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On the Projective Schur Group of a Field*

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If k is a field, the projective Schur group $\mathrm{PS}(k)$ of k is the subgroup of the Brauer group $\mathrm{Br}(k)$ consisting of those classes which contain a projective Schur algebra, i.e., a homomorphic image of a twisted group algebra $k^{\alpha}G$ with G finite, $\alpha \in H^2(G, k^*)$. It has been conjectured by Nelis and Van Oystaeyen (*J. Algebra* **137** (1991), 501–518) that $\mathrm{PS}(k) = \mathrm{Br}(k)$ for all fields k. We disprove this conjecture by showing that $\mathrm{PS}(k) \neq \mathrm{Br}(k)$ for rational function fields $k_0(x)$ where k_0 is any infinite field which is finitely generated over its prime field.

0. INTRODUCTION

Let k be any field. A k-central simple algebra B is called a *Schur algebra* (over k) if it is the homomorphic image of a group algebra kG for some finite group G. Equivalently, a Schur algebra over k is a finite-dimensional k-central simple algebra B which is spanned over k by a finite subgroup of the units of B. The *Schur group* $S(k)$ of k is the subgroup of the Brauer group $\mathrm{Br}(k)$ of k generated by (in fact consisting of) classes in $\mathrm{Br}(k)$ that are represented by Schur algebras. The Schur group is trivial for fields of positive characteristic [CR81, p. 148, Proof of Cor. 7.11], whereas for fields of characteristic zero there is the fundamental

THEOREM 0.1 (Brauer–Witt) [Y70, Chap. 3]. *Every Schur algebra B over a field k of characteristic zero is Brauer equivalent to a cyclotomic algebra.*

(Recall that a crossed product algebra $K_t^{\alpha}Q/k = (K/k, \alpha)$ is a cyclotomic algebra if $K = k(\zeta)$ is a cyclotomic extension (ζ a root of unity) and $\alpha \in H^2(Q, K^*)$ has a representative 2-cocycle whose values lie in a finite

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subgroup of K^*; t signifies the action of $Q = G(K/k)$ on K. In particular, Theorem 0.1 implies that every Schur algebra has a cyclotomic splitting field.)

The notions of Schur algebra and Schur group were generalized by Lorenz and Opolka [LO78], by replacing an ordinary group algebra by a *twisted* group algebra $k^\alpha G$, where $\alpha \in H^2(G, k^*)$ with G acting trivially on k^*, and obtaining the analogous notions of *projective Schur algebra* and the *projective Schur group* of k. More precisely, let B be a finite-dimensional k-central simple algebra. Then B is a projective Schur algebra over k iff it is the homomorphic image of a twisted group algebra $k^\alpha G$ for some finite group G and some $\alpha \in H^2(G, k^*)$. In characteristic zero $k^\alpha G$ is semisimple, so a projective Schur algebra is a direct summand of a twisted group algebra. In general, a projective Schur algebra may be characterized as a k-central simple algebra B that contains a group Γ in the group of units of B that is finite modulo k^* and spans B over k.

The basic example of a projective Schur algebra is a symbol algebra. If $B = (a, b, \zeta)_n$, where ζ is a primitive nth root of unity in k; u, v are generators of B; and $u^n = a$, $v^n = b$, $uv = \zeta vu$, then B is isomorphic to a twisted group algebra $k^\alpha(\mathbb{Z}/n \oplus \mathbb{Z}/n)$, where α is defined by the given relations.

The projective Schur group $\mathrm{PS}(k)$ of k is defined as the subgroup of $\mathrm{Br}(k)$ generated by (and again, consisting of) classes containing projective Schur algebras.

The projective Schur group is much larger than the Schur group. For example, if k is a number field then $\mathrm{PS}(k) = \mathrm{Br}(k)$ [LO78]. This is a consequence of the fact that every central simple algebra over a number field k has a cyclic splitting field which is contained in a cyclotomic extension of k and therefore is similar to a crossed product $L_t^\alpha G$ where $L = k(\zeta)$ is a cyclotomic extension, $G = G(L/k)$, and α has a representing cocycle with values in k^*. The subgroup E of $L_t^\alpha G^*$ given by the extension

$$\alpha: 1 \to k^* \to E \to G \to 1$$

is finite mod k^*, hence so is the composite $\Gamma = E\langle \zeta \rangle$, which spans $L_t^\alpha G$ over k. It follows that $L_t^\alpha G$ is a projective Schur algebra over k.

It has been conjectured [NVO91] that $\mathrm{PS}(k) = \mathrm{Br}(k)$ for all fields k. The main purpose of this paper is to disprove this conjecture. We will show for example that $\mathrm{PS}(k) \neq \mathrm{Br}(k)$ for rational function fields $k_0(x)$ with k_0 any infinite field which is finitely generated over its prime field. In a previous paper [AS93] the authors showed that every projective Schur algebra has an abelian splitting field L (i.e., L/k is an abelian extension). This result was not sufficient, however, to prove that $\mathrm{PS}(k) \neq \mathrm{Br}(k)$, since it is still an outstanding open question whether or not every central simple algebra has an abelian splitting field. Using the results of [AS93] on the structure of projective Schur algebras, we prove in Section 1:

Every projective Schur algebra over a field k has an abelian splitting field which is contained in a (finite) radical extension of k.

By a *radical extension* of k we mean an extension of the form $K = k(A)$, where A is a subgroup of K^* such that Ak^*/k^* is a torsion group, i.e., $K = k(\sqrt[n]{a}, \sqrt[m]{b}, \dots)$ with $a, b, \dots \in k^*$. In Section 2 we prove that an abelian extension of k which is contained in a radical extension of k is contained in the composite of a cyclotomic extension and a Kummer extension of k. Finally, in Section 3 we give examples of (cyclic) algebras over various fields k which do not have such splitting fields, which shows that, for such k, $\mathrm{PS}(k) \neq \mathrm{Br}(k)$.

1. SPLITTING FIELDS OF PROJECTIVE SCHUR ALGEBRAS

Let $B = k(\Gamma)$ be a projective Schur algebra over k, where the group Γ spans B over k and is finite modulo k^*. Let H be a (normal) subgroup of Γ that contains the commutator subgroup Γ' of Γ. By [AS93] the subalgebra spanned by H over k is a semisimple algebra. Moreover, if it is not simple one shows that $B \cong M_n(S)$, where S is a projective Schur algebra over k and n is the number of simple components in $k(H)$. Consequently we shall assume that $k(H)$ is a simple algebra for every $\Gamma' \subseteq H \subseteq \Gamma$. Next, since Γ is center-by-finite, the group Γ' is finite and so $k(\Gamma')$ is a Schur algebra over its center. Consider a maximal Schur algebra (over its center) of the form $k(\Gamma_0)$, where $\Gamma' \subseteq \Gamma_0 \subseteq \Gamma$. Denote by L the center of $k(\Gamma_0)$ and set $G =$ (the abelian group) Γ/Γ_0.

THEOREM 1.1 [AS93].

(a) *Conjugation of $k(\Gamma_0)$ by Γ induces an isomorphism $G \cong G(L/k)$, where L is the center of $k(\Gamma_0)$. In particular, L/k is an abelian extension.*

(b) *$k(\Gamma)$ is isomorphic to a ring-theoretic crossed product $k(\Gamma_0)^* G$.*

(c) *If F/L is a finite extension that splits $k(\Gamma_0) \cong M_r(D)$, then it also splits $k(\Gamma)$.*

Now since $k(\Gamma_0)$ is a Schur algebra over L, it is split by a cyclotomic extension $L(\zeta)$ of L. (In characteristic $p > 0$, $k(\Gamma_0)$ is already split.) Furthermore, since L/k is an abelian extension, $L(\zeta)$ is abelian over k and we obtain

THEOREM 1.2 [AS93]. *Every projective Schur algebra $k(\Gamma)$ is split by an abelian extension of k.*

As explained in the Introduction, our objective is to show that the above splitting field of $k(\Gamma)$ ($L(\zeta)$ in characteristic zero, L in positive characteristic) can be embedded in a radical extension $k(A)$ of k.

THEOREM 1.3. *Let k be any field, and let $B = k(T)$ be a simple k-algebra which is spanned over k by a group T such that T is finite modulo k^*. If $L = Z(k(T)) \supseteq k$ is Galois over k then it is contained in a radical extension $K = k(A)$ of k.*

Proof. Write $S = T/k^*$ and let $\alpha \in H^2(S, k^*)$ correspond to the extension

$$\alpha: 1 \to k^* \to T \to S \to 1.$$

Then the algebra B is a homomorphic image of the twisted group algebra $k^\alpha S$. We consider first the case $\mathrm{char}(k) = 0$. Then $k(T)$ is a direct summand of the semisimple algebra $k^\alpha S$. If K/k is an extension, then $K \otimes_k k(T) \cong K(T)$ is a direct summand (not necessarily simple) of $K \otimes_k k^\alpha S \cong K^{\alpha'} S$, where α' is the image of α under the map $H^2(S, k^*) \to H^2(S, K^*)$. There is a radical (Galois) extension K_1/k such that the image α' of α in $H^2(S, K_1^*)$ is represented by a 2-cocycle with values in the group of mth roots of unity μ_m, where m is a positive integer such that $\alpha^m = 1$ (e.g., $m = |S|$). (The following argument goes back to [B32] (see [NVO91]): let $f(\sigma, \tau)$ be a cocycle representing α. Then $f(\sigma,\tau)^m = g(\sigma)g(\tau)g(\sigma\tau)^{-1}$, where g: $S \to k^*$ is a map (1-cochain). Let $K_1 = k(\{\sqrt[m]{g(\sigma)} : \sigma \in S\}, \mu_m)$. Set $f'(\sigma, \tau) = f(\sigma, \tau)g(\sigma)^{-1/m}g(\tau)^{-1/m}g(\sigma\tau)^{1/m}$. Then $f'(\sigma, \tau)$ represents $\alpha' \in H^2(S, K_1^*)$ and $f'(\sigma, \tau)^m = 1$, i.e., $f'(\sigma, \tau) \in \mu_m$, for all $\sigma, \tau \in S$). Then $K_1(T) = K_1(S_1)$, where S_1 is the finite group in the exact sequence

$$1 \to \mu_m \to S_1 \to S \to 1$$

defined by α'. $K_1(S_1)$ is a direct summand of the group algebra $K_1 S_1$. By the Brauer splitting theorem [CR81] there exists a cyclotomic extension $K = K_1(\zeta)$ which splits $K_1 S_1$, i.e., $KS_1 = K \otimes_{K_1} K_1 S_1 \cong \oplus_i M_{r_i}(K)$. Now K is a radical (Galois) extension of k and since $k(T)$ is a direct summand of $k^\alpha S$, $K_1(T) = K_1(S_1)$ is a direct summand of $K_1 S_1$, so $K(T) = K(S_1)$ is a direct summand of KS_1. Now $K(T) = K \otimes_{K_1} K_1(T) \cong K \otimes_{K_1} (K_1 \otimes_k k(T)) \cong K \otimes_k k(T) \cong K \otimes_k M_r(D)$ is a direct summand of $KS_1 \cong \oplus_i M_{r_i}(K)$. It follows from the uniqueness part of the Wedderburn Theorem that $K \otimes_k M_r(D)$ is k-isomorphic to a direct sum of some of the $M_{r_i}(K)$. The center of $\oplus_{i=1}^{l} M_{r_i}(K)$ is k-isomorphic to $\oplus_{i=1}^{l} K$. On the other hand, the center of $K \otimes_k M_r(D)$ is k-isomorphic to $K \otimes_k L \cong \oplus KL$, where KL is the composite of K and L in an algebraic closure of k. We therefore have a k-embedding of KL into $\oplus_{i=1}^{l} K$, which is possible only if $KL = K$, i.e., $L \subseteq K$.

If $\mathrm{char}(k) = p \neq 0$, the twisted group algebra $k^{\alpha}S$ is not necessarily semisimple. However, since $L = Z(k(T))$ is Galois over k, the algebra $K \otimes_k k(T)$ is semisimple for every finite extension K/k. Furthermore, it is a homomorphic image of $K \otimes_k k^{\alpha}S \cong K^{\alpha'}S$. Now an argument similar to the above shows that there exists a radical extension K_1 of k, possibly nonseparable, such that $K_1 \otimes_k k(T)$ is a homomorphic image of a group algebra K_1S_1 with S_1 finite. Since $K_1 \otimes_k k(T)$ is semisimple, it is a homomorphic image of $K_1S_1/\mathrm{rad}(K_1S_1)$ and again by the Brauer splitting theorem there exists a cyclotomic extension $K = K_1(\zeta)$ which splits K_1S_1, i.e., $KS_1/\mathrm{rad}(KS_1) \cong M_{r_1}(K) \oplus \cdots \oplus M_{r_s}(K)$. Now $K \otimes_{K_1} K_1 \otimes_k k(T) \cong K \otimes_k k(T)$ is semisimple, so it is a homomorphic image of $KS_1/\mathrm{rad}(KS_1)$. Since L/k is Galois, the center of $K \otimes_k k(T)$ is k-isomorphic to $\oplus KL$ (direct sum of copies of the composite KL). The last part of the argument is the same as in characteristic zero. ∎

COROLLARY 1.4. *Let $k(\Gamma)$ be a projective Schur algebra over k. Then $k(\Gamma)$ has an abelian splitting field F contained in a radical extension K of k.*

Proof. Let Γ_0 be as in Theorem 1.1 above, and take $T = \Gamma_0$ in Theorem 1.3. Then the center L of $k(\Gamma_0)$ is embedded in a radical extension of K of k. It follows that the splitting field $L(\zeta)$ of $k(\Gamma)$ in Theorem 1.2 is embedded in the radical extension $K(\zeta)$ of k. ∎

2. SCHINZEL EXTENSIONS

In Section 1 we proved that every projective Schur algebra over a field k has a splitting field L which is abelian over k and is embedded in a radical extension of k. We prove next a purely field-theoretic result about such field extensions. If k is a field containing the mth roots of unity, an extension of the form $k(\sqrt[m]{U})/k$, where U is a subgroup of k^*, is called a *Kummer extension*. Let $b \in k^*$, n a positive integer prime to $\mathrm{char}(k)$, and m the number of nth roots of unity in k. A theorem of Schinzel [Sc77, Theorem 2; K88, p. 235] states that if the Galois group of $x^n - b$ over k is abelian, then $b^m = c^n$ for some $c \in k^*$. Extracting mnth roots of both sides of this equation yields $\sqrt[n]{b} = \zeta\sqrt[m]{c}$, where ζ is an mnth root of unity. Thus Schinzel's theorem implies that $k(\sqrt[n]{b}) \subseteq k(\zeta, \sqrt[m]{c})$, which is the composite of a cyclotomic extension of k with a Kummer extension of k. Let us call a finite abelian extension L of k a *Schinzel extension* of k if L is contained in the composite of a cyclotomic extension of k with a Kummer extension of k.

PROPOSITION 2.1. *Let k be a field, L a finite abelian extension of k which is contained in a finite radical extension $E = k(\sqrt[m]{U})$, and U a subgroup of k^*. Then L is a Schinzel extension of k.*

Proof. First we factor $m = p_0^r s$, where $p_0 = \mathrm{char}(k)$ is prime to s. Then L is contained in the maximal separable subextension of E, which is $k(\sqrt[s]{U})$. Hence we may assume m is prime to p_0.

Without loss of generality L is a cyclic extension of prime power degree p^r of k. Let $k_1 = k(\mu_m)$, with μ_m the group of mth roots of unity, $L_1 = Lk_1$, and $E_1 = Ek_1$. Then $E_1 = k_1(\sqrt[m]{U})$ is a finite Kummer extension of k_1. By Kummer theory, the cyclic subextension L_1 of E_1 is of the form $k_1(\sqrt[m]{b})$, where $b \in Uk_1^{*m}$ and hence without loss of generality $b \in U$. Now $L_1 = Lk_1$ is an abelian extension of k, hence its subextension $k(\sqrt[m]{b})$ is also abelian over k. By Schinzel's theorem above, $k(\sqrt[m]{b})$ is a Schinzel extension of k, hence so is the composite $L_1 = k_1(\sqrt[m]{b})$ and hence its subfield L. ∎

COROLLARY 2.2. *Let $k \subseteq L \subseteq k(\sqrt[m]{U})$ be as in Proposition 2.1, and assume that for every prime $p \| [L:k]$, k does not contain the pth roots of unity. Then L is contained in a cyclotomic extension of k.*

Proof. By Proposition 2.1, $L \subseteq CK$, with C/k cyclotomic, K/k Kummer. K is a direct composite of Kummer extensions K_q of prime power degree q^s, its "q-primary components". If $q \nmid [L:k]$, then we can erase K_q from the composite CK. On the other hand, for every $p \| [L:k]$, the component K_p is trivial, since k does not contain μ_p. Thus $K = k$ and $L \subseteq C$. ∎

COROLLARY 2.3. *Every projective Schur algebra over k is split by a Schinzel extension of k. In other words,* $\mathrm{PS}(k) \subseteq \mathrm{Br}(\Omega/k)$, *where Ω is the maximal Schinzel extension of k.*

COROLLARY 2.4. *Let p be a prime such that k does not contain the pth roots of unity. Then* $\mathrm{PS}(k)_p \subseteq \mathrm{Br}(k(\mu)/k)_p$, *where μ is the group of all roots of unity in the algebraic closure of k, and the subscript p denotes the p-primary component.*

Proof. Let B be a projective Schur algebra over k of order p^r in $\mathrm{Br}(k)$. By Corollary 2.3, B is split by a Schinzel extension L/k, and we may assume $[L:k]$ is a power of p. By Corollary 2.2 and the assumption that k does not contain the pth roots of unity, L is contained in a cyclotomic extension of k. ∎

3. $\mathrm{PS}(K) \neq \mathrm{Br}(K)$

We will show that $\mathrm{PS}(K) \neq \mathrm{Br}(K)$ if $K = k(t)$ is a rational function field in one variable over any field k satisfying a mild hypothesis, which holds for example if k is any infinite field which is finitely generated over its prime field.

LEMMA 3.1. *Let k be a field, p a prime $\neq \mathrm{char}(k)$. Suppose there exists a finite extension F of k and a cyclic extension L/F of degree $p^r > 1$ such that $L \cap F(\mu) = F$, where μ is the group of all roots of unity in the algebraic closure of k. Then $\mathrm{Br}(k(t))$ has an element of order p^r which is of order p^r also modulo $\mathrm{Br}(k(\mu, t)/k(t))$.*

Proof. By the theorem of Faddeev-Auslander-Brumer [FS81], we have an isomorphism

$$\mathrm{Br}(k(t))_p \cong \mathrm{Br}(k)_p \oplus \left\{\oplus_{\mathscr{P}} \mathrm{Hom}(G_k(\mathscr{P}), \mathbb{Q}/\mathbb{Z})_p\right\}$$

where $\mathscr{P} = \mathscr{P}(t)$ runs through the set $k_s[t]_{\mathrm{irr}}$ of monic irreducible polynomials in $k_s[t]$, where k_s is the separable closure of k, $G_k = G(k_s/k)$ is the absolute Galois group of k and $G_k(\mathscr{P})$ denotes the subgroup of G_k fixing $\mathscr{P}$. Note that $\mathscr{P}$ is of the form $t^{p^n} - \alpha$ for some $\alpha \in k_s$, so $G_k(\mathscr{P}) = G_{k(\alpha)}$. Write $k' = k(\mu)$. There is a commutative diagram

$$\begin{array}{ccc}
\mathrm{Br}(k(t))_p & \xrightarrow{\sim} & \mathrm{Br}(k)_p \oplus \{\oplus_{\mathscr{P}} \mathrm{Hom}(G_k(\mathscr{P}), \mathbb{Q}/\mathbb{Z})_p\} \\
\downarrow \mathrm{res} & & \downarrow \mathrm{res} \\
\mathrm{Br}(k'(t))_p & \xrightarrow{\sim} & \mathrm{Br}(k')_p \oplus \{\oplus_{\mathscr{Q}} \mathrm{Hom}(G_{k'}(\mathscr{Q}), \mathbb{Q}/\mathbb{Z})_p\}
\end{array}$$

where on the right side the summand $\mathrm{Hom}(G_k(\mathscr{P}), \mathbb{Q}/\mathbb{Z})_p$ is mapped to $\oplus_\sigma \mathrm{Hom}(G_{k'}(\sigma(\mathscr{P})), \mathbb{Q}/\mathbb{Z})_p$, where σ runs through a set of representatives of the double cosets of $(G_{k'}, G_k(\mathscr{P}))$. (See [So90, Lemma 1]. We note here that the statement of the Faddeev–Auslander–Brumer theorem and Lemma 1 and its proof in [So90] applies only in characteristic zero. They must be modified as above to apply to finite characteristic.) Choose $\alpha \in k_s$ so that the extension F given in the statement of Lemma 3.1 is $k(\alpha)$, and set $\mathscr{P} = t - \alpha$. Let $\chi \in \mathrm{Hom}(G_F, \mathbb{Q}/\mathbb{Z})$ such that the fixed field of $\ker(\chi)$ is L. Then the condition $L \cap k'(\alpha) = L \cap F(\mu) = F = k(\alpha)$ implies that res maps χ faithfully to $\oplus_\sigma \mathrm{Hom}(G_{k'}(\sigma(\mathscr{P})), \mathbb{Q}/\mathbb{Z})_p$. Thus the element of $\mathrm{Br}(k(t))_p$ corresponding to χ is of order p^r modulo $\ker(\mathrm{res}) = \mathrm{Br}(k'(t)/k(t))_p$. ∎

We therefore have

THEOREM 3.2. *Let p be a prime, k a field of characteristic $\neq p$ not containing the pth roots of unity. Suppose there exists a finite extension F of k and a cyclic extension L/F of degree $p^r > 1$ such that $L \cap F(\mu) = F$. Then*

$\mathrm{Br}(k(t))_p$ *has an element of order* p^r *which is of order* p^r *also modulo* $\mathrm{PS}(k(t))_p$. *In particular,* $\mathrm{PS}(k(t))_p \neq \mathrm{Br}(k(t))_p$.

Proof. By Corollary 2.4 above, $\mathrm{PS}(k(t))_p \subseteq \mathrm{Br}(k'(t)/k(t))_p$, so the assertion follows from Lemma 3.1. ∎

Recall that a field F is *Hilbertian* iff given any irreducible polynomial $f(x, y) \in F[x, y]$, there exists $c \in F$ such that $f(c, y) \in F[y]$ is irreducible. It is known [FJ86, Chap. 12] that any infinite field which is finitely generated over its prime field is Hilbertian.

COROLLARY 3.3. *Let k be a Hilbertian field, p a prime* $\neq \mathrm{char}(k)$, *and assume that k does not contain the pth roots of unity. Then* $\mathrm{PS}(k(t))_p \neq \mathrm{Br}(k(t))_p$; *in fact,* $\mathrm{Br}(k(t))_p/\mathrm{PS}(k(t))_p$ *is infinite.*

Proof. By Theorem 3.2, it suffices to show that for any positive integer r, there exists a finite extension F of k and a cyclic extension L/F of degree p^r such that $L \cap F(\mu) = F$. Since k is Hilbertian, the symmetric group S_{p^r} is realizable as a Galois group over k, say $G(L/k) \cong S_{p^r}$. Let F be the fixed field of an element of order p^r. Then L/F is cyclic of order p^r. Since $[L \cap k(\mu): k] = 1$ or 2 and p is odd by hypothesis, it follows that $L \cap F(\mu) = F$. ∎

COROLLARY 3.4. *Let k be an infinite field which is finitely generated over its prime field. Then* $\mathrm{Br}(k(t))_p/\mathrm{PS}(k(t))_p$ *is infinite for all but finitely many primes p.*

Proof. For almost all p, k does not contain the pth roots of unity. Since k is Hilbertian, the result follows from Theorem 3.3. ∎

It is interesting to consider also formal power series fields $k((t))$. Here we have Witt's theorem [Se79, p. 186]

$$\mathrm{Br}(k((t)))_p \cong \mathrm{Br}(k)_p \oplus \mathrm{Hom}(G_k, \mathbb{Q}/\mathbb{Z})_p$$

where p is a prime $\neq \mathrm{char}(k)$. In this direct sum, $\mathrm{Br}(k)$ represents the algebras $D \otimes_k k((t))$, where D is a k-central division algebra. In the second summand $\mathrm{Hom}(G_k, \mathbb{Q}/\mathbb{Z})$, the elements are represented by cyclic algebras of the form $(L((t))/k((t)), \sigma, t)$, where L is the fixed field of $\ker(\chi)$, $\chi \in \mathrm{Hom}(G_k, \mathbb{Q}/\mathbb{Z})$, and σ is a generator of the cyclic group $G(L/k)$ such that $\chi(\sigma) \equiv 1/n \pmod{\mathbb{Z}}$, where n is the order of χ.

PROPOSITION 3.5. *Let k be a field,* $p \neq \mathrm{char}(k)$, *and assume that k does not contain all p-power roots of unity. Let s be maximal such that k contains the* p^s*th roots of unity. Let* $r > 2s$ *and assume that there exists a cyclic extension L/k of degree* p^r *such that* $L \cap k(\mu) = k$. *Then* $\mathrm{Br}(k((t)))$ *contains an element of order* p^r *which is of order* $\geq p^{r-2s} \bmod \mathrm{Br}(\Omega/k((t)))$, *where Ω is the maximal Schinzel extension of $k((t))$. This element lies in the subgroup of* $\mathrm{Br}(k((t)))$ *corresponding to* $\mathrm{Hom}(G_k, \mathbb{Q}/\mathbb{Z})$ *in Witt's Theorem. In particular,* $\mathrm{Br}(k((t)))_p/\mathrm{PS}(k((t)))_p$ *is of order* $\geq p^{r-2s} > 1$.

Remark. If k contains all p-power roots of unity, then by the Merkurev–Suslin theorem $\mathrm{Br}(k((t)))_p = \mathrm{PS}(k((t)))_p$.

Proof of Proposition 3.5. Set $k' = k(\mu)$. Let K be the maximal Kummer p-extension of $k((t))$. Then $G(K/k((t)))$ has exponent p^s, and $\Omega = K(\mu)$ by Section 2 above. Write $K = k((t))((k((t))^{*1/p^s})$. Decomposing $k((t))^* = k^* \times \langle t \rangle \times U_1$, where $U_1 = 1-\text{units} = \{1 + \sum_{n \geq 1} a_n t^n : a_n \in k\}$, and observing that $U_1^{p^s} = U_1$ for $p \neq \mathrm{char}(k)$ (by Hensel's Lemma), we have $k((t))^*/k((t))^{*p^s} \cong k^*/k^{*p^s} \times \langle t \rangle / \langle t^{p^s} \rangle$. Set $K_1 = k((t))(k^{*1/p^s})$, so that $K = K_1(t^{1/p^s})$. We may write $K_1 = k_1 k((t))$, where $k_1 = k(k^{*1/p^s})$. Let $D = (L((t))/k((t)), \sigma, t)$. Tensoring up to $K_1' = k_1 k' k((t))$, we get $D_1' = D \otimes_{k((t))} K_1' = (k_1 k' L((t))/k_1 k' k((t)), \sigma^{p^s}, t)$ since $[L \cap k_1 : k] = p^s$. We claim D_1' is of order $= p^{r-s}$ in $\mathrm{Br}(K_1')$. Indeed, it is enough to consider finite extensions F of k which are contained in $k_1 k'$. There is a commutative diagram [Se79, p. 187] (Witt's Theorem)

$$\begin{array}{ccc} \mathrm{Br}(k((t)))_p & \longrightarrow & \mathrm{Br}(k)_p \oplus \mathrm{Hom}(G_k, \mathbb{Q}/\mathbb{Z})_p \\ \downarrow & & \downarrow \\ \mathrm{Br}(F((t)))_p & \longrightarrow & \mathrm{Br}(F)_p \oplus \mathrm{Hom}(G_F, \mathbb{Q}/\mathbb{Z})_p \end{array}$$

where the component $\mathrm{Hom}(G_k, \mathbb{Q}/\mathbb{Z})_p$ is mapped into $\mathrm{Hom}(G_F, \mathbb{Q}/\mathbb{Z})_p$ (restriction to G_F). Then D corresponds to a character $\chi: G_k \to \mathbb{Q}/\mathbb{Z}$ with kernel G_L. Since $[L \cap F : k] \leq p^s$, $\chi|_{G_F}$ is of order $\geq p^{r-s}$, i.e., $D \otimes_{k((t))} F((t))$ is of order $\geq p^{r-s}$ in $\mathrm{Br}(F((t)))$. Now the fields $F((t))$, $k \subseteq F \subseteq k_1 k'$, $[F : k] < \infty$, form a direct system and $K_1' = k_1 k' k((t))$ is the direct limit. Consequently, $D_1' = D \otimes_{k((t))} K_1'$ is of order $\geq p^{r-s}$ in $\mathrm{Br}(K_1')$. Since we already know that this order is $\leq p^{r-s}$, we have equality, proving the claim. Finally, since $\Omega = K_1'(t^{1/p^s})$ is of degree $= p^s$ over K_1', we have $D \otimes_{k((t))} \Omega = D_1' \otimes_{K_1'} \Omega$ is of order $\geq$ (in fact, $=$) p^{r-2s} in $\mathrm{Br}(\Omega)$. ∎

Remark 3.6. For many fields k, the hypothesis in Proposition 3.5 holds for all r, e.g., k a proper finite extension of $\mathbb{Q}_p$ or k a number field which is not totally real or $k = \mathbb{Q}(t)$, etc., so in such cases $\mathrm{Br}(k((t)))_p/\mathrm{PS}(k((t)))_p$ is infinite, even for all p in the latter two examples.

In contrast:

EXAMPLE 3.7. $\mathrm{PS}(\mathbb{Q}((t))) = \mathrm{Br}(\mathbb{Q}((t)))$, $\mathrm{PS}(\mathbb{Q}_p((t))) = \mathrm{Br}(\mathbb{Q}_p((t)))$.

Proof. By Witt's Theorem [Se79, p. 186], $\mathrm{Br}(\mathbb{Q}((t))) \cong \mathrm{Br}(\mathbb{Q}) \oplus \mathrm{Hom}(G_{\mathbb{Q}}, \mathbb{Q}/\mathbb{Z})$. In this direct sum, $\mathrm{Br}(\mathbb{Q})$ represents the algebras $D \otimes_{\mathbb{Q}}$

$\mathbb{Q}((t))$, where D is a $\mathbb{Q}$-central division algebra. Hence $\mathrm{Br}(\mathbb{Q}) = \mathrm{PS}(\mathbb{Q}) \subseteq \mathrm{PS}(\mathbb{Q}((t)))$. In the second summand $\mathrm{Hom}(G_{\mathbb{Q}}, \mathbb{Q}/\mathbb{Z})$, the elements are represented by cyclic algebras of the form $(L((t))/\mathbb{Q}((t)), \sigma, t)$, where $L/\mathbb{Q}$ is cyclic. By the Kronecker–Weber Theorem, L is contained in a cyclotomic extension K of $\mathbb{Q}$, so this cyclic algebra is similar to a crossed product $(K((t))/\mathbb{Q}((t)), G, \alpha)$ where α is representable by a cocycle with values in $\mathbb{Q}((t))^*$, which is a projective Schur algebra. Hence this second summand is also contained in $\mathrm{PS}(\mathbb{Q}((t)))$. The proof for $\mathbb{Q}_p$ is identical. ∎

From the results in this paper it seems that the projective Schur group of a field k is in general a proper subgroup of $\mathrm{Br}(k)$. The natural question arises: what (proper) subgroup is $\mathrm{PS}(k)$? For example, is it a relative Brauer group, i.e., is there an algebraic extension K/k such that $\mathrm{PS}(k) = \mathrm{Br}(K/k)$? Here is an example where this happens.

EXAMPLE 3.8. Let $k = \mathbb{F}_2$, $K = k(x, y)$, a rational function field in two variables over k. By Corollary 3.4, $\mathrm{Br}(K(\mu)/K) \neq \mathrm{Br}(K)$. By Corollary 2.4, $\mathrm{PS}(K) \subseteq \mathrm{Br}(K(\mu)/K)$, since K does not contain any roots of unity other than 1. On the other hand, $\mathrm{Br}(K(\mu)/K) \subseteq \mathrm{PS}(K)$ since (see Introduction) every element of $\mathrm{Br}(K)$ which is split by a cyclic cyclotomic extension of K is represented by a projective Schur algebra. (Since $\mathrm{char}(K)$ is finite, every cyclotomic extension of K is cyclic.) Therefore $\mathrm{PS}(K) = \mathrm{Br}(K(\mu)/K)$.

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We are grateful to the referee for pointing out that if one does not restrict oneself to algebraic extensions, every subgroup of $\mathrm{Br}(k)$ is a relative Brauer group $\mathrm{Br}(K/k)$ for some extension K of k (e.g., one can take K to be a composite of generic splitting fields).

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