilon$) by $G'_\epsilon(u_\epsilon)$, where

$$G_\epsilon(s) = -\int_s^{\tilde{A}} g_\epsilon(r)\, dr, \quad g_\epsilon(s) = -\int_s^{\tilde{A}} \frac{dr}{f_\epsilon(r)},$$

where $\tilde{A} > \max |u_\epsilon|$ for all small positive ϵ, and integrate to obtain

$$\int_\Omega G_\epsilon(u_\epsilon(t)) + \int_{Q_t} [u_{\epsilon xx}^2 - (\nu {u_\epsilon}^{m-n} - A{u_\epsilon}^{M-n})u_{\epsilon x}^2] = \int_\Omega G_\epsilon(u_\epsilon(0)). \tag{2.10}$$

If $m - n \neq -1$, then using (2.10) and integrating the term $-\int_{Q_T} \nu u_\epsilon^{m-n} u_{\epsilon x}^2$ by parts,

$$\int_\Omega G_\epsilon(u_\epsilon(t)) + \int_{Q_t} u_{\epsilon xx}^2 + A \int_{Q_t} {u_\epsilon}^{M-n} u_{\epsilon x}^2$$
$$\leq \frac{1}{2} \int_{Q_t} u_{\epsilon xx}^2 + \frac{1}{2(m-n+1)^2} \int_{Q_t} {u_\epsilon}^{2m-2n+2} + \int_\Omega G_\epsilon(u_\epsilon(0)). \tag{2.11}$$

And therefore recalling (2.9)

$$\int_\Omega G_\epsilon(u_\epsilon(t)) + \int_{Q_t} \frac{1}{2} u_{\epsilon xx}^2 \leq c_{12} \int_{Q_t} {u_\epsilon}^{-2} + C_1(t) + \int_\Omega G_\epsilon(u_\epsilon(0)). \tag{2.12}$$

If $m - n = -1$, the term $(\ln u_\epsilon)^2$ replaces $\frac{u_\epsilon^{2m-2n+2}}{(m-n+1)^2}$ in (2.11), and noting that

$$\ln^2(s) \leq c_{13} s^{-2} + c_{14}, \qquad \text{for } 0 < s < \tilde{A} < \infty,$$

the estimate (2.12) again follows.

Noting that for $0 < s$,

$$\epsilon s^{-4} \leq \frac{1}{f_\epsilon(s)},$$

it now follows easily that

$$\int_\Omega G_\epsilon(u_\epsilon) + \int_{Q_t} \frac{1}{2} u_{\epsilon xx}^2 \leq \frac{c_{15}}{\epsilon} \int_{Q_t} G_\epsilon(u_\epsilon(t)) + C_2(t) + \int_\Omega G_\epsilon(u_\epsilon(0)).$$

Hence by Gronwall's inequality,

$$\int_\Omega G_\epsilon(u_\epsilon(t)) \leq D_\epsilon(t) < \infty, \quad t \in [0, \tau_\epsilon). \tag{2.13}$$

where for all $0 < \epsilon \ll 1$, $D_\epsilon(t)$ is an increasing function defined on $(0, \infty)$. As in [2], (2.13) can be seen to imply positivity.

The solution, $u_\epsilon(x,\,t)$, may now be extended to exist globally, as in [2, 10]. Select $\tilde{f}_\epsilon(s) \in \mathcal{C}^2(\mathbb{R})$ such that $\tilde{f}_\epsilon(s) \equiv f_\epsilon(s)$ for $s \geq 2\sigma$, and $\tilde{f}_\epsilon(s) \geq f_\epsilon(\sigma)$ for all $s \in \mathbb{R}$. Thus $u_\epsilon(x,\,t)$ also constitutes a weak solution of

$$u_{\epsilon t} + \{\tilde{f}_\epsilon(u_\epsilon)(u_{\epsilon xxx} + h'(u_\epsilon)u_{\epsilon x})\}_x = 0,$$

satisfying the same initial and boundary conditions as before. For $x_\epsilon \in [-a,\,a)$, set

$$v_\epsilon(x,\,t) = \int_{x_\epsilon}^{x} u_\epsilon(\xi,\,t)\,d\xi - \int_0^t \tilde{f}_\epsilon(u_\epsilon(x_\epsilon,\,\theta))\{u_{\epsilon xxx} + h'(u_\epsilon)u_{\epsilon x}\}(x_\epsilon,\,\theta)\,d\theta.$$

The regularity and positivity of $u_\epsilon(x,\,t)$ imply that $v_\epsilon(x,\,t)$ is well defined in $D = \bar{\Omega} \times (0,\,\tau_\epsilon)$ and satisfies

$$\begin{cases} v_{\epsilon t} + \tilde{f}_\epsilon(u_\epsilon(x,t))\{v_{\epsilon xxxx} + \nu {u_\epsilon}^{m-n} v_{\epsilon xx} - A{u_\epsilon}^{M-n} v_{\epsilon xx}\} = 0, \\ v_\epsilon(\pm\, a, t) = v_{\epsilon xx}(\pm\, a, t) = 0. \end{cases} \tag{2.14}$$

Using parabolic regularity results for v_ϵ, enhanced regularity may be obtained for u_ϵ and hence for the u_ϵ-dependent coefficients in (2.14). Returning again to (2.14), additional regularity is obtained for v_ϵ, which allows us to conclude that $u_\epsilon \in \mathcal{C}^{4+\lambda}(\overline{\Omega} \times [0,\,\tau_\epsilon])$. Therefore the solution may be continued, in contradiction to the assumed maximality of the solution.

One may now argue as in [2], using (2.4), (2.8) to conclude that there exists a sequence $u_{\epsilon k}$ converging uniformly to a solution of $(\mathbb{P})$ on $\overline{Q}_T$ for all $0 < T < \infty$ as $\epsilon_k \to 0$. □

Remark 2.3. *The existence of weak solutions for the exceptional cases, (1.4), (1.6 i), with $h'(u)u_x$ replacing $\nu u^{m-n}u_x - Au^{M-n}u_x$ in accordance with (a),(b) in Definition 1, can be easily concluded for arbitrary initial data satisfying (2.1), (2.3). This may be accomplished by verifying that (2.6) holds for (1.4), that (2.7) holds for (1.6 i), and that (2.12) holds for both (1.4) and (1.6 i), then arguing as above.*

3. Strong entropy-energy solutions

To get strong entropy-energy solutions, we derive entropy estimates and use the approach of Beretta, Bertsch & Dal Passo [1] to get strong solutions, then derive an energy estimate. In accordance with the conditions in Theorem 1, we shall assume throughout this section that $\nu = \pm 1$, $0 \leq A$, $0 < n$, $-2 < m - n < M - n$, and $m - n < 2$ if $A = 0$, $\nu = 1$. Moreover, in referring to solutions of $(\mathbb{P})$ and $(\mathbb{P}_\epsilon)$, we shall assume that u_0 satisfies (2.1) and $u_{0\epsilon}$ satisfies (2.3). Some further restrictions shall be introduced in the sequel.

Before deriving the entropy estimates, we present a lemma, which is essentially a refinement of Theorem 3.1 in [1], which is useful for concluding regularity results from entropy estimates.

Lemma 3.1. *Let $u(x,t)$ be a weak solution of $(\mathbb{P})$ obtained as the limit of a subsequence of solutions $u_\epsilon(x,\,t)$ of $(\mathbb{P}_\epsilon)$. Suppose that $0 < n$ and that for some $\alpha \in (\frac{1}{2} - n,\, 2 - n)$, there exist constants c_1, c_2, and $\delta > 0$ which do not depend on ϵ, such that*

$$\int_{Q_T} u_\epsilon^{\alpha+n-2\gamma+1}(u_\epsilon^\gamma)_{xx}^2 \leq c_1, \tag{3.1}$$

and

$$\int_{Q_T} u_\epsilon^{\alpha+n-3} {u_\epsilon}_x^4 \leq c_2, \tag{3.2}$$

for all γ satisfying

$$\frac{1+n+\alpha}{3} \leq \gamma \leq \frac{1+n+\alpha}{3} + \delta, \tag{3.3}$$

then $u^{1/\beta}(\cdot,\, t) \in \mathcal{C}^1([-a,\, a])$ for all $\beta \in (0,\, \frac{3}{n+\alpha+1})$ for almost every $t > 0$.

Proof. For any $0 < \beta < \frac{3}{n+\alpha+1}$, we may choose γ satisfying (3.3) such that $0 < \beta\gamma < 1$. Setting $q = 4 - \frac{(1+n+\alpha)}{\gamma}$ and arguing as in the proof of [1, Lemma 3.1], it follows from (3.1),(3.2) that for almost every $t > 0$ there exists a $C_1(t) < \infty$ such that

$$\text{if } u(y,\, t) = 0 \text{ for some } y \in [-a,\, a], \text{ then}$$
$$|(u^\gamma)_x|^{(4-q)/q}(x,\, t) \leq C_1(t)|x-y|^{(q-1)/q} \quad \text{for } x \in [-a, a]. \tag{3.4}$$

From (3.4), we find by integrating that for almost every $t > 0$, there exists a $C_2(t) < \infty$ such that

$$\text{if } u(y,\, t) = 0 \text{ for some } y \in [-a,\, a], \text{ then}$$
$$u(x,t) \leq C_2(t)|x-y|^{\frac{3}{\alpha+n+1}} \quad \text{for } x \in [-a, a]. \tag{3.5}$$

Since $0 < \beta\gamma < 1$, we may combine (3.5) and (3.4) to obtain that for almost every $t > 0$, there exists a $C_3(t) < \infty$ such that

$$\text{if } u(y,\, t) = 0 \text{ for some } y \in [-a,\, a], \text{ then for } x \in [-a, a],$$
$$|(u^{1/\beta})_x(x,\, t)| \leq C_3(t)|x-y|^{\frac{3}{4-q}}|x-y|^{\frac{q-1}{4-q}} \leq C_3(t)|x-y|^\mu, \tag{3.6}$$

where $\mu = \frac{1}{\beta}\frac{3}{\alpha+n+1} - 1 > 0$ and $C_3(t) < \infty$. □

From Lemma 3.1, two simple but useful corollaries follow.

Corollary 3.2. *Let $u(x,t)$ be a weak solution of $(\mathbb{P})$ obtained as the limit of a subsequence of solutions $u_\epsilon(x,\, t)$ of $(\mathbb{P}_\epsilon)$. Suppose that $0 < n$ and that for some $\alpha \in (\frac{1}{2} - n,\, 2-n)$, there exist constants c_1, c_2, and $\delta > 0$, which do not depend on ϵ, such that for all γ satisfying (3.3), the estimates (3.1) and (3.2) hold. Then $u(\cdot,\, t) \in \mathcal{C}^1([-a, a])$ for almost every $t > 0$.*

Proof. Note that if $\alpha \in (\frac{1}{2} - n,\, 2-n)$, then $\frac{3}{\alpha+n+1} \in (1,\, 2)$. Hence $1 \in (0,\, \frac{3}{n+\alpha+1})$. □

Remark 3.3. *Note that if $u(\cdot,\, t) \in \mathcal{C}^1([-a, a])$ for almost every $t > 0$, then $u(x,\, t)$ is a strong solution in the sense of Bernis & Friedman* [2].

Corollary 3.4. *Let $u(x,t)$ be a weak solution of $(\mathbb{P})$ obtained as the limit of a subsequence of solutions $u_\epsilon(x,\, t)$ of $(\mathbb{P}_\epsilon)$. Let $0 < n$ and let Ψ denote a subset of $(\frac{1}{2} - n,\, 2-n)$. If for all $\alpha \in \Psi$, there exist constants c_1, c_2, and $\delta > 0$, which do not depend on ϵ, such that for all γ satisfying (3.3), the estimates (3.1) and (3.2) hold, then $u^{1/\beta}(\cdot,\, t) \in \mathcal{C}^1([-a,\, a])$ for all $\beta \in (0,\, \frac{3}{n+\inf\Psi+1})$ for almost every $t > 0$.*

Proof. The result is an immediate consequence of Lemma 3.1. □

We now derive our primary entropy estimates.

Let

$$\zeta \in \mathcal{C}^4([-a,\, a]) \text{ with support in } (-a,\, a) \text{ and } \zeta \geq 0, \tag{3.7}$$

or $\zeta = 1$, and let

$$G_\epsilon(s) = \frac{\epsilon s^{\alpha+n-3}}{(\alpha+n-4)(\alpha+n-3)} + \frac{s^{\alpha+1}}{\alpha(\alpha+1)},$$

where $\alpha \in (1/2-n,\, 2-n)\backslash\{0,\, -1\}$. Using $\zeta^4 G'_\epsilon(u_\epsilon)$ to test $(\mathbb{P}_\epsilon)$ on $Q_T = \Omega \times (0,\, T)$, $0 < T < \infty$, and treating the terms which also appear in the classical thin film equation as they were treated in [1, 8], we obtain that for any γ satisfying

$$\frac{t+1-\sqrt{(t-2)(1-2t)}}{3} < \gamma < \frac{t+1+\sqrt{(t-2)(1-2t)}}{3}, \tag{3.8}$$

where $t = \alpha + n$, there exist positive constants, c_3, c_4, which do not depend on ϵ, such that

$$\int_\Omega \zeta^4 G_\epsilon(u_\epsilon(x,\, T))\, dx + \\ c_3 \left[\int_{Q_T} \zeta^4 {u_\epsilon}^{\alpha+n-2\gamma+1} (u_\epsilon^\gamma)^2_{xx} + \int_{Q_T} \zeta^4 {u_\epsilon}^{\alpha+n-3} {u_{\epsilon x}}^4 \right] \leq \\ \int_\Omega \zeta^4 G_\epsilon(u_{0\epsilon}(x))\, dx + c_4 \int_{Q_T} (|\zeta_x|^4 + |\zeta\zeta_{xx}|^2) u_\epsilon^{n+\alpha+1} + I, \tag{3.9}$$

where $g_\epsilon(u_\epsilon) := G'_\epsilon(u_\epsilon)$, and

$$I := -\int_{Q_T} \zeta^4 g_\epsilon(u_\epsilon) \{ f_\epsilon(u_\epsilon)(\nu {u_\epsilon}^{m-n} u_{\epsilon x} - A {u_\epsilon}^{M-n} u_{\epsilon x}) \}_x.$$

Integrating I by parts,

$$I = \int_{Q_T} \zeta^4 g'_\epsilon(u_\epsilon) f_\epsilon(u_\epsilon)(\nu {u_\epsilon}^{m-n} {u_{\epsilon x}}^2 - A {u_\epsilon}^{M-n} {u_{\epsilon x}}^2) + \\ \int_{Q_T} 4\zeta^3 \zeta_x g_\epsilon(u_\epsilon) f_\epsilon(u_\epsilon)(\nu {u_\epsilon}^{m-n} u_{\epsilon x} - A {u_\epsilon}^{M-n} u_{\epsilon x}) := I_a + I_b.$$

The term I_a may be written as

$$I_a = \int_{Q_T} \zeta^4 (\nu {u_\epsilon}^{\alpha+m-1} - A {u_\epsilon}^{\alpha+M-1}) {u_{\epsilon x}}^2. \tag{3.10}$$

For $\nu = -1$, note that both terms in (3.10) are non-positive. For $\nu = +1$, we estimate

$$I_a \leq \delta \int_{Q^T} \zeta^4 {u_\epsilon}^{\alpha+n-3} {u_{\epsilon x}}^4 + c_5(\delta) \int_{Q_T} \zeta^4 {u_\epsilon}^{\alpha+2m-n+1} - \\ A \int_{Q_T} \zeta^4 {u_\epsilon}^{\alpha+M-1} {u_{\epsilon x}}^2, \tag{3.11}$$

where $\delta > 0$ is arbitrary.

With regard to I_b, integration by parts gives that

$$I_b = -\int_{Q_T} 4(\zeta^3\zeta_x)_x \Big[\int_0^{u_\epsilon} g_\epsilon(s) f_\epsilon(s) [\nu s^{m-n} - A s^{M-n}]\, ds \Big].$$

As in [1] we find that

$$|g_\epsilon(u_\epsilon)f_\epsilon(u_\epsilon)| \leq c_6 u_\epsilon^{n+\alpha}.$$

Thus, recalling (2.9) and that $M > m$,

$$I_b \leq c_7 \int_{Q_T} |(\zeta^3\zeta_x)_x| u_\epsilon^{\alpha+m+1}. \tag{3.12}$$

If $\nu = -1$, we may combine the estimates on I_a and I_b to obtain

$$\begin{aligned}
&\int_\Omega \zeta^4 G_\epsilon(u_\epsilon(x,\,T))\,dx + \\
&\qquad c_3\left[\int_{Q_T} \zeta^4 {u_\epsilon}^{\alpha+n-2\gamma+1}(u_\epsilon^\gamma)_{xx}^2 + \int_{Q_T} \zeta^4 {u_\epsilon}^{\alpha+n-3} {u_\epsilon}_x^4\right] + \\
&\qquad \int_{Q_T} \zeta^4 u_\epsilon^{\alpha+m-1} {u_\epsilon}_x^2 + A\int_{Q_T} \zeta^4 u_\epsilon^{\alpha+M-1} {u_\epsilon}_x^2 \leq \\
&\qquad c_4 \int_{Q^T} (|\zeta_x|^4 + |\zeta\zeta_{xx}|^2) u_\epsilon^{n+\alpha+1} + c_7 \int_{Q_T} |(\zeta^3\zeta_x)_x| u_\epsilon^{\alpha+m+1} + \\
&\qquad\qquad \int_\Omega \zeta^4 G_\epsilon(u_{0\epsilon}(x))\,dx.
\end{aligned} \tag{3.13}$$

Similarly, if $\nu = +1$, the estimates yield

$$\begin{aligned}
&\int_\Omega \zeta^4 G_\epsilon(u_\epsilon(x,\,T))\,dx + A\int_{Q_T} \zeta^4 {u_\epsilon}^{\alpha+M-1} {u_\epsilon}_x^2 + \\
&\qquad c_8\left[\int_{Q_T} \zeta^4 {u_\epsilon}^{\alpha+n-2\gamma+1}(u_\epsilon^\gamma)_{xx}^2 + \int_{Q_T} \zeta^4 {u_\epsilon}^{\alpha+n-3} {u_\epsilon}_x^4\right] \leq \\
&\qquad c_4 \int_{Q^T} (|\zeta_x|^4 + |\zeta\zeta_{xx}|^2) u_\epsilon^{n+\alpha+1} + c_7 \int_{Q_T} |(\zeta^3\zeta_x)_x| u_\epsilon^{\alpha+m+1} \\
&\qquad\qquad + c_5 \int_{Q_T} \zeta^4 {u_\epsilon}^{\alpha+2m-n+1} + \int_\Omega \zeta^4 G_\epsilon(u_{0\epsilon}(x))\,dx.
\end{aligned} \tag{3.14}$$

To obtain bounds from (3.13), (3.14), we impose certain conditions on α and on the initial data.

Remark 3.5. *Suppose that u_0 satisfies (2.1) and $0 < n < 3$. Defining*

$$\alpha^* = \begin{cases} \frac{1}{2} - n, & 0 < n \leq \frac{3}{2}, \\ -1, & \frac{3}{2} < n < 3, \end{cases} \tag{3.15}$$

we see that $\alpha^* + 1 \geq 0$ and $\alpha^* \in [\frac{1}{2} - n,\, 2-n)$. Hence, there exists $\alpha_* \in (\frac{1}{2} - n,\, 2-n)$ such that for all $\alpha \in [\alpha_*,\, 2-n)$,

$$\begin{aligned}
&\textstyle\int_\Omega \zeta^4 u_0^{\alpha+1}(x)\,dx < +\infty \qquad \text{if } \alpha \neq -1, \\
&\textstyle\int_\Omega \zeta^4 |\ln u_0(x)|\,dx < +\infty \qquad \text{if } \alpha = -1,
\end{aligned}$$

for all $\zeta \in \mathcal{C}^4([-a,\,a])$.

In consideration of the above remark, we define

Definition 2. *Suppose that* u_0 *satisfies (2.1), and* $\zeta \in \mathcal{C}^4([-a, a])$. *Then we define* $\alpha_0(\zeta) \equiv \inf \alpha$ *such that* $\alpha > \frac{1}{2} - n$ *and*

$$\begin{array}{ll} \int_\Omega \zeta^4 u_0^{\alpha+1}(x)\,dx < +\infty & \text{if } \alpha \neq -1, \\ \int_\Omega \zeta^4 |\ln u_0(x)|\,dx < +\infty & \text{if } \alpha = -1. \end{array}$$

Note that Remark 3.5 and the definition of $\alpha_0(\zeta)$ imply that if $0 < n < 3$, then

$$\frac{1}{2} - n \leq \alpha_0(\zeta) \leq \alpha^* < 2 - n. \tag{3.16}$$

The theorem below with regard to the stable case follows essentially from [1, 8], but is included for the sake of completeness.

Theorem 2. (The stable case.) *Suppose that* $\nu = -1$, $0 \leq A$, $0 < n$, $-2 < m-n$, *with* $m < M$ *if* $A > 0$.

i) Let $\beta \in (0, \beta_0)$ *where* $\beta_0 = \frac{3}{n+\alpha_0+1}$ *and* $\alpha_0 = \alpha_0(\zeta = 1)$, *and suppose that* $\alpha_0 < 2 - n$. *Then* $u^{1/\beta}(\cdot, t) \in \mathcal{C}^1([-a, a])$ *for almost every* $t > 0$.

ii) Let ζ *satisfy (3.7),* $\alpha_0 = \alpha_0(\zeta)$, *and suppose that* $\alpha_0 < 2 - n$. *Then, for any* $\alpha \in (\max\{\alpha_0, -m-1\}, 2-n)/\{0, -1\}$ *and for any* γ *satisfying (3.8),*

$$\begin{aligned} &\frac{1}{\alpha(\alpha+1)} \int_\Omega \zeta^4 u^{1+\alpha}(x, T)\,dx + \\ &\qquad c_1 \left[\int_P \zeta^4 u^{\alpha+n-2\gamma+1} (u^\gamma)^2_{xx} + \int_{Q_T} \zeta^4 u^{\alpha+n-3} u_x^4 \right] + \\ &\qquad \left[\int_{Q_T} \zeta^4 u^{\alpha+m-1} u_x^2 + A \int_{Q_T} \zeta^4 u^{\alpha+M-1} u_x^2 \right] \leq \\ &c_2 \int_{Q^T} (|\zeta_x|^4 + |\zeta\zeta_{xx}|^2) u^{n+\alpha+1} + c_3 \int_{Q_T} |(\zeta^3\zeta_x)_x| u^{\alpha+m+1} + \\ &\qquad\qquad \frac{1}{\alpha(\alpha+1)} \int_\Omega \zeta^4 {u_0}^{\alpha+1}(x)\,dx. \end{aligned} \tag{3.17}$$

Proof. Part *i)* follows by setting $\zeta = 1$ in (3.13), then implementing Lemma 3.1. Part *ii)* follows easily from (3.13) by letting $\epsilon \to 0$ and noting that $\alpha + m + 1$ and $\alpha + n + 1$ are positive in the indicated parameter range. □

Remark 3.6. *It follows from (3.16) that*

$$\beta_0 \geq \begin{cases} 2, & 0 < n \leq \frac{3}{2}, \\ \frac{3}{n}, & \frac{3}{2} < n < 3. \end{cases} \tag{3.18}$$

These are the bounds which were given in [1].

Theorem 3. (The unstable case.) *Let* $\nu = 1$, $0 \leq A$, $0 < n$, $-\frac{3}{2} < m - n$, $m - n < 2$ *if* $A = 0$, *and* $m < M$ *if* $0 < A$.

i) Let $\alpha_0 = \alpha_0(\zeta \equiv 1)$, $\alpha_1 = \max\{\alpha_0, -2m+n-1\}$, *and* $\beta_1 = \frac{3}{n+\alpha_1+1}$, *and suppose that* $\alpha_0 < 2 - n$. *Then* $u^{1/\beta}(\cdot, t) \in \mathcal{C}^1([-a, a])$, *for all* $\beta \in (0, \beta_1)$ *for almost every* $t > 0$.

ii) For any ζ satisfying (3.7), let $\alpha_0 = \alpha_0(\zeta)$ and $\alpha_2 = \max\{\alpha_0,\, -2m+n-1,\, -m-1\}$, and suppose that $\alpha_0 < 2-n$. Then, for any $\alpha \in (\alpha_2,\, 2-n)/\{0,-1\}$ and for any γ satisfying (3.8),

$$\frac{1}{\alpha(\alpha+1)}\int_\Omega \zeta^4 u^{1+\alpha}(x,\,T)\,dx + A\int_{Q_T}\zeta^4 u^{\alpha+M-1}u_x^2 +$$
$$c_1\left[\int_P \zeta^4 u^{\alpha+n-2\gamma+1}(u^\gamma)_{xx}^2 + \int_{Q_T}\zeta^4 u^{\alpha+n-3}u_x^4\right] \le$$
$$c_2\int_{Q^T}(|\zeta_x|^4 + |\zeta\zeta_{xx}|^2)u^{n+\alpha+1} + c_3\int_{Q_T}|(\zeta^3\zeta_x)_x|u^{\alpha+m+1} +$$
$$c_4\int_{Q_T}\zeta^4 u^{\alpha+2m-n+1} + \frac{1}{\alpha(\alpha+1)}\int_\Omega \zeta^4 {u_0}^{\alpha+1}(x)\,dx. \quad (3.19)$$

Proof. To prove part *i*), note that $\alpha_1 \in (\frac{1}{2}-n,\, 2-n)$, $\alpha_1 \ge \alpha_0$, $\alpha_1+2m-n+1 \ge 0$, then implement Lemma 3.1 with $\zeta = 1$. Part *ii*) follows by noting that $\alpha_2 \in (\frac{1}{2}-n,\, 2-n)$, $\alpha > \alpha_0$, $\alpha+m+1>0$, and $\alpha+2m-n+1 \ge 0$, then letting $\epsilon \to 0$ in (3.14). □

Remark 3.7. *The results given in Theorem 3 also hold for the exceptional cases (1.4), (1.6 i), with $A = 0$ and with $m-n$ assuming the values $m-n=-1$ and $m-n=0$, respectively. This can be easily demonstrated by following the arguments above, once one notices that estimates (3.11), (3.12) also hold for (1.4), (1.6 i) when the value of $m-n$ is taken as -1 or 0, respectively, by utilizing the bounds (1.9), (1.10).*

Remark 3.8. *Note that if $0<n<3$, then Corollary 3.2 and Remarks 3.3 and 3.5 imply that the solutions obtained in Theorems 2 and 3 are strong solutions in the sense of Bernis & Friedman* [2].

In the case of "strong slippage," in which $0<n<2$, the local entropy estimates provided in Theorems 2, 3 can be used to prove the finite speed propagation property for the strong solutions obtained there; see §4. However, in the case of "weak slippage" in which

$$2 \le n < 3, \quad (3.20)$$

these local entropy estimates are insufficient. In this latter case, to demonstrate the finite speed of propagation property, we shall rely on certain local energy estimates, which we now derive.

By testing the equation in the approximating problem, $(\mathbb{P}_\epsilon)$, with $-(\zeta^6 u_{\epsilon x})_x$, we easily deduce that

$$\int_\Omega \frac{\zeta^6}{2}|u_{\epsilon x}(x,T)|^2\,dx + \int_{Q_T}\zeta^6 f_\epsilon(u_\epsilon)|u_{\epsilon xxx}|^2 =$$
$$-\int_{Q_T} f_\epsilon(u_\epsilon)u_{\epsilon xxx}[2u_{\epsilon xx}(\zeta^6)_x + u_{\epsilon x}(\zeta^6)_{xx}] -$$
$$\int_{Q_T} f_\epsilon(u_\epsilon)(\nu u_\epsilon^{m-n}u_{\epsilon x} - Au_\epsilon^{M-n}u_{\epsilon x})(u_{\epsilon x}\zeta^6)_{xx} +$$
$$\int_\Omega \frac{\zeta^6}{2}|u_{0\epsilon x}(x)|^2\,dx. \quad (3.21)$$

Assuming (3.20) and noting (3.15), (3.16), it is easy to check that in Part *ii*) of Theorem 2, $\max\{\alpha^*,\ -m-1\} < 0$, and in Part *ii*) of Theorem 3, $\max\{\alpha^*,\ -2m+n-1,\ -m-1\} < 0$. Hence if n satisfies (3.20), α may be chosen to be fixed and to satisfy

$$-1 < \alpha < 0, \tag{3.22}$$

in addition to satisfying the constraints indicated in Part *ii*) of either theorem, for arbitrary ζ satisfying (3.7) or $\zeta = 1$. This choice for α suffices for proving the finite speed of propagation property for the case of weak slippage, although potentially local energy inequalities could be derived for a wider set of values of α.

For α satisfying (3.22), we have due to (1.13)

$$\int_\Omega u_\epsilon(x,T)^{1+\alpha}\,dx \le |\Omega|^{-\alpha}\biggl(\int_\Omega u_\epsilon(x,T)\,dx\biggr)^{1+\alpha} = |\Omega|^{-\alpha}\biggl(\int_\Omega u_{0\epsilon}(x)\,dx\biggr)^{1+\alpha} \le d_1. \tag{3.23}$$

Setting $\zeta = 1$ in (3.13) and employing (2.3),(3.23), we obtain the following inequality when $\nu = -1$,

$$c_3\biggl[\int_{Q_T} u_\epsilon^{\alpha+n-2\gamma+1}(u_\epsilon^\gamma)_{xx}^2 + \int_{Q_T} u_\epsilon^{\alpha+n-3}u_{\epsilon x}^4\biggr] + \int_{Q_T}(u_\epsilon^{\alpha+m-1} + Au_\epsilon^{\alpha+M-1})u_{\epsilon x}^2 \le \int_\Omega G_\epsilon(u_{0\epsilon}(x))\,dx + d_2 \le d_3. \tag{3.24}$$

Similarly, setting $\zeta = 1$ in (3.14), employing (2.3),(3.23), and noting that the assumptions on α imply that $\alpha > \alpha_2 \ge -2m+n-1$, we obtain the following inequality when $\nu = 1$,

$$c_8\biggl[\int_{Q_T} u_\epsilon^{\alpha+n-2\gamma+1}(u_\epsilon^\gamma)_{xx}^2 + \int_{Q_T} u_\epsilon^{\alpha+n-3}u_{\epsilon x}^4\biggr] + A\int_{Q_T} u_\epsilon^{\alpha+M-1}u_{\epsilon x}^2 \le \int_\Omega G_\epsilon(u_{0\epsilon}(x))\,dx + c_5\int_{Q_T} u_\epsilon^{\alpha+2m-n+1} + d_4 \le d_5. \tag{3.25}$$

Next we pass to the limit $\epsilon \to 0$ in (3.21). First, setting $\zeta = 1$ in (3.21) yields that for any $\delta > 0$,

$$\begin{aligned} 2^{-1}\int_\Omega |u_{\epsilon x}(x,T)|^2\,dx + \int_{Q_T} f_\epsilon(u_\epsilon)|u_{\epsilon xxx}|^2 &\le 2^{-1}\int_\Omega |u_{0\epsilon x}|^2\,dx + \\ (1 + A|\sup u_\epsilon|^{M-m})\int_{Q_T} f_\epsilon(u_\epsilon)u_\epsilon^{m-n}|u_{\epsilon x}||u_{\epsilon xxx}| &\le 2^{-1}\int_\Omega |u_{0\epsilon x}|^2\,dx + \\ \delta\int_{Q_T} f_\epsilon(u_\epsilon)|u_{\epsilon xxx}|^2 + \int_{Q_T} u_\epsilon^{\alpha+n-3}u_{\epsilon x}^4 &+ c(\delta)\int_{Q_T} u_\epsilon^{4m-2n-(\alpha+n)+3}. \end{aligned} \tag{3.26}$$

Suppose that

$$4m - 2n - (\alpha+n) + 3 \ge 0, \tag{3.27}$$

then setting $\delta = 1/2$ in (3.26) and using the estimates (3.24), (3.25), we deduce that

$$\int_\Omega |u_{\epsilon x}(x,T)|^2\,dx + \int_{Q_T} f_\epsilon(u_\epsilon)|u_{\epsilon xxx}|^2 < d_6. \tag{3.28}$$

Remark 3.9. *In the context of the assumptions of Theorem 2, when* n *satisfies (3.20),* $m > n-2 \geq 0$, *hence* $\max\{\alpha^*, -m-1\} = \alpha^* = -1$. *Thus for arbitrary initial data satisfying (2.1),(2.3), (3.27) is satisfied for some admissible* α, *if*

$$m - \frac{3}{4}n \geq -1,$$

which is stronger than the previous constraint, $m - n > -2$.

In the context of the assumptions of Theorem 3, when n *satisfies (3.20),* $m > n-2 \geq 0$; *hence* $\max\{\alpha^*, -m-1\} = \alpha^* = -1$. *But* α *must also satisfy* $\alpha > -2m+n-1$. *It is easy to check that* (3.27) *holds for some admissible* α, *if and only if* m, n *satisfy the condition*

$$3m - 2n > -2. \tag{3.29}$$

Recalling the constraint $m - n > -\frac{3}{2}$ *in Theorem 3, it is easy to check that* (3.29) *constitutes an additional constraint if* $n < \frac{5}{2}$.

Using the estimates (3.28), (3.24), (3.25), it is easy to check that the integrals on the right-hand side of (3.21) are uniformly bounded with respect to ϵ if (3.27) is satisfied. For arbitrary $\eta > 0$, $u_\epsilon \to u$ strongly in the space $C^{4,1}(\{u > \eta\})$. Therefore passage to the limit $\epsilon \to 0$ in all of the integrals in (3.21) over the domain $\{u > \eta\}$ is straightforward. As to integrals over the domain $\{u < \eta\}$, we have, for example, by virtue of (3.27),

$$\left| \int_{Q_T \cap \{u<\eta\}} f_\epsilon(u_\epsilon) u_{\epsilon xxx} u_{\epsilon x} {u_\epsilon}^{m-n} \zeta^6 \, dx\, dt \right| \leq \\ \left(\int_{Q_T \cap \{u<\eta\}} f_\epsilon(u_\epsilon) |u_{\epsilon xxx}|^2 \right)^{1/2} \left(\int_{Q_T \cap \{u<\eta\}} u_\epsilon^{\alpha+n-3} {u_{\epsilon x}}^4 \right)^{1/4} \times \\ \left(\int_{Q_T \cap \{u<\eta\}} u^{4m-2n-(\alpha+n)+3} \right)^{1/4} \leq c\eta^{\frac{4m-2n-(\alpha+n)+3}{4}} \to 0 \text{ as } \eta \to 0.$$

Analogously, it is easy to check that all of the other integrals over $\{u < \eta\}$ on the right-hand side of (3.21) are bounded from above by some continuous function, $h(\eta)$, such that $h(\eta) \to 0$ as $\eta \to 0$. Therefore, first passing to the limit $\epsilon \to 0$, and afterwards letting $\eta \to 0$, we easily obtain

$$2^{-1} \int_\Omega \zeta^6 |u_x(x,T)|^2 \, dx + \int_{Q_T \cap \{u>0\}} \zeta^6 u^n u_{xxx}^2 \leq \\ 2^{-1} \int_\Omega \zeta^6 |u_{0x}(x)|^2 \, dx - \int_{Q_T \cap \{u>0\}} u^n u_{xxx} [2u_{xx}(\zeta^6)_x + u_x(\zeta^6)_{xx}] - \\ \int_{Q_T \cap \{u>0\}} (\nu u^m u_x - A u^M u_x)(u_x \zeta^6)_{xx}. \tag{3.30}$$

Since $u(\cdot, t) \in C^1(\bar{\Omega})$ for almost $t \in [0,T]$, it is possible to estimate from below the second term on the left hand side of (3.30) by using Lemma B.1 from [11] (generalized Bernis inequalities). As a result we obtain,

$$2^{-1} \int_\Omega \zeta^6 |u_x(x,T)|^2 \, dx + d_7 \int_{Q_T} \zeta^6 \big((u^{\frac{n+2}{6}})_x^6 + |(u^{\frac{n+2}{3}})_{xx}|^3 + (u^{\frac{n+2}{2}})_{xxx}^2 \big) + \\ d_7 \int_{Q_T \cap \{u>0\}} \zeta^6 u^n u_{xxx}^2 \leq 2^{-1} \int_\Omega \zeta^6 |u_{0x}(x)|^2 \, dx + d_8 \int_{Q_T} |\zeta_x|^6 u^{n+2} -$$

$$\int_{Q_T\cap\{u>0\}} u^n u_{xxx}[2u_{xx}(\zeta^6)_x + u_x(\zeta^6)_{xx}] -$$
$$\int_{Q_T\cap\{u>0\}} (\nu u^m u_x - Au^M u_x)(u_x\zeta^6)_{xx}, \quad (3.31)$$

Next we estimate the terms in third integral on the right-hand side following idea proposed in [13]:

$$\int_{Q_T\cap\{u>0\}} u^n u_{xxx}u_x(\zeta^6)_{xx}$$
$$= 6\int_{Q_T\cap\{u>0\}} (u^{\frac{n}{2}}u_{xxx}\zeta^3)(u^{\frac{n-4}{6}}u_x\zeta)(u^{\frac{n+2}{3}}(5\zeta_x^2 + \zeta\zeta_{xx}))$$
$$\leq 6\Big(\int_{Q_T\cap\{u>0\}} u^n u_{xxx}^2\zeta^6\Big)^{1/2}\Big(\int_{Q_T\cap\{u>0\}} u^{n-4}u_x^6\zeta^6\Big)^{1/6}$$
$$\times\Big(\int_{Q_T} u^{n+2}(5\zeta_x^2+\zeta\zeta_{xx})^3\Big)^{1/3}$$
$$\leq \delta\int_{Q_T\cap\{u>0\}} (u^n u_{xxx}^2 + (u^{\frac{n+2}{6}})_x^2)\zeta^6$$
$$+ c(\delta)\int_{Q_T} u^{n+2}(\zeta_x^6 + (\zeta\zeta_{xx})^3) \qquad \forall\,\delta>0;$$

$$\int_{Q_T\cap\{u>0\}} u^n u_{xxx}u_{xx}(\zeta^6)_x = 6\int_{Q_T\cap\{u>0\}} (u^{\frac{n}{2}}u_{xxx}\zeta^3)$$
$$\times\Big[\Big(u^{\frac{n-1}{3}}u_{xx} + \frac{n-1}{3}u^{\frac{n-4}{3}}u_x^2 - \frac{n-1}{3}u^{\frac{n-4}{3}}u_x^2\Big)\zeta^2\Big](u^{\frac{n}{2}-\frac{n-1}{3}}\zeta_x)$$
$$\leq 6\Big(\int_{Q_T\cap\{u>0\}} u^n u_{xxx}^2\zeta^6\Big)^{1/2}\Big[\int_{Q_T\cap\{u>0\}} \Big(\frac{3}{n+2}|(u^{\frac{n+2}{3}})_{xx}|$$
$$+ \frac{n-1}{3}\Big(\frac{6}{n+2}\Big)^2(u^{\frac{n+2}{6}})_x^2\Big)^3\zeta^6\Big]^{1/3}\Big(\int_{Q_T} u^{n+2}\zeta_x^6\Big)^{1/3}$$
$$\leq \delta\int_{Q_T\cap\{u>0\}} (u^n u_{xxx}^2 + |(u^{\frac{n+2}{3}})_{xx}|^3 + (u^{\frac{n+2}{6}})_x^6)\zeta^6$$
$$+ c(\delta)\int_{Q_T} u^{n+2}\zeta_x^6 \qquad \forall\,\delta>0.$$

Using these estimates in inequality (3.31) with $\delta = d_7/6$ we obtain validity of the following statement.

Theorem 4. *Let $\nu=\pm1$, $0\leq A$, $2\leq n<3$, $m-\frac{2}{3}n > -\frac{2}{3}$ if $2\leq n<\frac{5}{2}$, and $m-n>-\frac{3}{2}$ if $\frac{5}{2}\leq n<3$, with the additional constraints that $m<M$ if $A>0$, and $m<n+2$ if $A=0$ and $\nu=1$. Then the strong solutions obtain in Theorems 2, 3 satisfy the following local energy estimate*

$$\int_\Omega \zeta^6|u_x(x,T)|^2\,dx + d_{10}\int_{Q_T}\zeta^6\big((u^{\frac{n+2}{6}})_x^6 + (u^{\frac{n+2}{3}})_{xx}^3 + (u^{\frac{n+2}{2}})_{xxx}^2\big) +$$
$$d_{10}\int_{Q_T\cap\{u>0\}}\zeta^6 u^n u_{xxx}^2 \leq \int_\Omega \zeta^6|u_{0x}(x)|^2\,dx + d_9\int_{Q_T} u^{n+2}(|\zeta_x|^6 + |\zeta\zeta_{xx}|^3) -$$

$$\int_{Q_T\cap\{u>0\}} (\nu u^m u_x - A u^M u_x)(u_x\zeta^6)_{xx}, \quad (3.32)$$

where $\zeta(x)$ is arbitrary nonnegative function from $C^4([-a,a])$.

4. Finite speed propagation (strong slippage: $0 < n < 2$)

In this section, we consider problem ($\mathbb{P}$) with initial data, u_0, which satisfies (2.1) and which also possesses the additional property,

$$\operatorname{supp} u_0 \subset \{x \le 0\}. \quad (4.1)$$

Let us introduce the following family of subdomains:

$$\Omega(s) = \Omega \cap \{x \,|\, x > s\} \quad \forall\, s \in (-a,a), \qquad Q_t(s) = \Omega(s) \times (0,t). \quad (4.2)$$

Theorem 5. *Let u_0 satisfy (2.1), (4.1), let $\nu = \pm 1$, $0 \le A$, $0 < n < 2$, $m < M$ if $0 < A$, $m < n+2$ if $\nu = 1$, $A = 0$, and*

$$m > 0 \quad \text{if} \quad \nu = -1, \qquad m > \frac{n}{2} \quad \text{if} \quad \nu = 1,$$

and let u denote an arbitrary strong nonnegative solution of problem ($\mathbb{P}$), obtained as in Theorem 1, which satisfies the local entropy estimate in Theorem 2 or 3. Then u possesses the finite speed of propagation property in the sense that there exists a continuous function, $s(t)$, such that $s(0) = 0$, and a positive time T_0, such that

$$\operatorname{supp} u(\cdot,t) \subset \overline{\Omega} \setminus \Omega(s(t)), \qquad s(t) < a \quad \forall\, t < T_0,\ s(T_0) = a. \quad (4.3)$$

Remark 4.1. *The analysis in the section applies also to (1.6 i).*

Proof. Since by assumption $0 < n < 2$, the local entropy estimate, (3.17) or (3.19), holds for some positive $\alpha < 2 - n$. The proof of the finite speed of propagation property is based on careful analysis of the properties of solutions satisfying these inequalities with positive α. In the stable case, $\nu = -1$, such analysis was performed in [8] with $A = 0$. If $\nu = -1$ and $A > 0$, a similar proof can be given. Therefore we restrict our attention here to the unstable case, $\nu = 1$. Although we set $A = 0$ for simplicity, all our estimates are also valid for the case $\nu = 1$, $A > 0$. Thus, the estimate (3.19) can be written in the form

$$\frac{1}{\alpha(\alpha+1)} \int_\Omega \zeta^4 u^{1+\alpha}(x,T)\,dx + c_1 \int_{Q_T} \zeta^4 \big(u^{\alpha+n-2\gamma+1}(u^\gamma)^2_{xx} + u^{\alpha+n-3}u_x^4\big)$$
$$\le \frac{1}{\alpha(\alpha+1)} \int_\Omega \zeta^4 u_0^{1+\alpha}(x)\,dx + cR, \qquad \alpha > 0, \quad (4.4)$$

where the constant $c = \max(c_2, c_3, c_4)$ and

$$R := R_1 + R_2 + R_3 = \int_{Q_T} (|\zeta_x^4| + |\zeta\zeta_{xx}|^2) u^{n+\alpha+1} +$$
$$\int_{Q_T} |(\zeta^3\zeta_x)_x| u^{\alpha+m+1} + \int_{Q_T} \zeta^4 u^{\alpha+2m-n+1}.$$

Let us define a cut-off function $\zeta_{s,\delta}(x)$, as follows

$$\zeta_{s,\delta}(x) = \varphi\Big(\frac{x-s}{\delta}\Big), \quad (4.5)$$

where $s \in \mathbb{R}, \delta > 0$ are free parameters, and $\varphi(r)$ is a nonnegative nondecreasing $C^2(\mathbb{R})$ function such that:

$$\varphi(r) = 0 \text{ for } r \le 0, \quad \varphi(r) = 1 \text{ for } r > 1. \tag{4.6}$$

We now introduce three energy functions for the solution u under consideration, connected with the terms on the right hand side of our entropy estimate (4.4),

$$\begin{aligned} J_T(s) &:= \int_{Q_T(s)} u^{\beta_1+\alpha+1}(x,t)\,dx\,dt, \\ E_T(s) &:= \int_{Q_T(s)} u^{\beta_2+\alpha+1}(x,t)\,dx\,dt, \\ I_T(s) &:= \int_{Q_T(s)} u^{\beta_3+\alpha+1}(x,t)\,dx\,dt, \end{aligned} \tag{4.7}$$

where $\beta_1 = n,\ \beta_2 = m,\ \beta_3 = 2m - n$. Using (4.4), we deduce three functional inequalities with respect to $J_T(s)$, $E_T(s)$, $I_T(s)$. Setting $\zeta(x) = \zeta_{s,\delta}(x)$ in (4.4), we obtain after some simple computations

$$\begin{aligned} \sup_{t\in(0,T)} \int_{\Omega(s+\delta)} u^{\alpha+1} + D_1 \int_{Q_T(s+\delta)} \Big[\big(u^{\frac{\alpha+n+1}{2}}\big)^2_{xx} + \big(u^{\frac{\alpha+n+1}{4}}\big)^4_x \Big] \\ \le D_2 \Bigg[\frac{1}{\delta^4} \int_{Q_T(s)\setminus Q_T(s+\delta)} u^{\alpha+n+1} + \frac{1}{\delta^2} \int_{Q_T(s)\setminus Q_T(s+\delta)} u^{\alpha+m+1} + \\ \int_{Q_T(s)} u^{\alpha+2m-n+1} \Bigg] + \int_{\Omega(s)} u_0(x)^{1+\alpha} dx. \end{aligned} \tag{4.8}$$

Here and throughout the proof, D_i denote positive constants which can depend on the problem parameters, α, n, m, but not on s, δ, and T. For arbitrary $\beta > 0,\ l > 4,\ s > 0, \delta > 0$, such that $s + 2\delta < a$, we obtain for $\sigma = \beta + \alpha + 1$ that

$$\int_{\Omega(s+2\delta)} u^\sigma dx \le \int_{\Omega(s+\delta)} (u\,\zeta^l_{s+\delta,\delta})^\sigma dx = \int_{\Omega(s+\delta)} v^{\frac{4\sigma}{n+\alpha+1}} dx, \tag{4.9}$$

where $v = v(x,t) = (u\,\zeta^l_{s+\delta,\delta})^{\frac{n+\alpha+1}{4}}$. By the Gagliardo–Nirenberg interpolation inequality,

$$\int_{\Omega(s+\delta)} v^{\frac{4\sigma}{n+\alpha+1}} dx \le D_3 \left[\int_{\Omega(s+\delta)} |v_x|^4 dx \right]^{\frac{\theta\sigma}{n+\alpha+1}} \left[\int_{\Omega(s+\delta)} v^{\frac{4(\alpha+1)}{n+\alpha+1}} dx \right]^{\frac{(1-\theta)\sigma}{\alpha+1}} \tag{4.10}$$

where $\theta = \frac{\beta(n+\alpha+1)}{\sigma(n+4(\alpha+1))}$. Combining (4.9) and (4.10),

$$\begin{aligned} \int_{\Omega(s+2\delta)} u^{\beta+\alpha+1} dx \le D_4 \left[\int_{\Omega(s+\delta)} u^{\alpha+1} dx \right]^{\frac{(1-\theta)\sigma}{\alpha+1}} \\ \times \left[\int_{\Omega(s+\delta)} \big|\big(u^{\frac{n+\alpha+1}{4}}\big)_x\big|^4 dx + \frac{1}{\delta^4} \int_{\Omega(s+\delta)\setminus\Omega(s+2\delta)} u^{n+\alpha+1} dx \right]^{\frac{\theta\sigma}{n+\alpha+1}}. \end{aligned} \tag{4.11}$$

Let us suppose that

$$\frac{\theta\sigma}{n+\alpha+1} < 1.$$

or, equivalently,

$$\beta < n + 4(\alpha + 1), \tag{4.12}$$

Then, integrating (4.11) with respect to t and using Hölder's inequality,

$$\int_{Q_T(s+2\delta)} u^{\beta+\alpha+1} \le D_5 T^{1-\frac{\beta}{n+4(\alpha+1)}} \sup_{t\in(0,T)} \left(\int_{\Omega(s+\delta)} u^{\alpha+1}(t) \right)^{\frac{(1-\theta)(\beta+\alpha+1)}{\alpha+1}}$$
$$\times \left[\frac{1}{\delta^4} \int_{Q_T(s+\delta)\setminus Q_T(s+2\delta)} u^{n+\alpha+1} + \int_{Q_T(s+\delta)} \left|\left(u^{\frac{n+\alpha+1}{4}}\right)_x\right|^4 \right]^{\frac{\beta}{n+4(\alpha+1)}}$$

$$\le D_6 T^{1-\frac{\beta}{n+4(\alpha+1)}} \left[\sup_{t\in(0,T)} \int_{\Omega(s+\delta)} u^{\alpha+1}(t) + \frac{1}{\delta^4} \int_{Q_T(s+\delta)\setminus Q_T(s+2\delta)} u^{n+\alpha+1} \right.$$
$$\left. + \int_{Q_T(s+\delta)} \left|\left(u^{\frac{n+\alpha+1}{4}}\right)_x\right|^4 \right]^{1+\mu}, \tag{4.13}$$

where $\mu = \frac{4\beta}{n+4(\alpha+1)} > 0$.

Using the definitions in (4.7) and the a priori estimate (4.8), we deduce from (4.13) that

$$\int_{Q_T(s+2\delta)} u^{\beta+\alpha+1} \le D_7 T^{1-\frac{\mu}{4}} \Big[\frac{J_T(s) - J_T(s+2\delta)}{\delta^4} +$$
$$\frac{E_T(s) - E_T(s+\delta)}{\delta^2} + I_T(s) + \int_{\Omega(s)} u_0^{\alpha+1}\, dx \Big]^{1+\mu}. \tag{4.14}$$

The inequality (4.14) holds for the three values of $\beta_i,\ i = 1, 2, 3$, prescribed in (4.7), if condition (4.12) holds with $\beta = \beta_i,\ i = 1, 2, 3$. These conditions may be written as

$$1)\ \alpha + 1 > 0, \quad 2)\ m < n + 4(\alpha + 1), \quad 3)\ 2m - n < n + 4(\alpha + 1).$$

It is easy to check that all of these conditions are satisfied for some $\alpha \in (\alpha_2, 2 - n)$ if and only if

$$m < 6 - n. \tag{4.15}$$

Thus, if inequality (4.15) holds, we obtain the following system of functional inequalities:

$$\begin{aligned}
J_T(s+\delta) &\le D_8 T^{\frac{4-\mu_1}{4}} \Big[\frac{J_T(s) - J_T(s+\delta)}{\delta^4} + \frac{E_T(s) - E_T(s+\delta)}{\delta^2} + I_T(s) + h_0(s) \Big]^{1+\mu_1}, \\
E_T(s+\delta) &\le D_9 T^{\frac{4-\mu_2}{4}} \Big[\frac{J_T(s) - J_T(s+\delta)}{\delta^4} + \frac{E_T(s) - E_T(s+\delta)}{\delta^2} + I_T(s) + h_0(s) \Big]^{1+\mu_2}, \\
I_T(s+\delta) &\le D_{10} T^{\frac{4-\mu_3}{4}} \Big[\frac{J_T(s) - J_T(s+\delta)}{\delta^4} + \frac{E_T(s) - E_T(s+\delta)}{\delta^2} + I_T(s) + h_0(s) \Big]^{1+\mu_3},
\end{aligned} \tag{4.16}$$

where $h_0(s) = \int_{\Omega(s)} u_0(x)^{1+\alpha}\,dx$, and

$$\mu_1 = \frac{4n}{n+4(\alpha+1)},\ \mu_2 = \frac{4m}{n+4(\alpha+1)},\ \mu_3 = \frac{4(2m-n)}{n+4(\alpha+1)}.$$

Due to the boundedness and nonnegativity of u, the following estimates are obvious

$$\begin{aligned} J_T(0) \le J_T &:= \int_{Q_T} u^{\beta_1+\alpha+1}\,dx\,dt < cT, \quad \forall\, T>0,\\ E_T(0) \le E_T &:= \int_{Q_T} u^{\beta_2+\alpha+1}\,dx\,dt < cT, \quad \forall\, T>0,\\ I_T(0) \le I_T &:= \int_{Q_T} u^{\beta_3+\alpha+1}\,dx\,dt < cT, \quad \forall\, T>0, \end{aligned} \tag{4.17}$$

where c is a constant which does not depend on T. The validity of the statement of Theorem 5 when inequality (4.15) holds, now follows from (4.16), (4.17), and Lemma A.2 in [11], since $h_0(s) = 0$ for any $s > 0$.

If $m \ge 6-n$, we proceed as follows. Fix $\overline{m}$ such that $\frac{n}{2} < \overline{m} < 6-n$. It is easy to see that due to the boundedness of the solution u, all of the previous estimates in the proof of Theorem 5 are remain true when m is replaced by $\overline{m}$. As result the system (4.16) is obtained with respect to new energy functions (4.7) defined by the values

$$\beta_1 = n,\ \beta_2 = \overline{m},\ \beta_3 = 2\,\overline{m} - m,$$

and where

$$\mu_1 = \frac{4n}{n+4(\alpha+1)}, \quad \mu_2 = \frac{4\,\overline{m}}{n+4(\alpha+1)}, \quad \mu_3 = \frac{4(2\,\overline{m}-n)}{n+4(\alpha+1)}.$$

In this manner, the validity of the statement of the theorem for the case $m \ge 6-n$ again follows from (4.16) and Lemma A.2 in [11]. $\square$

5. Finite speed propagation (weak slippage: $2 \le n < 3$)

In this section we shall again consider problem $(\mathbb{P})$ with initial data, u_0, which satisfies (2.1) as well as the additional property, (4.1). The subdomains, $\Omega(s)$ and $Q_t(s)$, will be understood here to be as defined in (4.2).

We first prove the following lemma, which provides control on the $L^1_{loc}(\Omega)$ norm of some minimal positive power of the solution under consideration, $u(x,t)$. For the sake of simplicity, the results in this section are proven for $\nu = 1$ and $A = 0$, though they remain valid for $\nu = -1$ and $A > 0$ as well. The results here can also be readily shown to apply to (1.4) if $2 < n < 3$ and to (1.6 i).

Lemma 5.1. *Let $\nu = \pm 1$, $0 \le A$, $1/2 < n < 3$, $m > n/2$, $\eta > \frac{1-n}{3}$, $\varepsilon > 0$, with $m < M$ if $A > 0$ and $m < n+2$ if $\nu = 1$, $A = 0$. Then there exists a positive constant c, depending on n, m, η, ε only, such that any nonnegative strong solution u of problem $(\mathbb{P})$ satisfies*

$$\int_\Omega u(x,T)^{\eta+1}\zeta^4 \le \varepsilon\left(\int_{Q_T\cap\{u>0\}} \zeta^6 u^n u_{xxx}^2 + \int_{Q_T} |\zeta_x|^6 u^{n+2}\right) +$$
$$c\left(\int_{Q_T} \left[u^{n+2\eta}|\zeta_x|^2 + u^{\frac{3m+3\eta+1-n}{2}}\zeta^3 + u^{m+\eta+1}|\zeta\zeta_x|^2\right]\right) +$$

$$\int_\Omega u_0^{\eta+1}\zeta^4 + c\int_{Q_T\cap \operatorname{supp}\zeta} u^{n+3\eta-1}, \quad (5.1)$$

for arbitrary nonnegative $\zeta \in C^2([-a,a])$.

Proof. The proof here follows that of Lemma 5.2 in [11]. Let us test the integral identity in (2.2) by the following test function

$$\varphi = -l_\delta \zeta^4 (u+\gamma)^\eta, \quad \gamma > 0,$$

where $\{l_\delta(t)\} \subset C_c^\infty(0,T)$ and $l_\delta \to \chi_{(0,T)}$ as $\delta \to 0$. After some simple computations, we obtain

$$-\int_{Q_T} (l_\delta)_t \zeta^4 \frac{(u+\gamma)^{\eta+1}}{\eta+1} = \int_{Q_T} u^m u_x l_\delta ((u+\gamma)^\eta \zeta^4)_x +$$
$$\int_{Q_T\cap\{u>0\}} l_\delta (\zeta^4)_x u^n (u+\gamma)^\eta u_{xxx} +$$
$$\eta \int_{Q_T\cap\{u>0\}} l_\delta \zeta^4 u^n (u+\gamma)^{\eta-1} u_x u_{xxx} := A_1 + A_2 + A_3. \quad (5.2)$$

The terms A_2, A_3 are estimated as in [11, Lemma 5.2]. For any $\epsilon > 0$,

$$|A_2| \le \epsilon \int_{Q_T\cap\{u>0\}} \zeta^6 u^n u_{xxx}^2 + C_1(\epsilon)\int_{Q_T} |\zeta_x|^2 u^n (u+\gamma)^{2\eta},$$

$$|A_3| \le \epsilon\left(\int_{Q_T\cap\{u>0\}} \zeta^6 u^n u_{xxx}^2 + \int_{Q_T} |\zeta_x|^6 u^{n+2}\right) +$$
$$C_2(\epsilon)\int_{Q_T\cap \operatorname{supp}\zeta} (u+\gamma)^{n+3\eta-1}.$$

Here C_i denote constants which may depend on m, n, η, and on ϵ if indicated, but which are independent of γ and δ.

Let us now estimate A_1.

$$A_1 = \int_{Q_T} \eta u^m (u+\gamma)^{\eta-1} u_x^2 \zeta^4 l_\delta +$$
$$\int_{Q_T} 4u^m (u+\gamma)^\eta u_x \zeta^3 \zeta_x l_\delta := A_1^{(1)} + A_1^{(2)}. \quad (5.3)$$

Since $m - \frac{n-4}{3} + \eta - 1 > 0$, it follows from Young's inequality and Lemma B.1 in [11] that for any $\epsilon > 0$,

$$|A_1^{(1)}| \le \epsilon \int_{Q_T\cap\{u>0\}} u_x^6 u^{n-4}\zeta^6 + C_3(\epsilon)\int_{Q_T} (u+\gamma)^{\frac{3}{2}(m-\frac{n-4}{3}+\eta-1)}\zeta^3 \le$$
$$\epsilon\left(\int_{Q_T\cap\{u>0\}} \zeta^6 u^n u_{xxx}^2 + C_4\int_{Q_T} \zeta_x^6 u^{n+2}\right) + C_3(\epsilon)\int_{Q_T} (u+\gamma)^{\frac{3m+3\eta+1-n}{2}}\zeta^3. \quad (5.4)$$

With regard to $A_1^{(2)}$, we have by Young's inequality

$$|A_1^{(2)}| \le |A_1^{(1)}| + C_5 \int_{Q_T} (u+\gamma)^{m+\eta+1}|\zeta\zeta_x|^2. \quad (5.5)$$

Thus all the integrals in (5.2) are uniformly bounded with respect to parameters $\delta > 0, \gamma > 0$. Therefore, collecting the estimates obtained for the terms $A_i, i =$

$1, \ldots, 3$, passing to the limit $\delta \to 0$, and then to the limit $\gamma \to 0$, the estimate (5.1) follows. $\square$

Theorem 6. *Let u_0 satisfy (2.1), (4.1), let $\nu = \pm 1$, $0 \le A$, $1/2 < n < 3$, $m < n/2$, $m < M$ if $A > 0$, and $m < n+2$ if $\nu = 1$, $A = 0$, and let u denote an arbitrary strong nonnegative solution of problem $(\mathbb{P})$, obtained as in Theorem 1, which satisfies the local entropy estimate in Theorem 2 or 3. Then u possesses the finite speed of propagation property in the sense of Theorem 5.*

Proof. Let us consider the local energy estimate (3.32) obtained in Theorem 4, and estimate the third term on the right hand side, setting $\nu = 1$ for simplicity,

$$B := \int_{Q_T \cap \{u>0\}} u^m u_x (u_x \zeta^6)_{xx} = \int_{Q_T \cap \{u>0\}} u^m u_x u_{xxx} \zeta^6 +$$
$$12 \int_{Q_T \cap \{u>0\}} u^m u_x u_{xx} \zeta^5 \zeta_x + 6 \int_{Q_T \cap \{u>0\}} u^m u_x^2 (\zeta^5 \zeta_x)_x := B_1 + B_2 + B_3.$$

Using Young's inequality, we obtain that for any $\epsilon > 0$,

$$|B_1| \le \left| \int_{Q_T \cap \{u>0\}} u^{\frac{n}{2}} u_{xxx} u_x u^{\frac{(n-4)}{6}} u^{m - \frac{n}{2} - \frac{n-4}{6}} \zeta^6 \right| \le$$
$$\epsilon \int_{Q_T \cap \{u>0\}} (u^n u_{xxx}^2 + u_x^6 u^{n-4}) \zeta^6 + D_1(\epsilon) \int_{Q_T} u^{3m-2n+2} \zeta^6.$$

Here and in the sequel D_i denote constants which may depend on m, n, η, and on ϵ if indicated, but which are independent of δ and s. Similarly, we may estimate

$$B_2 \le \epsilon \int_{Q_T \cap \{u>0\}} (u_{xx}^3 u^{n-1} + u_x^6 u^{n-4}) \zeta^6 + D_2(\epsilon) \int_{Q_T} u^{2m-n+2} |\zeta^2 \zeta_x|^2,$$
$$B_3 \le \epsilon \int_{Q_T \cap \{u>0\}} u_x^6 u^{n-4} \zeta^6 + D_3(\epsilon) \int_{Q_T} u^{\frac{3m-n+4}{2}} |(\zeta^3 \zeta_x)_x|^{\frac{3}{2}}.$$

Using these estimates in (3.32), we obtain that for $\epsilon > 0$ sufficiently small

$$\int_\Omega |u_x(x,T)|^2 \zeta^6 + 2^{-1} d_{10} \int_{Q_T} \zeta^6 [(u^{\frac{n+2}{6}})_x^6 + (u^{\frac{n+2}{3}})_{xx}^3 + (u^{\frac{n+2}{2}})_{xxx}^2] +$$
$$2^{-1} d_{10} \int_{Q_T \cap \{u>0\}} \zeta^6 u^n u_{xxx}^2 \le \int_\Omega |u_{0x}|^2 \zeta^6 + d_9 \int_{Q_T} (|\zeta_x|^6 + |\zeta \zeta_{xx}|^3) |u|^{n+2} +$$
$$D_4 \int_{Q_T} u^{3m-2n+2} \zeta^6 + D_4 \int_{Q_T} u^{2m-n+2} |\zeta^2 \zeta_x|^2 + D_4 \int_{Q_T} u^{\frac{3m-n+4}{2}} |(\zeta^3 \zeta_x)_x|^{\frac{3}{2}}. \quad (5.6)$$

Assuming that $\eta > \frac{1-n}{3}$, summing the inequalities (5.6), (5.1), and taking $\epsilon > 0$ to be sufficiently small, we obtain:

$$\int_\Omega |u(x,T)|^{\eta+1} \zeta^4 + \int_\Omega |u_x(x,T)|^2 \zeta^6 + 4^{-1} d_{10} \int_{Q_T \cap \{u>0\}} \zeta^6 |(u^{\frac{n+2}{2}})_{xxx}|^2$$
$$\le \int_\Omega |u_{0x}|^2 \zeta^6 + \int_\Omega |u_0|^{\eta+1} \zeta^4 + D_5 R,$$
$$R := \int_{Q_T} u^{3m-2n+2} \zeta^6 + \int_{Q_T} u^{\frac{3m+3\eta+1-n}{2}} \zeta^3 +$$
$$\int_{Q_T \cap \operatorname{supp} \zeta} u^{n+3\eta-1} + \int_{Q_T} (|\zeta_x|^6 + |\zeta \zeta_{xx}|^3) u^{n+2} + \int_{Q_T} u^{2m-n+2} |\zeta^2 \zeta_x|^2 +$$

$$\int_{Q_T} u^{n+2\eta}|\zeta_x|^2 + \int_{Q_T} u^{m+\eta+1}|\zeta\zeta_x|^2 + \int_{Q_T} u^{\frac{3m-n+4}{2}}|(\zeta^3\zeta_x)_x|^{\frac{3}{2}}. \quad (5.7)$$

Let us take $\zeta(x)$ as the function $\zeta_{s,\delta}$ from (4.5). It then follows from (5.7) that

$$\sup_{t\in(0,T)} \int_{\Omega(s+\delta)} |u(x,t)|^{\eta+1}dx + \sup_{t\in(0,T)} \int_{\Omega(s+\delta)} |u_x(x,t)|^2dx$$
$$+ 4^{-1}d_{10} \int_{Q_T(s+\delta)} |(u^{\frac{n+2}{2}})_{xxx}|^2 \leq \int_{\Omega(s)} (|u_{0x}|^2 + |u_0|^{\eta+1})\,dx + D_6\tilde{R},$$
$$\tilde{R} := \int_{Q_T(s)} u^{3m-3n+2} + \int_{Q_T(s)} u^{\frac{3m+3\eta+1-n}{2}} + \int_{Q_T(s)} u^{n+3\eta-1}+$$
$$\delta^{-6}\int_{Q_T(s)} u^{n+2} + \delta^{-2}\int_{Q_T(s)} u^{2m-n+2} + \delta^{-2}\int_{Q_T(s)} u^{n+2\eta}+$$
$$\delta^{-2}\int_{Q_T(s)} u^{m+\eta+1} + \delta^{-3}\int_{Q_T(s)} u^{\frac{3m-n+4}{2}} := \sum_{i=1}^{8}\delta^{-\chi_i}\int_{Q_T(s)} u^{\xi_i}. \quad (5.8)$$

We now wish to guarantee the validity of the inequalities

$$\xi_i > 1+\eta, \qquad i = 1,2,\ldots,8. \quad (5.9)$$

First we ensure that

$$\xi_3 = n+3\eta-1 > 1+\eta \Leftrightarrow \eta > 1-\frac{n}{2} := \eta_{min}. \quad (5.10)$$

Next we deduce a restriction for m by considering

$$\xi_1 = 3m-2n+2 > 1+\eta \Leftrightarrow \frac{3}{2}(2m-n)+1+\Big(1-\frac{n}{2}\Big) > 1+\eta. \quad (5.11)$$

Together, (5.10) and (5.11) yield that

$$\eta_{min} = 1-\frac{n}{2} < \eta < \eta_{min} + \frac{3}{2}(2m-n). \quad (5.12)$$

There exists η satisfying (5.12) iff

$$2m-n > 0. \quad (5.13)$$

Next it is easy to see that

$$\xi_2 = \frac{3m+3\eta+1-n}{2} = \frac{(n+3\eta-1)+(3m-2n+2)}{2}.$$

Therefore the inequality

$$\xi_2 > 1+\eta \quad (5.14)$$

follows from (5.10) and (5.11). It is easy to check that the other inequalities in (5.9) can be satisfied by an appropriate choice of η if conditions (5.10) and (5.11) are satisfied.

As result of (5.9), the following Gagliardo–Nirenberg interpolation inequalities holds for $i = 1,2,\ldots,8$,

$$\int_{\Omega(s+2\delta)} u^{\xi_i}dx \leq D_7\bigg(\int_{\Omega(s+\delta)} \big|\big((\zeta_{s+\delta,\delta}^{\frac{6}{n+2}}u)^{\frac{n+2}{2}}\big)_{xxx}\big|^2dx\bigg)^{\frac{\theta_i\xi_i}{n+2}}$$
$$\times\bigg(\int_{\Omega(s+\delta)} (\zeta_{s+\delta,\delta}^{\frac{6}{n+2}}u)^{\eta+1}dx\bigg)^{\frac{(1-\theta_i)\xi_i}{\eta+1}}, \quad (5.15)$$

where $\theta_i = \frac{(n+2)(\xi_i-\eta-1)}{\xi_i(5\eta+n+7)}$, $i = 1, 2, \ldots, 8$. We now wish to guarantee that

$$\frac{\theta_i \xi_i}{n+2} < 1, \quad i = 1, 2, \ldots, 8. \tag{5.16}$$

From the definition of θ_i, it follows that (5.16) holds iff

$$\eta > \frac{\xi_i - n - 8}{6}, \quad i = 1, 2, \ldots, 8. \tag{5.17}$$

It is easy to check that all the inequalities in (5.9) and (5.17) hold for some η in the interval (5.12) if condition (5.13) is satisfied.

Therefore, as in [11], we may deduce from (5.8), (5.15), the following inequalities

$$\int_{Q_T(s+\delta)} u^{\xi_i} \leq D_8 T^{1-\frac{\theta_i \xi_i}{n+2}} \left(\sum_{j=1}^{8} \delta^{-\chi_j} \int_{Q_T(s)} u^{\xi_j} + H_0(s) \right)^{1+\frac{6(\xi_i-\eta-1)}{5\eta+n+7}}, \tag{5.18}$$

where $H_0(s) := \int_{\Omega(s)} (u_0^{1+\eta}(x) + |u_{0x}(x)|^2)\, dx = 0$, for all $0 < s < s+\delta < a$. From (5.18) and Lemma A.2 from [11], the conclusion of Theorem 6 now follows. $\square$

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