UPPER BOUNDS FOR COARSENING FOR THE DEEP QUENCH OBSTACLE PROBLEM

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Abstract. The deep quench obstacle problem models phase separation at low temperatures. During phase separation, domains of high and low concentration are formed, then *coarsen* or grow in average size. Of interest is the time dependence of the dominant length scales of the system. Relying on recent results by Novick-Cohen & Shishkov [16], we demonstrate upper bounds for coarsening for the deep quench obstacle problem, with either constant or degenerate mobility. For the case of constant mobility, we obtain upper bounds of the form $t^{1/3}$ at early times as well as at times t for which $E(t) \leq \frac{(1-\overline{u}^2)}{4}$, where $E(t)$ denotes the free energy. For the case of degenerate mobility, we get upper bounds of the form $t^{1/3}$ or $t^{1/4}$ at early times, depending on the value of $E(0)$, as well as bounds of the form $t^{1/4}$ whenever $E(t) \leq \frac{(1-\overline{u}^2)}{4}$.

1. Introduction

The deep quench obstacle free boundary problem

$$(\mathbf{DQ}) \qquad \begin{cases} \frac{\partial u}{\partial t} = \nabla \cdot M(u) \nabla w, & (x,t) \in \Omega_T, \\ w + \epsilon^2 \triangle u + u \in \partial \Gamma(u), & (x,t) \in \Omega_T, \end{cases}$$

was apparently first proposed by Oono and Puri [17] as a phenomenological model for phase separation. In (DQ), $\epsilon > 0$, $\partial\Gamma(\cdot)$ is the subdifferential of the indicator function $I_{[-1,1]}(\cdot)$, and $u(x,t)$, which represents the concentration of one of the two components of a binary mixture, should satisfy $\partial_\nu u = 0$ on the "free boundary" where $u = \pm 1$, and $\partial_n u = 0$ on $\partial\Omega$. We shall assume that $\Omega_T = (0,\, T) \times \Omega$, where $0 < T < \infty$ and Ω is a bounded convex domain. We shall focus on a degenerate mobility variant of (DQ) in which $M(u) = 1 - u^2$, as well as on a constant mobility non-degenerate variant in which $M(u) = 1$. Although there has been more study of phase separation models with non-degenerate, or more specifically, constant mobilities, degenerate mobilities reflect somewhat more careful modeling and should capture some of the underlying physics more accurately.

The deep quench obstacle problem (DQ) also corresponds to the zero temperature or *deep quench* limit of the Cahn-Hilliard equation [10],

$$
(\mathbf{CH}) \quad \begin{cases} u_t = \nabla \cdot M(u) \nabla w, & (x,t) \in \Omega_T, \\ w = \frac{\Theta}{2}\{\ln(1+u) - \ln(1-u)\} - u - \epsilon^2 \triangle u, & (x,t) \in \Omega_T, \\ n \cdot \nabla u = n \cdot \nabla w = 0, & (x,t) \in \partial\Omega_T, \end{cases}
$$

where n denotes the unit exterior normal to $\partial\Omega$, Θ is a scaled temperature, and $M(u) = 1 - u^2$ or $M(u) = 1$. See [6, 7, 15]. Indeed existence results for (DQ) can be obtained by considering appropriate limits of solutions to (CH). Existence results were first proven for $M(u) = 1$ by Blowey & Elliott [4], and later for $M(u) = 1 - u^2$ by Elliott & Garcke [9]. See the discussion in [3]. Numerical schemes for (DQ) have been developed for the constant mobility case by Blowey & Elliott [5] and for the degenerate mobility case by Bănas & Nürnberg [1, 2].

Recently, numerical simulations have been undertaken to explore and compare the dynamic properties of the constant mobility and the degenerate mobility deep quench obstacle problem by Bănas, Novick-Cohen & Nürnberg [3]. For initial conditions corresponding to a perturbation of a spatially uniform state, the dynamics for the deep quench obstacle problem (DQ) is roughly similar to that of the Cahn-Hilliard equation (CH), if the initial conditions u_0 satisfy $\frac{1}{|\Omega|}\int_\Omega u_0\,dx \in (-1,1)$, and thus lie in the linearly unstable regime (below the *spinodal*). The basic stages of evolution include an initial period of linear instability during the onset of phase separation (spinodal decomposition or the linear regime), followed by an intermediate period during which local saturation to $u_\pm = \pm 1$, which correspond to near equilibrium phases, occurs throughout most of the domain, followed finally by *coarsening* during which the characteristic dimensions of the support of the equilibrium phases grow. The initial conditions described above are physically reasonable and easily implemented, and the various qualitative stages in the evolution of the dynamics have often been reported experimentally.

A way to follow phase separation as it progresses through the various stages is to track some indicator of the dominant length scale of the system as a function of time. At early times, the length scale should roughly reflect the most unstable modes of the linear unstable regime, and during coarsening the average length scale for the system should be seen to grow. In the present paper we adopt the approach of Kohn & Otto [12] and Novick-Cohen & Shishkov [16], using two underlying length scales, to be denoted by $E^{-1}(t)$ and $L(t)$, and define the parametric set of length scales, $S^{-1}(\bar{t},\, t;\, r,\, \varphi)$ for $0 \le \bar{t} < t$ and $(r,\, \varphi) \in \overline{\Lambda}$, where

$$
S(\bar{t},\, t; r, \varphi) := (t - \bar{t})^{-1/r} ||E^{\varphi} L^{-(1-\varphi)}||_{L^r([\bar{t},\, t])},
$$

and $\bar{t} = 0$, t_* or t^*, where t_*, t^* are transitional values to be prescribed, and $\overline{\Lambda}$ is a suitably prescribed set. By the combined use of an algebraic bound and a differential inequality, rigorous time dependent upper bounds are obtained in terms of $S^{-1}(\bar{t},\, t;\, r,\, \varphi)$. This general approach has been implemented in recent years in various applications where coarsening occurs. See for example [8, 13, 18].

Our approach differs from that of [12] in that we obtain upper bounds both at early and late times. Our approach also differs from that of [16] in that we carefully delineate the coefficients and exponents appearing in the upper bounds, which in turn reflect our choice of the parameters $(r,\, \varphi)$ as well as the initial values of $E(t)$ and $L(t)$, and indicate the realm of applicability of our results in terms of the initial conditions and the value of the free energy functional $E(t)$. This entails estimating $E(t)$ and $L(t)$, and identifying certain transitional values [16]. This also allows our results to be readily comparable with the predictions of Bănas, Novick-Cohen & Nürnberg [3]. To facilitate comparison, the scalings and notations employed in [3] are used throughout. We emphasize that though the results presented here are for (DQ), the deep quench obstacle problem, the methodology and the qualitative conclusions carry over directly to the (CH) framework with either constant or degenerate mobility.

The first length scale, $L(t)$, is based on the norm of $u(x,t)$ in a space whose dual is $W^{1,\infty}(\Omega)$. Since the mean mass, $\bar{u} = \frac{1}{|\Omega|}\int_\Omega u(x,\, t)\, dx$, is time invariant for (DQ) [4, 9], it is convenient to define

$$L(t) := \sup_{\xi \in Y} \frac{1}{|\Omega|} \int_\Omega (u(x,\, t) - \bar{u})\, \xi(x)\, dx, \tag{1.1}$$

where

$$Y := \Big\{ \xi \in W^{1,\infty} \mid \sup_\Omega \epsilon |\nabla \xi| = 1 \Big\}. \tag{1.2}$$

That $L(t)$ acts as a length scale for spatial variation in the concentration can be seen directly from (1.1)-(1.2).

The second length scale, $E^{-1}(t)$, shall be based on the free energy

$$E(t) := \frac{1}{2\,|\Omega|} \int_\Omega \Big\{ \epsilon^2 |\nabla u|^2 + \Big[\frac{\partial W}{\partial u}\Big]^2 \Big\}_{|u=u(x,\,t)}\, dx, \tag{1.3}$$

where

$$\frac{\partial W}{\partial u} = (1-u^2)^{1/2}. \tag{1.4}$$

During the later stages of coarsening the system is approximately partitioned into regions in which $u = \pm 1$, hence

$$\begin{aligned} E(t) \geq \frac{1}{|\Omega|}\int_\Omega \epsilon |\nabla W(u)|\, dx &= \frac{1}{|\Omega|}\int_\Omega \epsilon (1-u^2)^{1/2} |\nabla u|\, dx, \\ &\approx \frac{\epsilon\pi}{2|\Omega|}\, |\,\text{perimeter of interfacial regions}\,|. \end{aligned} \tag{1.5}$$

The transitional times t_*, t^* are defined in terms of the time at which $E(t)$ attains the values $\frac{1}{2}$ and $\frac{(1-\overline{u}^2)}{4}$, respectively.

We summarize our main results as follows:

1. If $E(0) \leq \frac{(1-\overline{u}^2)}{4}$, then for $t > 0$ there are upper bounds of the form $[\frac{t}{\epsilon^2}]^{1/3}$ for (DQ) with constant mobility and upper bounds of the form $[\frac{t}{\epsilon^2}]^{1/4}$ for (DQ) with degenerate mobility.

2. For (DQ) with constant mobility, if $\frac{(1-\overline{u}^2)}{4} < E(0)$, then for $0 < t \leq \min\{\epsilon^2, t^*\}$ there are upper bounds of the form $[\frac{t}{\epsilon^2}]^{1/3}$, and when $t > t^*$ there are bounds of the form $[\frac{t-t^*}{\epsilon^2}]^{1/3}$.

3. For (DQ) with degenerate mobility, if $\frac{(1-\overline{u}^2)}{4} < E(0) < \frac{1}{2}$, then for $0 < t \leq \min\{\epsilon^2, t^*\}$ there are upper bounds of the form $[\frac{t}{\epsilon^2}]^{1/4}$, and when $t > t^*$, (and therefore $E(t) \leq \frac{(1-\overline{u}^2)}{4}$), there are bounds of the form $[\frac{t-t^*}{\epsilon^2}]^{1/4}$.

4. For (DQ) with degenerate mobility, if $\frac{1}{2} < E(0)$, then for $0 < t \leq \min\{\epsilon^2, t_*\}$ there are upper bounds of the form $[\frac{t}{\epsilon^2}]^{1/3}$, and when $t > t^*$ there are upper bounds of the form $[\frac{t-t^*}{\epsilon^2}]^{1/4}$.

See Theorems 4 and 5 in §5 for further details. Note that the upper bounds listed above in 2.-4. may contain temporal gaps. While the bounds above which are prescribed to hold on semi-infinite intervals should hold when the length scale is evaluated based on $E(t)$ only, i.e. based on $S^{-1}(\bar{t}, t; r, \varphi)$ with $\varphi = 1$, the upper bounds which are stated as being valid on short intervals are based on evaluation of both $E(t)$ and $L(t)$, i.e. they are based on $S^{-1}(\bar{t}, t; r, \varphi)$ with $\varphi < 1$.

In §2, algebraic bounds, differential inequalities, and implied upper bounds are obtained, which rely heavily on the results of [12, 16]. In §3, as in [16], time dependent transitions in the upper bounds are prescribed which depend on the initial conditions and on the times t_*, t^*. In §4, some comments are made regarding the values that $L(0)$ and $E(0)$ may assume, in particular for initial conditions which reflect a perturbation of a spatially uniform state. Afterwards, the predicted growth rates, waiting times and coefficients in the upper bounds outlined in §3 are analyzed. In §5, the information from §4 is incorporated into the results from §3, and the resultant upper bounds are outlined in Theorems 4 and 5.

2. Estimates, inequalities, and bounds

In this section we present three lemmas and a corollary, which provide the basis for the conclusions in the remainder of the paper. The general framework here follows [12, 16] closely.

The first lemma gives an algebraic bound from below in terms of $E(t)$ and $L(t)$.

Lemma 1. Suppose that $|\bar{u}| < 1$, then for $t \geq 0$,

$$1 \leq \frac{2^{5/2}}{(1-\bar{u}^2)}[\,3\,(E(t) + \frac{\epsilon|\partial\Omega|}{|\Omega|})\,L(t)]^{1/2} + \frac{1}{(1-\bar{u}^2)}2E(t). \tag{2.1}$$

Proof. See [16, Lemma 4.1]. □

The formulation here is a little simpler than in [16], since an alternative given there has been eliminated which does not eventually provide useful information in terms of the upper bounds.

Remark 2.1. *If the domain Ω is sufficiently small, phase separation does not occur and the uniform state $u = \bar{u}$ is stable,* [11]*. In this case $E_{\min} = \frac{1}{2|\Omega|}(1-\overline{u}^2)$. Typically, though, phase separation does occur and referring to (1.5), and recalling the asymptotic Γ-convergence estimate* [15, 19]*,*

$$E_{\min} \sim \frac{\epsilon\pi}{2|\Omega|}\,|\,\text{perimeter of interfacial regions}|. \tag{2.2}$$

In either case, we have that $E_{\min} > 0$ if $|\bar{u}| < 1$. If phase separation followed by coarsening occurs, which is the case of interest which we focus on here, as we are considering convex domains, the minimal amount of surface area for a completely partitioned system is $\mathcal{O}(|\partial\Omega|)$ and typically much smaller. So (2.2) implies the lower bound $\mathcal{O}(\epsilon|\partial\Omega|/|\Omega|)$ for $E_{\min}$.

Remark 2.2. *Although the term $\frac{\epsilon|\partial\Omega|}{|\Omega|}$ in Lemma 1 scales as $\frac{1}{\text{length}}$ and becomes arbitrarily small in appropriate large domain limits, for finite domains this term can be seen from (2.2) to compete with $E_{\min}$ in the long time limit. Note this term does not arise at all if periodic boundary conditions are considered, and a bound like (2.1) but without boundary terms proportional to $\frac{\epsilon|\partial\Omega|}{|\Omega|}$ can also be obtained for rectangular domains with Neumann boundary conditions.*

Neglecting the boundary term in (1.5) for the sake of simplicity, the following corollary (see [16, Corollary 4.4]) is an immediate consequence of Lemma 1.

Corollary 1. Suppose that $|\bar{u}| < 1$, then for any $t \geq 0$, either

$$\begin{array}{ll} (i) & E > \frac{(1-\bar{u}^2)}{4}, \ \text{or} \\ (ii) & E \leq \frac{(1-\bar{u}^2)}{4} \ \text{and} \ EL \geq \frac{1}{384}(1-\bar{u}^2)^2. \end{array}$$

Corollary 1, as well as Lemma 1 and Lemmas 2-4 which follow, can all be generalized to incorporate the boundary term by restating the results in terms of

$$\widetilde{E} = E + \frac{\epsilon|\partial\Omega|}{|\Omega|}. \tag{2.3}$$

For simplicity, we shall not write down this generalization explicitly.

Lemma 2 and Lemma 3 provide a differential inequality in terms of $E(t)$, $L(t)$, and their time derivatives for (DQ) with constant and degenerate mobility, respectively. The analysis for the degenerate and for the nondegenerate cases is a little different. For (DQ) with constant mobility, we have the following result. See [12, Lemma 2] and [16, Lemma 4.2].

Lemma 2. Suppose that $u(x,t)$ denotes a solution to (DQ) with constant mobility and $|\bar{u}| < 1$. Then

$$\epsilon^2 |\dot{L}|^2 \leq -\dot{E}. \tag{2.4}$$

For (DQ) with degenerate mobility, we first note that (1.3)-(1.4) imply

$$\frac{1}{|\Omega|} \int_\Omega (1 - u^2)\, dx \leq \min\{1, 2E(t)\}. \tag{2.5}$$

Using (2.5), we may proceed as in the proof of [16, Lemma 4.3] to obtain the following, slightly stronger, result.

Lemma 3. Suppose that $u(x,t)$ denotes a solution to (DQ) with degenerate mobility and $|\bar{u}| < 1$. Then

$$\epsilon^2 |\dot{L}|^2 \leq -\min\{1,\, 2E\,\}\dot{E}. \tag{2.6}$$

Proof. Arguing as in [12, Lemma 2], it follows that

$$\epsilon |\dot{L}| \leq \frac{1}{|\Omega|} \int_\Omega |J|\, dx,$$

and

$$-\dot{E} \geq \frac{1}{|\Omega|} \int_\Omega \frac{1}{(1-u^2)} |J|^2\, dx,$$

where

$$J = -(1-u^2)\nabla[u + \triangle u],$$

and hence by the Cauchy-Schwartz inequality,

$$\epsilon^2 |\dot{L}|^2 \leq -\dot{E}\, \frac{1}{|\Omega|} \int_\Omega (1-u^2)\, dx. \tag{2.7}$$

Now (2.5),(2.7) imply (2.6). $\square$

The lemma below provides upper bounds based on the results of our earlier lemmas. It follows directly from [16, Lemma 3.3] upon rescaling time $t \to t/\epsilon^2$ in accordance with the scaling of time adopted here. See also [12, Lemma 3].

Lemma 4. Suppose that

$$\epsilon^2 |\dot{L}|^2 \leq -AE^\alpha \dot{E}, \quad 0 < t, \tag{2.8}$$

where $\alpha = 0$ or 1, and A is a positive constant. Let $(r, \varphi) \in \Lambda(\alpha)$, where $\Lambda(\alpha)$ is defined by the inequalities

$$0 \leq \varphi \leq 1, \quad r < 3 + \alpha, \quad \varphi r > 1 + \alpha, \quad (1-\varphi)r < 2. \tag{2.9}$$

Set

$$S(0,\,t;\,r,\,\varphi) = t^{-1/r}||E^{\varphi}L^{-(1-\varphi)}||_{L^r([0,\,t])}, \tag{2.10}$$

and

$$\begin{aligned}
&\gamma_1 = (1+\alpha) - \varphi r,\ \gamma_2 = 1 - \tfrac{1}{2}(1-\varphi)r,\\
&\sigma_1 = -(1+\alpha) + \varphi(3+\alpha),\ \sigma_2 = -(1+\alpha) + \tfrac{2\varphi}{1-\varphi},\\
&\xi = -A^{-1}\gamma_1\gamma_2^{-2}(1-\rho^{-\gamma_2})^2,\ \text{ where } \rho > 1 \text{ is arbitrary.}
\end{aligned} \tag{2.11}$$

i) If there exists a positive constant B such that

$$LE \geq B, \quad t \geq 0, \tag{2.12}$$

then

$$S^r(0,\,t;\,r,\,\varphi) + \epsilon^2 t^{-1} L(0)^{(3+\alpha)-r} \geq \epsilon^{\frac{2r}{3+\alpha}}\,\vartheta_1 t^{-\frac{r}{(3+\alpha)}}, \quad t > 0, \tag{2.13}$$

where

$$\vartheta_1 = \left[\frac{3+\alpha}{2(3+\alpha)-2r}\left[(\xi B^{\sigma_1})^{\frac{r}{(3+\alpha)-r}} + B^{\varphi r}\rho^{-r}\right]\right]^{\frac{(3+\alpha)-r}{(3+\alpha)}}. \tag{2.14}$$

ii) If there exists a positive constant C such that

$$E \geq C, \quad t \geq 0, \tag{2.15}$$

then

$$S^r(0,\,t;\,r,\,\varphi) + \epsilon^2 t^{-1} L(0)^{2-(1-\varphi)r} \geq \epsilon^{(1-\varphi)r}\,\vartheta_2 t^{-\frac{(1-\varphi)r}{2}}, \quad t > 0, \tag{2.16}$$

where

$$\vartheta_2 = \left[\frac{1}{2-(1-\varphi)r}\left[(\xi C^{\sigma_2})^{\frac{(1-\varphi)r}{2-(1-\varphi)r}} + C^{\varphi r}\rho^{-(1-\varphi)r}\right]\right]^{\frac{2-(1-\varphi)r}{2}}. \tag{2.17}$$

3. Transitional upper bounds

The results in this section demonstrate how various different upper bounds can hold over the course of phase separation. Theorem 1, for (DQ) with constant mobility, which appears in [16] as Theorem 4.5, is presented here for the sake of completeness. Theorem 2, which then follows for (DQ) with degenerate mobility, is similar to [16, Theorem 4.6], but relies on Lemma 3 which is slightly stronger than [16, Lemma 4.3] which was used in proving Theorem 4.6 in [16].

For (DQ) with either constant or degenerate mobility, if $t_1 < t_2$, then $E(t_1) \geq E(t_2)$. See [4, 9]. Hence we may define

$$\begin{aligned}
t_* &= \sup\Big\{\{0\} \cup \{t \in (0,\,\infty)\,|\,E(t) > 1/2\,\}\Big\},\\
t^* &= \sup\Big\{\{0\} \cup \{t \in (0,\,\infty)\,|\,E(t) > (1-\bar{u}^2)/4\,\}\Big\}.
\end{aligned}$$

Note that $t_* \leq t^*$, since $1/2 > (1-\bar{u}^2)/4$.

For (DQ) with constant mobility, assuming the boundary contribution $\frac{\epsilon|\partial\Omega|}{|\Omega|}$ to be negligible for the sake of simplicity and noting that (2.4) is autonomous, Lemma 2, Lemma 4, and Corollary 1 imply the following:

Theorem 2. Let $|\bar{u}| < 1$ and $(r, \varphi) \in \Lambda(0)$.
I. If $t^* = 0$, then

$$S^r(0,\, t;\, r, \varphi) \geq \epsilon^{\frac{2r}{3}} \vartheta_1 t^{-\frac{r}{3}} - \epsilon^2 t^{-1} L(0)^{3-r}, \quad t > 0,$$

where $\vartheta_1 = \vartheta_1(A, B, \alpha, r, \varphi)$ with $A = 1$, $B = \frac{1}{384}(1 - \bar{u}^2)^2$, $\alpha = 0$.
II. If $0 < t^* < \infty$, let $t_1 > t^*$ be arbitrary. Then

$$S^r(0,\, t;\, r, \varphi) \geq \epsilon^{r(1-\varphi)} \vartheta_2 t^{-\frac{r(1-\varphi)}{2}} - \epsilon^2 t^{-1} L(0)^{2-(1-\varphi)r}, \quad 0 < t \leq t_1,$$

where $\vartheta_2 = \vartheta_2(A, C, \alpha, r, \varphi)$ with $A = 1$, $C = E(t_1)$, $\alpha = 0$, and

$$S^r(t^*,\, t;\, r, \varphi) \geq \epsilon^{\frac{2r}{3}} \vartheta_1 (t - t^*)^{-\frac{r}{3}} - \epsilon^2 (t - t^*)^{-1} L(t^*)^{3-r}, \quad t > t_1,$$

where $\vartheta_1 = \vartheta_1(A, B, \alpha, r, \varphi)$ with $A = 1$, $B = \frac{1}{384}(1 - \bar{u}^2)^2$, $\alpha = 0$.
III. If $t^* = \infty$, then

$$S^r(0,\, t;\, r, \varphi) \geq \epsilon^{r(1-\varphi)} \vartheta_2 t^{-\frac{r(1-\varphi)}{2}} - \epsilon^2 t^{-1} L(0)^{2-(1-\varphi)r}, \quad t > 0,$$

where $\vartheta_2 = \vartheta_2(A, C, \alpha, r, \varphi)$ with $A = 1$, $C = E(\infty)$, $\alpha = 0$.

Since $S^r(t_1,\, t_2;\, r, \varphi) \geq 0$ for $0 \leq t_1 \leq t_2$, the estimates given above provide nontrivial bounds only if the sum of the two terms on the right hand side of the various estimates is positive. Similar considerations hold for the estimates which are obtained for (DQ) with degenerate mobility given below in Theorem 3.

To obtain estimates for (DQ) with degenerate mobility, note that $\min\{1,\, 2E\} = 1$ when $E > 1/2$ and $\min\{1,\, 2E\} = 2E$ when $E < 1/2$. Hence, assuming the boundary effects to be negligible for the sake of simplicity and noting that (2.6) is autonomous, it follows from Lemma 3, Lemma 4, and Corollary 1 that

Theorem 3. Let $|\bar{u}| < 1$.
I. If $0 = t_* = t^*$, then for $(r,\, \varphi) \in \Lambda(1)$,

$$S^r(0,\, t;\, r, \varphi) \geq \epsilon^{\frac{r}{2}} \vartheta_1 t^{-\frac{r}{4}} - \epsilon^2 t^{-1} L(0)^{4-r}, \quad t > 0,$$

where $\vartheta_1 = \vartheta_1(A, B, \alpha, r, \varphi)$ with $A = 2$, $B = \frac{1}{384}(1 - \bar{u}^2)^2$, $\alpha = 1$.
II. If $0 = t_* < t^* < \infty$, let $t_1 > t^*$ be arbitrary. Then for $(r,\, \varphi) \in \Lambda(1)$,

$$S^r(0,\, t;\, r, \varphi) \geq \epsilon^{r(1-\varphi)} \vartheta_2 t^{-\frac{r(1-\varphi)}{2}} - \epsilon^2 t^{-1} L(0)^{2-(1-\varphi)r}, \quad 0 < t \leq t_1,$$

where $\vartheta_2 = \vartheta_2(A, C, \alpha, r, \varphi)$ with $A = 2$, $C = E(t_1)$, $\alpha = 1$, and

$$S^r(t^*,\, t;\, r, \varphi) \geq \epsilon^{\frac{r}{2}} \vartheta_1 (t - t^*)^{-\frac{r}{4}} - \epsilon^2 (t - t^*)^{-1} L(t^*)^{4-r}, \quad t > t_1,$$

where $\vartheta_1 = \vartheta_1(A, B, \alpha, r, \varphi)$ with $A = 2$, $B = \frac{1}{384}(1 - \bar{u}^2)^2$, $\alpha = 1$.

III. If $0 < t_* < t^* = \infty$, let $t_1 > t_*$ be arbitrary. Then

$$S^r(0,\,t;\,r,\varphi) \geq \epsilon^{r(1-\varphi)}\vartheta_2 t^{-\frac{r(1-\varphi)}{2}} - \epsilon^2 t^{-1} L(0)^{2-(1-\varphi)r}, \quad 0 < t \leq t_1,$$

for $(r,\,\varphi) \in \Lambda(0)$, and $\vartheta_2 = \vartheta_2(A, C, \alpha, r, \varphi)$ with $A = 1$, $C = E(t_1)$, $\alpha = 0$, and

$$S^r(t_*,\,t;\,r,\varphi) \geq \epsilon^{r(1-\varphi)}\vartheta_1 (t - t_*)^{-\frac{r(1-\varphi)}{2}} - \epsilon^2 (t - t_*)^{-1} L(t_*)^{2-(1-\varphi)r},$$
$$t > t_1,$$

for $(r,\,\varphi) \in \Lambda(1)$, and $\vartheta_2 = \vartheta_2(A, C, \alpha, r, \varphi)$ with $A = 2$, $C = E(\infty)$, $\alpha = 1$.

IV. If $0 < t_* < t^* < \infty$, let $t_2 > t^* > t_1 > t_*$ be arbitrary. Then

$$S^r(0,\,t;\,r,\varphi) \geq \epsilon^{r(1-\varphi)}\vartheta_2 t^{-\frac{r(1-\varphi)}{2}} - \epsilon^2 t^{-1} L(0)^{2-(1-\varphi)r}, \quad 0 < t <\leq t_1,$$

for $(r,\,\varphi) \in \Lambda(0)$, and $\vartheta_2 = \vartheta_2(A, C, \alpha, r, \varphi)$ with $A = 1$, $C = E(t_1)$, $\alpha = 0$, and

$$S^r(t_*,\,t;\,r,\varphi) \geq \epsilon^{r(1-\varphi)}\vartheta_1 (t - t_*)^{-\frac{r(1-\varphi)}{2}} - \epsilon^2 (t - t_*)^{-1} L(t_*)^{2-(1-\varphi)r},$$
$$t_1 < t \leq t_2,$$

for $(r,\,\varphi) \in \Lambda(1)$, and $\vartheta_2 = \vartheta_2(A, C, \alpha, r, \varphi)$ with $A = 2$, $C = E(t_2)$, $\alpha = 1$, and

$$S^r(t^*,\,t;\,r,\varphi) \geq \epsilon^{\frac{r}{2}}\vartheta_1 (t - t^*)^{-\frac{r}{4}} - \epsilon^2 (t - t^*)^{-1} L(t^*)^{4-r}, \quad t > t_2,$$

for $(r,\,\varphi) \in \Lambda(1)$, and $\vartheta_1 = \vartheta_1(A, B, \alpha, r, \varphi)$ with $A = 2$, $B = \frac{1}{384}(1 - \bar{u}^2)^2$, $\alpha = 1$.

V. If $t_* = \infty$, then for $(r,\,\varphi) \in \Lambda(0)$,

$$S^r(0,\,t;\,r,\varphi) \geq \epsilon^{r(1-\varphi)}\vartheta_2 t^{-\frac{r(1-\varphi)}{2}} - \epsilon^2 t^{-1} L(0)^{2-(1-\varphi)r}, \quad 0 < t, \qquad (3.1)$$

where $\vartheta_2 = \vartheta_2(A, C, \alpha, r, \varphi)$ with $A = 1$, $C = E(\infty)$, $\alpha = 0$.

4. The implications of Theorems 2 and 3

In looking at Theorems 2 and 3, it can be seen that in order to understand their implications, a few considerations need to be taken into account. First of all, these theorems contain a number of initial condition dependent quantities, namely $E(0)$, $L(0)$, $L(t_*)$, $L(t^*)$, $E(\infty)$, in addition to $\overline{u}$. Thus it would be good to have some intuition with regard to the values that these quantities can assume. Moreover, the results are stated in terms of $(r,\,\varphi) \in \Lambda(\alpha)$. Thus a panoply of upper bounds is provided, which leads naturally to the question of identifying upper bounds which are optimal in some sense. Furthermore, while the bounds given are nonnegative and hence nontrivial at sufficiently large times, it is not immediately clear as to whether there exists some waiting time for the bounds to become nontrivial. Finally, as the bounds contain various coefficients, it is good to have some handle on these values. In this section, these issues are addressed.

4.1. **Estimating** $E(0), L(0), L(t_*), L(t^*),\ E(\infty)$**.** Note that $E(0)$, $L(0)$, $L(t^*)$, $E(\infty)$ appearing in Theorem 2, as well as $E(0)$, $L(0)$, $L(t_*)$, $L(t^*)$, $E(\infty)$ appearing in Theorem 3 depend on the choice of the initial data. Clearly $\overline{u}$ also depends on the initial data, since $\overline{u} = \overline{u}_0$. With regard to $E(0)$ and $E(\infty)$, since $E(t)$ is a nonincreasing function of time, we have that

$$E_{\min} \le E(\infty) \le E(0),$$

where estimates on $E_{\min}$ were mentioned in Remark 2.1. From the definition of $E(t^*)$, it follows that if $0 < t^* < \infty$, then $E(t^*) = \frac{1}{4}(1-\overline{u}^2)$ and hence $L(t^*) \ge \frac{1}{96}(1-\overline{u}^2)$ according to (ii) in Corollary 1. It is less straightforward to estimate $L(t_*)$ and $L(\infty)$, though $L(0)$ as well as $E(0)$ can be calculated given specific initial conditions.

Some intuition can be gained by considering initial conditions which constitute *perturbations of* $\overline{u}$,

$$u_0(x) = \overline{u} + \tilde{u}_0(x), \tag{4.1}$$

where $\frac{1}{|\Omega|}\int_\Omega \tilde{u}_0\, dx = 0$. Linearization of (DQ) about $\overline{u}$ indicates the existence of a *fastest growing mode* given by

$$u_0(x) = \overline{u} + a\cos(k_{\max}x), \tag{4.2}$$

where $k_{\max} = (\sqrt{2}\epsilon)^{-1}$, which grows as $e^{\sigma_{\max}t}$ with $\sigma_{\max} \approx M(\overline{u})/(4\epsilon^2)$.

Note that for systems reflecting perturbations of $\overline{u}$, (1.3),(4.2) imply that $E(0) = \frac{(1-\overline{u}^2)}{2} + \mathcal{O}(a^2)$, where $|a| \ll 1$. Thus, in this context, $\frac{(1-\overline{u}^2)}{2} + \mathcal{O}(a^2)$ provides an upper bound on $E(\infty)$. Within this context, it is not unreasonable to suppose that $t_* = 0$ and that $0 < t^*$, which corresponds to Cases (II) and (III) of Theorems 2 and 3, and makes the other cases unnecessary to consider. Moreover, if $t_* = 0$, then $L(t_*) = L(0)$. Assuming $L(0)$ to reflect (4.2) would imply that $L(0) = \mathcal{O}(|a|)$, where a is the amplitude of the fastest growing mode. If $t_* = 0$, is seems reasonable to expect that $t^* = \mathcal{O}(1)$, so that Case (II) in Theorems 2 and 3 should be considered. If, say, $|\Omega|$ is sufficiently small and complete phase separation does not ensue, then conceivably $t^* = \infty$, making the Cases (III) relevant to consider. As $|\Omega|$ becomes small, boundary effects which have been neglected can become important, and the results outlined above need to be suitably revised.

4.2. **Estimating the predicted growth rates.** We turn now to the implications of the inequalities appearing in Lemma 4 in terms of upper bounds on the coarsening rates. The inequalities in (2.9) defining $\Lambda(\alpha)$ imply that $\Lambda(\alpha)$ is the convex region bounded by

$$\begin{aligned} I &= \{(r,\varphi) = (r,1)\,|\, 1+\alpha < r < 3+\alpha\},\\ II &= \{(r,\varphi) = (3+\alpha,\varphi)\,|\, \tfrac{1+\alpha}{3+\alpha} \le \varphi \le 1\}, \text{ and}\\ III &= \{(r,\varphi) = (r,\tfrac{1+\alpha}{r})\,|\, 1+\alpha \le r \le 3+\alpha\}, \end{aligned} \tag{4.3}$$

with $I \subset \Lambda(\alpha)$, and we set $(r^*, \varphi^*) := (3+\alpha, \frac{1+\alpha}{3+\alpha}) = II \cap III$. See Figures 1, 2 in [12].

Since the left hand side of (2.13) is nonnegative, for (2.13) to have nontrivial content, one needs that $\epsilon^2 L(0)^{(3+\alpha)-r} < \vartheta_1 \epsilon^{\frac{2r}{3+\alpha}} t^{\frac{(3+\alpha)-r}{3+\alpha}}$. Since $r < 3+\alpha$ within Λ, this implies the constraint

$$t > \epsilon^2 L(0)^{(3+\alpha)} \vartheta_1^{-\frac{3+\alpha}{(3+\alpha)-r}}. \tag{4.4}$$

Noting that $2-(1-\varphi)r > 0$ within Λ, (2.16) similarly implies the constraint

$$t > \epsilon^2 L(0)^2 \vartheta_2^{-\frac{2}{2-(1-\varphi)r}}. \tag{4.5}$$

Analogous considerations in the context of the inequalities appearing in Theorems 2 and 3, imply the constraints

$$t - t^* > \epsilon^2 L(t^*)^{(3+\alpha)} \vartheta_1^{-\frac{3+\alpha}{(3+\alpha)-r}}, \tag{4.6}$$

and

$$t - t_* > \epsilon^2 L(t_*)^2 \vartheta_2^{-\frac{2}{2-(1-\varphi)r}}. \tag{4.7}$$

From (4.4)-(4.7), it follows that an understanding of the values assumed by $\vartheta_1^{-\frac{3+\alpha}{(3+\alpha)-r}}$, $\vartheta_2^{-\frac{2}{2-(1-\varphi)r}}$ is required, in order to ascertain whether or not there is a *waiting time* for (2.13),(2.16) to become positive and hence to assume nontrivial content.

Before addressing this question, in the result that follows, we shall assume (4.4)-(4.7) to hold as necessary and estimate the implied growth rates.

Claim 1. Let α, ϵ, $L(0)$, $L(t_*)$, $L(t^*)$ be given, and assume ϑ_1, ϑ_2 to be prescribed and bounded.

If $t > t^*$, then (2.13), (4.4), (4.6) imply upper bounds that are proportional to $[\frac{t}{\epsilon^2}]^{\frac{1}{3+\alpha}}$, if $t^* = 0$, and to $[\frac{t-t^*}{\epsilon^2}]^{\frac{1}{3+\alpha}}$, if $t^* > 0$.

Similarly, if $t_* < t \le \min\{t_* + \epsilon^2, t^*\}$, then (2.16), (4.5), (4.7) imply upper bounds that are proportional to $[\frac{t}{\epsilon^2}]^{\frac{1}{3+\alpha}}$, if $t_* = 0$, and to $[\frac{t-t_*}{\epsilon^2}]^{\frac{1}{3+\alpha}}$, if $t_* > 0$.

Proof. If $t > t^* = 0$ and (4.4) holds, then (2.13) implies that

$$S^{-1}(0, t;\, r, \varphi) \le \vartheta_1^{-\frac{1}{r}} \Big(\frac{t}{\epsilon^2}\Big)^{\frac{1}{(3+\alpha)}} \Big[1 - \vartheta_1{}^{-1} \Big[\frac{L^{3+\alpha}(0)\epsilon^2}{t}\Big]^{\frac{(3+\alpha)-r}{3+\alpha}}\Big]^{-\frac{1}{r}}, \tag{4.8}$$

for $(r, \varphi) \in \Lambda(\alpha)$, and an upper bound with growth proportional to $[\frac{t}{\epsilon^2}]^{\frac{1}{3+\alpha}}$ is implied. If $t > t^* > 0$ and (4.6) holds, then exploiting the autonomous nature of Lemmas 2 and 3, a similar inequality based on $S^{-1}(t^*, t;\, r, \varphi)$ implies an upper bound proportional to $[\frac{t-t^*}{\epsilon^2}]^{\frac{1}{3+\alpha}}$.

If $0 = t_* < t \le \min\{\epsilon^2, t^*\}$ and (4.5) holds, then (2.16) implies that

$$S^{-1}(0, t;\, r, \varphi) \le \vartheta_2^{-\frac{1}{r}} \Big(\frac{t}{\epsilon^2}\Big)^{\frac{(1-\varphi)}{2}} \Big[1 - \vartheta_2{}^{-1} \Big[\frac{L^2(0)\epsilon^2}{t}\Big]^{\frac{2-(1-\varphi)r}{2}}\Big]^{-\frac{1}{r}}, \tag{4.9}$$

for $(r,\varphi) \in \Lambda(\alpha)$, and an upper bound with growth proportional to $[\frac{t}{\epsilon^2}]^{\frac{(1-\varphi)}{2}}$ is implied. If $0 < t_* < t \le \min\{t_* + \epsilon^2, t^*\}$ and (4.7) holds, a similar inequality based on $S^{-1}(t_*, t;\, r,\, \varphi)$ implies an upper bound proportional to $[\frac{t-t_*}{\epsilon^2}]^{\frac{1}{3+\alpha}}$.

Examining $\Lambda(\alpha)$, we see that

$$0 < \frac{(1-\varphi)}{2} < \frac{(1-\varphi^*)}{2} = \frac{1}{3+\alpha}, \tag{4.10}$$

with φ assuming the value φ^* in the upper left hand corner of $\Lambda(\alpha)$, where $(r,\varphi) = (r^*, \varphi^*)$. Since $0 < t - t_* < \epsilon^2$, within the present framework, the best upper bound is achieved by taking the exponent as large as possible, yielding $[\frac{t}{\epsilon^2}]^{\frac{1}{3+\alpha}}$ (or $[\frac{t-t_*}{\epsilon^2}]^{\frac{1}{3+\alpha}}$) as the upper bound. $\square$

Remark 4.1. *The procedure used here is to fix ϑ_1, ϑ_2, and to examine the implied growth rates, then to allow $(r,\varphi) \in \Lambda(\alpha)$ and hence ϑ_1, ϑ_2 to vary in minimizing the waiting time and in examining the rate coefficients. With regard to (2.13), (4.4), (4.6), the choice of $(r,\varphi) \in \Lambda(\alpha)$ does not effect the growth rate, so this approach is benign. With regard to (2.16), (4.5), (4.7), the choice of $(r,\varphi) \in \Lambda(\alpha)$ minimizing waiting time is also the best choice in terms of growth rates according to (4.10), thus justifying our approach.*

Remark 4.2. *Note that if $\epsilon^2 < t^*$ or if $t_* + \epsilon^2 < t^*$, then there are temporal gaps in the upper bounds provided by Claim 1. If, for example, $\epsilon^2 < t < t^*$, seemingly we should like to take the exponent as small as possible, which would imply taking $\varphi = 1$, yielding the (trivial) bound t^0. This conclusion is also rather redundant as $t < t^*$ implies that $E \ge C$, or equivalently $E^{-1} \le C^{-1}$, which is approximately the estimate attained. Thus, the bound (2.16) appears to most obviously helpful when $t < \epsilon^2$. Details regarding the case $t^* > \epsilon^2$ will be amplified shortly.*

4.3. **Estimating waiting times.** The following claim demonstrates that the waiting times for (4.4)–(4.7) to hold can be made arbitrarily small.

Claim 2. Let α, ϵ, $L(0)$ (and $L(t_*)$, $L(t^*)$, if applicable) be fixed, and let the values of A, B, and C appearing in the statement of Lemma 4 refer to the values in Theorems 2 and 3. Then by choosing the parameter $\rho > 1$ in (2.11) sufficiently close to 1, and by taking $(r,\varphi) \in \Lambda(\alpha)$ sufficiently near to II or to (r^*, φ^*), respectively, the coefficients $\vartheta_1^{-\frac{3+\alpha}{(3+\alpha)-r}}$ and $\vartheta_2^{-\frac{2}{2-(1-\varphi)r}}$ appearing in (4.4)-(4.7) can be taken to be arbitrarily small.

Proof. From (2.14) and (2.17), we have that

$$\vartheta_1^{\frac{3+\alpha}{(3+\alpha)-r}} = \frac{3+\alpha}{2(3+\alpha)-2r}\Big[(\xi B^{\sigma_1})^{\frac{r}{(3+\alpha)-r}} + B^{\varphi r}\rho^{-r}\Big], \tag{4.11}$$

and that

$$\vartheta_2^{\frac{2}{2-(1-\varphi)r}} = \frac{1}{2-(1-\varphi)r}\Big[(\xi C^{\sigma_2})^{\frac{(1-\varphi)r}{2-(1-\varphi)r}} + C^{\varphi r}\rho^{-(1-\varphi)r}\Big]. \tag{4.12}$$

From the definitions in (2.11) and the definition of $\Lambda(\alpha)$, it follows that

(1) Along I, $\gamma_1 = 1+\alpha-r$, $\gamma_2 = 1$, $\gamma_1/(\gamma_2)^2 = \gamma_1$, $\sigma_1 = 2$, $\sigma_2 = \infty$,
(2) Along II, $\gamma_1 = (1+\alpha)-(3+\alpha)\varphi$, $\gamma_2 = -\gamma_1/2$, $\gamma_1/(\gamma_2)^2 = 4/\gamma_1$, $\sigma_1 = -(1+\alpha)+\varphi(3+\alpha)$, $\sigma_2 = -(1+\alpha)+\frac{2\varphi}{1-\varphi}$,
(3) Along III, $\gamma_1 = 0$, $\gamma_2 = (3+\alpha-r)/2$, $\gamma_1/(\gamma_2)^2 = 0$, $\sigma_1 = \frac{(1+\alpha)(3+\alpha-r)}{r}$, $\sigma_2 = \frac{(1+\alpha)(3+\alpha-r)}{r-1-\alpha}$.

Now

$$B^{\sigma_1}\xi = \frac{-B^{\sigma_1}\gamma_1}{A\gamma_2^2}(1-\rho^{-\gamma_2})^2, \quad C^{\sigma_2}\xi = \frac{-C^{\sigma_2}\gamma_1}{A\gamma_2^2}(1-\rho^{-\gamma_2})^2.$$

In Theorems 2 and 3, $A = 1$, $2^{1/2}$, or 2, $B = \frac{1}{384}(1-\overline{u}^2)^2$, and thus $0 < A^{-1} \leq 1$ and $0 < B < 1$. In Theorems 2 and 3, $C = E(t_1)$, $E(t_2)$, or $E(\infty)$, where $t_1 > t_*$ or t^* and $t_2 > t^*$. Therefore $E(t_1), E(t_2) \leq \frac{1}{2}$. If Theorems 2, 3 hold with $C = E(\infty) > \frac{1}{2}$, they also hold with C replaced by $\frac{1}{2}$, since $C = \frac{1}{2}$ also constitutes as a lower bound for $E(t)$ which is nonincreasing. Thus, without loss of generality, we may assume that $0 < C < 1$. Noting that $\gamma_1 < 0$ and $\gamma_2 > 0$ throughout $\Lambda(\alpha)$, it follows that for any $(\varphi, r) \in \Lambda(\alpha)$, by choosing $\rho > 1$ sufficiently close to 1, we may guarantee that

$$0 < B^{\sigma_1}\xi < 1, \quad 0 < C^{\sigma_2}\xi < 1. \tag{4.13}$$

The factor $\frac{3+\alpha}{2(3+\alpha)-2r}$ in (4.11) is bounded within $\Lambda(\alpha)$, and becomes unbounded as II is approached. Similarly the factor $\frac{1}{2-(1-\varphi)r}$ in (4.12) is bounded within $\Lambda(\alpha)$, and becomes unbounded as (r^*, φ^*) is approached. Since $0 < B, C < 1$ and $\rho > 1$, the terms $B^{\varphi r}\rho^{-r}$ and $C^{\varphi r}\rho^{-(1-\varphi)r}$ in (4.11), (4.12) are bounded throughout $\Lambda(\alpha)$

Hence $\vartheta_1^{\frac{3+\alpha}{(3+\alpha)-r}}$ is positive and bounded within $\Lambda(\alpha)$ and becomes unbounded as II is approached. Similarly, $\vartheta_2^{\frac{2}{2-(1-\varphi)-r}}$ is positive and bounded within $\Lambda(\alpha)$ and becomes unbounded as (r^*, φ^*) is approached. □

Remark 4.3. *For $\epsilon^2 < t < t^*$ when (4.5) holds, an upper bound can be ascertained via*

$$\min_{(r,\varphi)\in\Lambda(\alpha)} S^{-1}(0,\, t;\, r, \varphi) \leq \min_{(r,\varphi)\in\Lambda(\alpha)} \left\{ \vartheta_2^{-1/r} \left(\tfrac{t}{\epsilon^2}\right)^{\frac{(1-\varphi)}{2}} \left[1 - \vartheta_2^{-1} \left[\tfrac{\epsilon^2 L^2(0)}{t}\right]^{\frac{2-(1-\varphi)r}{2}} \right]^{-\frac{1}{r}} \right\}. \tag{4.14}$$

The precise growth rate implied by (4.14) is not so clear and it is a bit of a question of how it is to be defined, though quite obviously it lies somewhere between 0 *and* $\frac{1}{3+\alpha}$. *An analogous statement may be made in regard to $S^{-1}(t_*,\, t; r,\, \varphi)$ when (4.7) holds and $t_* < t < t^*$.*

4.4. **Evaluation of $\vartheta_1^{-\frac{1}{r}}, \vartheta_2^{-\frac{1}{r}}$.** For given $(r,\, \phi) \in \Lambda(\alpha)$, we see that, as in (4.8), (4.9), the upper bounds which are obtained contain the factors $\vartheta_1^{-\frac{1}{r}}$, $\vartheta_2^{-\frac{1}{r}}$, where ϑ_1 and ϑ_2 are prescribed in (2.14), (2.17), respectively.

Thus it is of some interest to evaluate these factors, particularly in the neighborhood of II and $(r^*,\ \varphi^*)$, where the upper bounds have been being evaluated.

Claim 3. Let α, ϵ, $L(0)$ (and $L(t_*)$, $L(t^*)$, if applicable) be fixed, let the values of A, B, and C appearing in the statement of Lemma 4 correspond to their values from Theorems 2 and 3, and let $\rho > 1$ from (2.11) be chosen sufficiently close to 1. Then $\vartheta_1^{-\frac{1}{r}}$ and $\vartheta_2^{-\frac{1}{r}}$ are bounded throughout $\Lambda(\alpha)$, and $\vartheta_1^{-\frac{1}{r}} \to 1$ as $(r,\ \varphi) \to$ II, and $\vartheta_2^{-\frac{1}{r}} \to 1$ as $(r,\ \varphi) \to (r^*,\ \varphi^*)$.

Proof. Looking at (2.14) and (2.17), we see that we may write $\vartheta_1^{\frac{1}{r}}$, $\vartheta_2^{\frac{1}{r}}$ as

$$\vartheta_1^{\frac{1}{r}} = A_1 B_1, \quad \vartheta_2^{\frac{1}{r}} = A_2 B_2,$$

where

$$A_1 = \left[\frac{3+\alpha}{2(3+\alpha-r)}\right]^{\frac{(3+\alpha)-r}{(3+\alpha)r}}, \quad A_2 = \left[\frac{1}{2-(1-\varphi)r}\right]^{\frac{2-(1-\varphi)r}{2r}},$$

and

$$B_1 = \left[(\xi B^{\sigma_1})^{\frac{r}{(3+\alpha)-r}} + B^{\varphi r}\rho^{-r}\right]^{\frac{(3+\alpha)-r}{(3+\alpha)r}},$$

$$B_2 = \left[(\xi C^{\sigma_2})^{\frac{(1-\varphi)r}{2-(1-\varphi)r}} + C^{\varphi r}\rho^{-(1-\varphi)r}\right]^{\frac{2-(1-\varphi)r}{2r}}.$$

For A_1 and A_2, we note that

(1) Along I, A_1 is bounded and $\lim_{(r,\varphi)\to(3+\alpha,1)} A_1 = 1$, $A_2 = 2^{-\frac{1}{3+\alpha}}$,
(2) Along II, $A_1 = 1$ (by taking limits from within $\Lambda(\alpha)$), A_2 is bounded with $\lim_{(r,\varphi)\to(r^*,\ \varphi^*)} A_2 = 1$,
(3) Along III, A_1 is bounded and $\lim_{(r,\varphi)\to(r^*,\ \varphi^*)} A_1 = 1$, A_2 is bounded with $\lim_{(r,\varphi)\to(r^*,\ \varphi^*)} A_2 = 1$.

Thus, looking at the definitions of A_1 and A_2, we see that A_1, A_2 are bounded from below throughout $\Lambda(\alpha)$, and $A_1 \to 1$ along II, and $A_2 \to 1$ as $(r,\varphi) \to (r^*,\ \varphi^*)$.

Next we consider B_1 and B_2. By choosing ρ so that (4.13) is satisfied, B_1 and B_2 can be seen to be bounded from below throughout $\Lambda(\alpha)$, and $B_1 \to 1$ along II, and $B_2 \to 1$ as $(r,\varphi) \to (r^*,\ \varphi^*)$.

Thus by choosing ρ so that (4.13) is satisfied, $\vartheta_1^{-\frac{1}{r}}$ $\vartheta_2^{-\frac{1}{r}}$ are bounded from below throughout $\Lambda(\alpha)$, $\vartheta_1^{-\frac{1}{r}} \to 1$ along II, and $\vartheta_2^{-\frac{1}{r}} \to 1$ as $(r,\varphi) \to (r^*,\ \varphi^*)$. □

Remark 4.4. *Throughout the discussion in this section, the upper bounds obtained have been based on choosing ρ so that $0 < B^{\sigma_1}\xi < 1$, and $0 < C^{\sigma_2}\xi < 1$. We note that it is possible to obtain upper bounds based on other choices of ρ. For example, the parameter ρ can also always be chosen so that the two terms appearing in B_1 or in B_2 are equal.*

5. In summary

In this section we outline the upper bounds which may be concluded by incorporating the results from Claims 1–3 into Theorems 2 and 3.

Recall that $t_* = \sup\{\{0\} \cup \{t > 0 \,|\, E(t) > \frac{1}{2}\}\}$ and $t^* = \sup\{\{0\} \cup \{t > 0 \,|\, E(t) > \frac{(1-\bar{u}^2)}{4}\}\}$. In the two theorems which follow, $l(t)$ denotes our measure of the length scale of the system which is based on $S^{-1}(\bar{t}, t; r, \varphi)$, where for notational simplicity, its dependence on some specific choice of the parameters (r, φ) has not been indicated.

Theorem 4. (DQ) with constant mobility. Let $|\bar{u}| < 1$.

I. If $t^* = 0$, then

$$l(t) \leq \left[\frac{t}{\epsilon^2}\right]^{\frac{1}{3}}, \quad t > 0.$$

II. If $0 < t^* < \infty$, let $t_1 > t^*$ be arbitrary. Then

$$l(t) \leq \left[\frac{t}{\epsilon^2}\right]^{\frac{1}{3}}, \quad 0 < t \leq \min\{t_1, \epsilon^2\},$$

and

$$l(t) \leq \left[\frac{t - t^*}{\epsilon^2}\right]^{\frac{1}{3}}, \quad t > t_1.$$

III. If $t^* = \infty$, then

$$l(t) \leq \left[\frac{t}{\epsilon^2}\right]^{\frac{1}{3}}, \quad 0 < t \leq \epsilon^2.$$

Theorem 5. (DQ) with degenerate mobility. Let $|\bar{u}| < 1$.

I. If $0 = t_* = t^*$, then

$$l(t) \leq \left[\frac{t}{\epsilon^2}\right]^{\frac{1}{4}}, \quad t > 0.$$

II. If $0 = t_* < t^* < \infty$, let $t_1 > t^*$ be arbitrary. Then

$$l(t) \leq \left[\frac{t}{\epsilon^2}\right]^{\frac{1}{4}}, \quad 0 < t \leq \min\{t_1, \epsilon^2\}.$$

and

$$l(t) \leq \left[\frac{t - t^*}{\epsilon^2}\right]^{\frac{1}{4}}, \quad t > t_1.$$

III. If $0 < t_* < t^* = \infty$, let $t_1 > t_*$ be arbitrary. Then

$$l(t) \leq \left[\frac{t}{\epsilon^2}\right]^{\frac{1}{3}}, \quad 0 < t \leq \min\{t_1, \epsilon^2\},$$

and

$$l(t) \leq \left[\frac{t - t_*}{\epsilon^2}\right]^{\frac{1}{4}}, \quad t_1 < t \leq t_* + \epsilon^2.$$

IV. If $0 < t_* < t^* < \infty$, let $t_2 > t^* > t_1 > t_*$ be arbitrary. Then

$$l(t) \leq \left[\frac{t}{\epsilon^2}\right]^{\frac{1}{3}}, \quad 0 < t < \min\{t_1, \epsilon^2\},$$

and

$$l(t) \leq \left[\frac{t - t_*}{\epsilon^2}\right]^{\frac{1}{4}}, \quad t_1 < t \leq \min\{t_2, t_* + \epsilon^2\},$$

and

$$l(t) \leq \left[\frac{t - t^*}{\epsilon^2}\right]^{\frac{1}{4}}, \quad t > t_2.$$

V. If $t_* = \infty$,

$$l(t) \leq \left[\frac{t}{\epsilon^2}\right]^{\frac{1}{3}}, \quad 0 < t \leq \epsilon^2.$$

Note that if $E(0) > \frac{(1-\overline{u}^2)}{4}$, then in accordance with Remark 4.3, certain temporal gaps may appear in the upper bounds. Cases (II) in Theorems 4 and 5 appear to most closely reflect the dynamics of coarsening for initial conditions which correspond to a perturbation of a spatially uniform state, in accordance with the discussion in § 4.1. Case (III) in Theorem 4 and Case(V) Theorem 5 could reflect systems, for example small systems, in which separation does not occur. Case (IV) in Theorem 5 could correspond, for example, to some less regular initial conditions.

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