



# On a conjecture of Moore

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## 1. Introduction

We address a conjecture of J.C. Moore which concerns a criterion for a module over a group ring to be projective. Throughout,  $G$  denotes a group,  $H$  is a subgroup of finite index and  $R$  denotes a ring. The primary object is to study modules over the group ring  $RG$ . In [3], Chouinard records the following.

**Moore's Conjecture.** *Suppose that for all  $x \in G \setminus H$ , either there is an integer  $n$  such that  $1 \neq x^n \in H$  or  $x$  has finite order invertible in  $R$ . Then every  $RG$ -module  $M$  which is projective over  $RH$  is also projective over  $RG$ .*

This can be regarded as a generalization of Serre's Theorem that every torsion-free group of finite virtual cohomological dimension has finite cohomological dimension (see [2]). For suppose that  $G$  is torsion-free and that  $H$  has cohomological dimension  $n < \infty$ . Then the  $n$ th syzygy in any projective resolution of  $\mathbb{Z}$  over  $\mathbb{Z}G$  is projective as  $\mathbb{Z}H$ -module, and Moore's Conjecture implies at once that it is also projective as  $\mathbb{Z}G$ -module. One of the special cases of the conjecture which we prove here also implies Serre's Theorem.

It is natural to generalize the conjecture, because, in addition to group rings, it makes perfect sense for crossed products and in fact for strongly group-graded rings. Recall that a  $G$ -graded ring is a ring  $A$  with a direct sum decomposition

$$A = \bigoplus_{g \in G} A_g$$

such that  $A_g A_h \subseteq A_{gh}$  for all  $g, h \in G$ . Such a ring is said to be *strongly*  $G$ -graded if and only if  $A_g A_h = A_{gh}$  for all  $g, h$ . If  $A$  is any  $G$ -graded ring and  $K$  is a subgroup of  $G$  we write

$$A_K := \bigoplus_{k \in K} A_k$$

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for the subring supported on  $K$ . Examples of strongly  $G$ -graded rings include the group ring  $RG$ , and, more generally, any crossed product  $R * G$ . The group ring  $A = RG$  is graded by setting  $A_g := Rg$ , and, in this case,  $A_K = RK$  for any subgroup  $K$ . We refer the reader to [1] and [4] for further information. Here, we shall consider the following strengthening of Moore's Conjecture.

**Conjecture 1.1.** *Let  $A$  be a strongly  $G$ -graded ring. Suppose that for all  $x \in G \setminus H$ , either there is an integer  $n$  such that  $1 \neq x^n \in H$  or  $x$  has finite order invertible in  $A$ . Then every  $A$ -module  $M$  which is projective over  $A_H$  is also projective over  $A$ .*

In this article we establish Conjecture 1.1 in two special cases. First, when  $G$  belongs to  $\mathbf{H}_1\mathfrak{F}$ , and secondly when  $G$  belongs to  $\mathbf{H}\mathfrak{F}$  and  $M$  is finitely generated as  $A$ -module. We refer the reader to [5] for an explanation of the classes  $\mathbf{H}_1\mathfrak{F}$  and  $\mathbf{H}\mathfrak{F}$ . They are large classes:  $\mathbf{H}_1\mathfrak{F}$  contains, but is strictly larger than, the class of groups of finite virtual cohomological dimension. The class  $\mathbf{H}\mathfrak{F}$ , which contains  $\mathbf{H}_1\mathfrak{F}$  as a subclass, is extension closed, subgroup closed, and contains, amongst others, all countable linear groups.

## 2. Background results

The arguments in this paper are based on two results. One, which follows from results in [4], allows us to reduce the question of projectivity of a module over a strongly  $G$ -graded ring to the question of its projectivity over  $A_H$  and over the family of subrings  $A_F$ , where  $F$  runs through the finite subgroups of  $G$ . The second result [1] allows us to reduce from finite subgroups to elementary abelian subgroups.

**Theorem 2.1** (Cornick and Kropholler [4]). *Let  $A$  be a strongly  $G$ -graded ring, and let  $M$  be an  $A$ -module which is projective as an  $A_H$ -module and which is also projective as an  $A_F$ -module for all finite subgroups  $F$  of  $G$ . Suppose that one of the following conditions is satisfied: either*

- (1)  $G$  belongs to  $\mathbf{H}_1\mathfrak{F}$ , or
- (2)  $G$  belongs to  $\mathbf{H}\mathfrak{F}$  and  $M$  is finitely generated as an  $A$ -module.

*Then  $M$  is projective as an  $A$ -module.*

**Proof.** Lemma 6.6 of [4] shows that  $M$  is projective over  $A$  if and only if it has finite projective dimension over  $A$ . Hence we need only show that  $M$  has finite projective dimension.

If  $G$  belongs to  $\mathbf{H}_1\mathfrak{F}$  then it follows immediately from Theorem A of [4] that  $M$  has finite projective dimension.

Now suppose that  $M$  is finitely generated. Since  $M$  is projective over  $A_H$ , it is certainly of type  $\mathbf{FP}_\infty$  over  $A_H$ , and this property is inherited over the larger ring, because  $H$  has finite index in  $G$ . Hence  $M$  is of type  $\mathbf{FP}_\infty$  over  $A$ . In this case, if

$G$  belongs to  $\mathbf{H}\mathfrak{F}$ , then Theorem B of [4] shows that  $M$  has finite projective dimension.

Thus  $M$  has finite projective dimension over  $A$  under either of conditions (1) and (2), and the result follows.  $\square$

**Theorem 2.2** (Aljadeff and Ginosar [1]). *Let  $F$  be a finite group and let  $A$  be a strongly  $F$ -graded ring. Then an  $A$ -module  $M$  is projective if and only if it is projective as an  $A_E$ -module for all elementary abelian subgroups  $E$  of  $F$ .*

The goal of [1] is to prove this in the case when  $A$  is a crossed product of  $A_1$  by  $F$ . However, it can be shown to hold in the strongly  $F$ -graded case with no greater difficulty.

### 3. The main theorem

In this section, we deduce the following special case of Conjecture 1.1 from Theorems 2.1 and 2.2.

**Theorem 3.1.** *Let  $A$  be a strongly  $G$ -graded ring. Suppose that for all  $x \in G \setminus H$ , either there is an integer  $n$  such that  $1 \neq x^n \in H$  or  $x$  has finite order invertible in  $A$ . Let  $M$  be an  $A$ -module which is projective as an  $A_H$ -module. Suppose that one of the following conditions holds: either*

- (1)  $G$  belongs to  $\mathbf{H}_1\mathfrak{F}$ , or
- (2)  $G$  belongs to  $\mathbf{H}\mathfrak{F}$  and  $M$  is finitely generated as an  $A$ -module.

*Then  $M$  is projective as an  $A$ -module.*

**Proof.** Since  $M$  is projective over  $A_H$ , it is a fortiori projective as an  $A_1$ -module. We first show that  $M$  is projective as an  $A_F$ -module for every finite subgroup  $F$  of  $G$ . In view of Theorem 2.2, it suffices to show that  $M$  is projective as an  $A_E$ -module for every elementary abelian subgroup  $E$  of  $G$ . Let  $p$  be a prime and let  $E$  be an elementary abelian  $p$ -subgroup of  $G$ . If  $p$  is invertible in  $A$  then  $M$  is projective over  $A_E$ , by a variation on Maschke's Theorem, because it is projective over  $A_1$ . On the other hand, if  $p$  is not invertible in  $A$  then the hypotheses imply that  $E$  is contained in  $H$ , and hence  $M$  is projective  $A_E$ , because it is projective over  $A_H$ .

Thus  $M$  is projective as an  $A_F$ -module for every finite subgroup  $F$  of  $G$  and now the result follows from Theorem 2.1.  $\square$

Notice that this result has Serre's Theorem as a corollary because  $\mathbf{H}_1\mathfrak{F}$  contains all groups of finite virtual cohomological dimension. The proof [5] that groups of finite virtual cohomological dimension belong to  $\mathbf{H}_1\mathfrak{F}$  is, not surprisingly, very closely related to Serre's original proof of his theorem.

**References**

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