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Induction from Elementary Abelian Subgroups

E. Aljadeff* and Y. Ginosar

Department of Mathematics, Technion-Israel Institute of Technology, 32000 Haifa, Israel

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INTRODUCTION

Let R be a ring with identity element 1 and G a finite group. RG denotes the group ring. Chouinard [C] showed that an RG-module M is weakly projective (projective) iff it is RH weakly projective (projective) for every elementary abelian subgroup H of G (see the definition of weak projective in Section 1). His proof is based on Serre's theorem on products of Bockstein operators. Chouinard's theorem can be generalized (using his original result) to arbitrary crossed products $R * G$. Recall that a crossed product $R * G$ is an associative ring which contains R and has an R basis $\{u_\sigma\}_{\sigma \in G}$. The multiplicative structure is given by the rules:

(1) ("Twisting") $u_\sigma u_\tau = \alpha(\sigma, \tau) u_{\sigma\tau}$, where $\alpha: G \times G \to U(R)$ (units of R).

(2) ("Action") $u_\sigma x = t_\sigma(x) u_\sigma$ where $t_\sigma \in \mathrm{Aut}(R)$.

(See also [P, Chap. 1, Sect. 1].)

One of the main points of this generalization is that results on group actions on rings R may be obtained from the structure of R as a module over the skew group ring $R_t G$.

For a given action of G on R (i.e., a homomorphism $t: G \to \mathrm{Aut}(R)$) we construct the trace map for $H \le G$

$$\mathrm{tr}_H : R \to R^H \qquad (H\text{-invariants})$$
$$r \to \sum_{\sigma \in H} \sigma(r).$$

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It is easy to show (Section 2) that if tr_G is surjective onto R^G then tr_H is surjective onto R^H for every subgroup H of G. In this paper we prove

THEOREM 1. *Let R and G be as above. If tr_H is surjective onto R^H for every elementary abelian subgroup H of G, then tr_G is surjective onto R^G.*

Let us point out that in case that R is a commutative ring, tr_G is surjective onto R^G iff tr_P is surjective onto R^P for every prime order subgroup P of G (see [A, Corollary 02]). This stronger result is known to fail for arbitrary non-commutative rings. Theorem 1 is a consequence of our main result.

THEOREM 2. *Let $R * G$ be a crossed product of G over R and let M be an $R * G$-module. Then M is weakly projective (projective) iff M is $R * H$ weakly projective (projective) for every elementary abelian subgroup H of G.*

To see that it implies Theorem 1 above, let $R * G = R_t G$ be the skew group algebra of G over R defined by the given action $t: G \to \mathrm{Aut}(R)$ (trivial twisting) and note that R, as a principal $R_t G$ module, is projective iff the trace map tr_G is surjective onto R^G (see [ARS, Proposition 1.7] or Section 2 below). From Theorem 2 we conclude also

THEOREM 3. *Let $R * G$ be a crossed product algebra and M a left $R * G$ module. Then*

$$\mathrm{proj.dim}_{R * G} M = \sup_{\substack{H \leq G \\ \text{elementary} \\ \text{abelian}}} \{\mathrm{proj.dim}_{R * H} M)$$

and so

$$\mathrm{gl.dim}\, R * G = \sup_{\substack{H \leq G \\ \text{elementary} \\ \text{abelian}}} \{\mathrm{gl.dim}\, R * H\}.$$

This generalizes a theorem of Yi [Y, Corollary 5.4].

1. WEAK PROJECTIVE MODULES OVER CROSSED PRODUCTS

DEFINITION. A module M over $R * G$ is said to be weakly projective if for every diagram of $R * G$ modules (and $R * G$ morphisms)

$$\begin{array}{ccc} & M & \\ & \downarrow \mu & \\ A & \xrightarrow[\varphi]{} B & \to 0 \end{array}$$

in which φ may be lifted to an R-map, it may be lifted also to an $R * G$ map.

For a left $R * G$-module M, a diagonal action of G on $\mathrm{Hom}_R(M, M)$ is defined ($\sigma(f) = u_\sigma f u_\sigma^{-1}$, $u_\sigma \in R * G$) and hence we have the (trace) map

$$\mathrm{tr}_G\colon \mathrm{Hom}_R(M, M) \to \mathrm{Hom}_{R*G}(M, M)$$
$$f \mapsto \sum_{\sigma \in G} u_\sigma f u_\sigma^{-1}.$$

LEMMA 4. *Let M be an $R * G$ module. Then M is $R * G$ weakly projective iff there exists an $f \in \mathrm{Hom}_R(M, M)$ with $\mathrm{tr}_G(f) = 1_M$.*

Proof. Construct the (left) $R * G$ module $\mathbb{Z}G \otimes_{\mathbb{Z}} M$ with the diagonal action

$$x u_\tau(\sigma \otimes m) = \tau\sigma \otimes x u_\tau m, \qquad m \in M, \tau, \sigma \in G, x \in R.$$

Consider the surjective $R * G$ map

$$\eta\colon \mathbb{Z}G \otimes_{\mathbb{Z}} M \to M$$
$$\sigma \otimes m \mapsto m.$$

Clearly, η splits as an R map (say, by $m \mapsto e \otimes m$, e the identity element in G). Suppose M is $R * G$ weakly projective, so η splits also as an $R * G$ map, i.e., there is an $R * G$ map $\psi\colon M \to \mathbb{Z}G \otimes_{\mathbb{Z}} M$ such that $\eta \circ \psi = \mathrm{id}_M$. Since G is a $\mathbb{Z}$-basis, $\psi(m) = \sum_{\sigma \in G} \sigma \otimes f_\sigma(m)$ where $f_\sigma\colon M \to M$, $\sigma \in G$ are uniquely defined. Furthermore, for every $x \in R$

$$\sum_{\sigma \in G} \sigma \otimes f_\sigma(xm) = \sum_{\sigma \in G} \sigma \otimes x f_\sigma(m) \qquad (\psi \textit{ is } R\text{-linear})$$

and so $f_\sigma \in \mathrm{Hom}_R(M, M)$ for every $\sigma \in G$. Since ψ is $R * G$ linear

$$u_\tau \psi(m) = \psi(u_\tau m), \qquad \tau \in G$$
$$\sum_{\sigma \in G} \tau\sigma \otimes u_\tau f_\sigma(m) = \sum_{\sigma \in G} \sigma \otimes f_\sigma(u_\tau m).$$

Comparing base elements, we obtain

$$\tau e \otimes u_\tau f_e(m) = \tau \otimes f_\tau(u_\tau m)$$

or

$$u_\tau f_e(m) = f_\tau(u_\tau m),$$

and denoting $u_\tau m = m'$ we get

$$u_\tau f_e u_\tau^{-1}(m') = f_\tau(m')$$

for every $\tau \in G$ and $m' \in M$. Since ψ splits η,

$$m' = \sum_{\tau \in G} f_\tau(m') = \sum_{\tau \in G} u_\tau f_e\left(u_\tau^{-1} m'\right).$$

Hence $f_e \in \operatorname{Hom}_R(M, M)$ satisfies $\operatorname{tr}_G(f_e) = 1_M$.

Conversely, let $f \in \operatorname{Hom}_R(M, M)$ with $\operatorname{tr}_G(f) = 1_M$ and let

$$\begin{array}{ccc} & M & \\ & \downarrow{\scriptstyle\mu} & \\ A \underset{\varepsilon}{\rightarrow} & B & \rightarrow 0 \end{array}$$

be a diagram of $R * G$ modules and $R * G$ maps. Suppose that μ can be lifted to an R-map $s: M \to A$, i.e., $\varepsilon \circ s = \mu$. Define the $R * G$ map

$$\begin{aligned} \bar{s}&: M \to A \\ m &\mapsto \sum_{\tau \in G} u_\tau s \circ f u_\tau^{-1}(m). \end{aligned}$$

Clearly, $\bar{s}$ is an $R * G$ map. Furthermore, $\varepsilon \circ \bar{s}(m) = \varepsilon \sum_{\tau \in G} u_\tau s \circ f u_\tau^{-1}(m) = \mu(m)$, so $\bar{s}$ is a lifting of μ, and M is weakly projective. ∎

We apply this to obtain Chouinard's theorem for crossed products.

Proof of Theorem 2. By Lemma 4, for every elementary abelian subgroup H of G there exists $f^{(H)} \in \operatorname{Hom}_R(M, M)$ with

$$\operatorname{tr}_H\left(f^{(H)}\right) = \sum_{\sigma \in H} u_\sigma f^{(H)} u_\sigma^{-1} = 1_M.$$

Every $f^{(H)}$ induces a map

$$\begin{aligned} f_*^{(H)}: \operatorname{Hom}_R(M, M) &\to \operatorname{Hom}_R(M, M) \\ g &\mapsto f^{(H)} g. \end{aligned}$$

We claim that $\operatorname{tr}_H(f_*^{(H)}) = 1_{\operatorname{Hom}_R(M, M)}$. Indeed, acting on $g \in \operatorname{Hom}_R(M, M)$

$$\operatorname{tr}_H(f_*^{(H)})(g) = \sum_{\sigma \in H} \sigma\left(f_*^{(H)} \sigma^{-1}(g)\right)$$

and evaluating at $m \in M$

$$\begin{aligned} \operatorname{tr}_H\big(f_*^{(H)}\big)(g)(m) &= \sum_{\sigma \in H} \sigma\big(f_*^{(H)}\sigma^{-1}(g)\big)(m) \\ &= \sum_{\sigma \in H} u_\sigma\big(f_*^{(H)}\sigma^{-1}(g)\big)\big(u_\sigma^{-1}m\big) \\ &= \sum_{\sigma \in H} u_\sigma\big(f^{(H)}u_\sigma^{-1}g\big(u_\sigma u_\sigma^{-1}m\big)\big) = g(m). \end{aligned}$$

Thus, $\operatorname{Hom}_R(M, M)$ is $\mathbb{Z}H$ weakly projective for every elementary abelian subgroup H of G. By Chouinard's theorem [C, Theorem 1], it follows that $\operatorname{Hom}_R(M, M)$ is $\mathbb{Z}G$ weakly projective and hence $\hat{H}^n(G, \operatorname{Hom}_R(M, M)) = 0$ (the n^{th} Tate cohomology). The vanishing of $\hat{H}^0(G, \operatorname{Hom}_R(M, M))$ says that the map $\operatorname{tr}_G\colon \operatorname{Hom}_R(M, M) \to \operatorname{Hom}_R(M, M)^G$ is surjective. In particular, there exists $f \in \operatorname{Hom}_R(M, M)$ with $\operatorname{tr}_G(f) = 1_M$. By Lemma 4 we conclude that M is $R * G$ weakly projective. Finally, the statement about projectivity (rather than weak projectivity) follows from the fact that projectivity of $R * G$ $(R * H)$ modules is the same as weak projectivity together with projectivity over R.

COROLLARY 5 (compare with [P, 18.10]). *$R * G$ is semisimple artinian iff $R * H$ is semisimple artinian for every elementary abelian subgroup H of G.*

Proof. This is clear.

Proof of Theorem 3. A projective resolution of an $R * G$ module M remains projective over $R * H$, and recall that $\operatorname{proj.dim}_{R * H} M \leq n$ is equivalent to $R * H$ projectivity of the n^{th} *syzygy* module K_n of any resolution of M. The statement on the global dimension follows at once.

REMARK. The results of this paper hold for G-strongly graded rings (generalizing crossed products $R * G$) (see [P, Chap. 1, Sect. 2]). The proofs are basically the same. Let us write the diagonal action in this case. Let S be a G-strongly graded ring, that is, $S = \sum_{g \in G} S_g$ (the sum being direct) such that for every $\sigma, \tau \in G$ $S_\sigma S_\tau = S_{\sigma\tau}$ (rather than just $S_\sigma S_\tau \subset S_{\sigma\tau}$ as required for G-graded rings). Let M, N be left S modules. Then the action of G on $\operatorname{Hom}_{S_e}(M, N)$ (e the identity element in G) is given by

$$(\sigma h)(m) = \sum_{i=1}^{n_\sigma} x_1 h(y_i m),$$

where $x_i \in S_\sigma$, $y_i \in S_{\sigma^{-1}}$, and $\sum_{i=1}^{n_\sigma} x_i y_i = 1$. One verifies that this action of G is well defined.

2. INDUCTION ON TRACE MAPS

We consider now a special case of crossed products—the skew group ring R_tG. R is an R_tG module (in particular a G-module) so the trace map

$$\operatorname{tr}_G\colon R \to R^G$$
$$x \mapsto \sum_{\sigma \in G} \sigma(x)$$

is defined.

It is obvious that $\operatorname{tr}_G\colon R \to R^G$ is surjective if and only if 1_R is in the image of this map, and so surjectivity of $\operatorname{tr}_G\colon R \to R^G$ implies surjectivity of $\operatorname{tr}_H\colon R \to R^H$, H any subgroup of G, for if $g_1, \dots, g_r$ are representatives of (left) cosets of $G{:}H$ and $x \in R$ is such that $\operatorname{tr}_G(x) = 1_R$ then the element $y = \sum_{i=1}^r g_i(x)$ satisfies $\operatorname{tr}_H y = 1_R$. Since $\operatorname{Hom}_R(R, R) \simeq R$ as G-modules, Lemma 4 implies that R is an R_tG projective module if and only if there exists an element $x \in R$ with $\operatorname{tr}_G(x) = 1_R$. Together with Theorem 2 we obtain again Theorem 1.

It might be interesting to obtain a formula for x_G with $\operatorname{tr}_G(x_G) = 1_R$ depending on all y_H with $\operatorname{tr}_H(y_H) = 1_R$ for every $H \leq G$ elementary abelian. The existence of such a formula was proved by S. Shelah.

PROPOSITION 6 (S. Shelah). *An expression for x_G, of the required form, does exist. Moreover, fixing the finite group G, the formula does not depend on the ring R. Such a formula for x_G has the form*

$$\sum_{i=1}^{n} a_1 \sigma_{i_1}(y_{H_{s_1}}) \sigma_{i_2}(y_{H_{s_2}}) \dots \sigma_{i_{j(i)}}(y_{H_{s_{j(i)}}}),$$

where $s = s(i)$, $a_i \in \mathbb{Z}$, $\sigma_{i_t} \in G$, $H_l \leq G$ elementary abelian, $y_{H_l} \in R$ with $\operatorname{tr}_{H_l}(y_{H_l}) = 1$.

Proof. Let K_G be the variety (i.e., defined by a set of equations)

$$\left\{ (R, \varphi_\sigma, y_H)_{\substack{\sigma \in G: \\ H \leq G \text{ elementary abelian}}} \quad \varphi\colon G \to \operatorname{Aut}(R),\, y_H \in R,\, \operatorname{tr}_H(y_H) = 1 \right\}$$

and let

$$F_G = \left(R^F, \varphi_G^F, y_H\right)_{\substack{\sigma \in G \\ H \leq G \text{ elementary abelian}}}$$

be the free algebra in K_G (i.e., it is generated by $\{0, 1, \dots, y_H, \dots\}_{H \leq G}$ and by the operations $\{+, -, \times, \varphi_\sigma\}_{\sigma \in G}$). By Theorem 1, there exists $x_G \in R^F$ with

$$\sum_{\sigma \in G} \varphi_\sigma^F(x_G) = 1. \qquad (*)$$

By algebraic manipulations x_G is equal to some term as described above. Finally, by the freeness, $(*)$ holds for every $(R, \varphi, y_H) \in K_G$ as required.

If R is commutative, an explicit formula for x_G is given in [A, Theorem 2.1] depending on all y_P, $P \leq G$ (cyclic) of prime order. Here we show how to compute x_G in the special case that R is an algebra over $\mathbb{F}_2$ and G is an *abelian* 2-group (or has such 2-Sylow subgroup).

First, let G be a cyclic group of order 2^l, $l > 1$ generated by σ, and $H = \{1, \sigma^{2^{l-1}}\}$ be its unique elementary abelian subgroup.

CLAIM. *If* $x \in R$ *satisfies* $\operatorname{tr}_H(x) = x + \sigma(x)^{2^{l-1}} = 1_R$, *then* $y = x\sigma(x)$ *satisfies* $\operatorname{tr}_G(y) = 1$.

Proof.

$$\begin{aligned}
\operatorname{tr}_G(x\sigma(x)) &= \sum_{i=0}^{2^l-1} \sigma^i(x\sigma(x)) \\
&= \sum_{i=0}^{2^{l-1}-1} \sigma^i\left[x\sigma(x) + \sigma^{2^{l-1}}(x\sigma(x))\right] \\
&= \sum_{i=0}^{2^{l-1}-1} \sigma^i\left[x\sigma(x) + \sigma^{2^{l-1}}(x)\sigma^{2^{l-1}+1}(x)\right] \\
&= \sum_{i=0}^{2^{l-1}-1} \sigma^i[x\sigma(x) + (1-x)(1-\sigma(x))] \\
&= \sum_{i=0}^{2^{l-1}-1} \sigma^i[1 - x - \sigma(x)] = -x - \sigma^{2^{l-1}}(x) = 1_R
\end{aligned}$$

as desired.

Next, let $G = \langle \sigma_1 \rangle \oplus \langle \sigma_2 \rangle \oplus \cdots \oplus \langle \sigma_r \rangle$ be an abelian 2-group. By induction it is enough to compute x_G with $\operatorname{tr}_G(x_G) = 1$ in case that $G = \langle \sigma \rangle \oplus N$, $|\langle \sigma \rangle| = 2^{s+1}$, $H = \langle \sigma^{2^s} \rangle \oplus N$, and $x_H \in R$ such that $\operatorname{tr}_H(x_H) = 1$. Indeed, let $z = \operatorname{tr}_N(x_H)$. Then $\operatorname{tr}_{\langle \sigma^{2^s} \rangle}(z) = 1$ and by the cyclic case, $\operatorname{tr}_{\langle \sigma \rangle}(z\sigma(z)) = 1$. Now, z and therefore $z\sigma(z)$ are N invariant, so there exists $y \in R$ with $\operatorname{tr}_N(y) = z\sigma(z)$. Consequently $\operatorname{tr}_G(y) = 1$. Following the steps above we have

$$y = \operatorname{tr}_N(x_H)\sigma(\operatorname{tr}_N(x_H))\operatorname{tr}_{\langle \sigma^{2^s} \rangle}(x_H).$$

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