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On the global dimension of multiplicative Weyl algebras

By

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0. Introduction. Let k be any field and let $k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be the Laurent polynomial ring in n indeterminates. It is isomorphic to the group ring kG where G is the free abelian group of rank n. Given an element $\alpha \in H^2(G, k^*)$ (G acting trivially on k^*) we can construct the twisted group ring $k^\alpha G$. As a k-vector space, it is isomorphic to the group ring kG and if $\{u_\sigma\}_{\sigma \in G}$ is a basis, we define the multiplication by the rule $u_{x_i} u_{x_j} = f(x_i, x_j) u_{x_i x_j}$ where $f: G \times G \to k^*$ is a 2-cocycle representing α. An element $\alpha \in H^2(G, k^*)$ can be interpreted as a system of $\binom{n}{2}$-commutators

$$\alpha_{ij} = u_{x_i} u_{x_j} u_{x_i}^{-1} u_{x_j}^{-1} = f(x_i, x_j) f^{-1}(x_j, x_i).$$

We will abuse the notation and simple write

$$\alpha_{ij} = x_i x_j x_i^{-1} x_j^{-1}.$$

In this paper we are concerned with the global dimension of these algebras. Recall from [2] two basic results on the global dimension of twisted group rings:

(1) $\operatorname{gl.dim} k^\alpha G \leqq \operatorname{gl.dim} kG (= n)$

(2) (monotonicity): if H is a subgroup of G then

$$\operatorname{gl.dim} k^{\bar\alpha} H \leqq \operatorname{gl.dim} k^\alpha G, \quad \text{where } \bar\alpha = \operatorname{res}_H^G \alpha.$$

Since the restriction of any element $\alpha \in H^2(G, k^*)$ to an infinite cyclic subgroup of G is trivial, it follows that $1 \leqq \operatorname{gl.dim} k^\alpha G \leqq n$.

In [5] Rosset has given an example where $\operatorname{gl.dim} k^\alpha G = 1$ (there $k = \mathbb{C}$). Manipulating with (2) above, one can construct an element $\alpha \in H^2(G, k^*)$ for which $\operatorname{gl.dim} k^\alpha G = r$ for any $1 \leqq r \leqq n$. From (2) it follows also that if $\operatorname{res}_H^G \alpha = 1$ for some subgroup H of rank r, then $\operatorname{gl.dim} k^\alpha G \geqq \operatorname{gl.dim} kH = r$. The natural question is whether the converse holds (see [4]) or, more precisely,

Question. If $\operatorname{gl.dim} k^\alpha G \geqq r (\leqq n)$, do there exist r multiplicatively independent monomials $\{m_s = x_1^{t_{1s}} \dots x_n^{t_{ns}}\}_{s=1}^r$ that commute in $k^\alpha G$? i.e., does there exist a subgroup $H \leqq G$ of rank r such that $\operatorname{res}_H^G \alpha = 1$?

The answer is positive for $r = 1$ (since $\operatorname{res}_{\langle x_1 \rangle}^G \alpha = 1$). The case $r = n$ follows as shown below from [1] or [4]. There one shows that the commutators α_{ij} are roots of unity and therefore there exists a subgroup H of finite index such that $\operatorname{res}_H^G \alpha = 1$.