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GLOBAL DIMENSIONS OF CROSSED PRODUCTS

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Suppose Γ is a group and a homomorphism $t:\Gamma\to \mathrm{Aut}(K)$ is given. Here K is a field and $\mathrm{Aut}(K)$ is the group of field automorphisms of K. Then we say that Γ acts on K. In such circumstances the multiplicative group K^* is a Γ-module and it is well known that elements of $H^2(\Gamma,K^*)$ give rise to 'crossed product' algebras. To recall this let $\alpha\in H^2(\Gamma,K^*)$ and let $f:\Gamma\times\Gamma\to K^*$ be a 2-cocycle representing α. One defines the crossed product, which we denote by

$$K_t^\alpha\Gamma$$

as follows. As left K vector space it is a direct sum $\coprod_{\sigma\in\Gamma}Ku_\sigma$. Multiplication is defined so as to satisfy the rule

$$(xu_\sigma)\cdot(yu_\tau)=x\sigma(y)f(\sigma,\tau)u_{\sigma\tau}.$$

Here $\sigma(y)$ is the action of $t(\sigma)$ on y. It is easy to see that this multiplication is associative (this follows from the cocycle condition) and that, up to isomorphism of rings, $K_t^\alpha\Gamma$ only depends on α, not on the choice of f. It is thus assumed sometimes, for convenience, that f is a 'normalized' cocycle, i.e., $f(1,\sigma)=f(\tau,1)=1$ for $\sigma,\tau\in\Gamma$. Then the unit element of $K_t^\alpha\Gamma$ is u_1.

Examples. (1) If Γ is finite and acts faithfully, then $K_t^\alpha\Gamma$ is the old crossed product. It is a central simple algebra with center the fixed field K^Γ.

(2) If t and α are both trivial, we get the ordinary group ring $K\Gamma$.

(3) This is a more complicated (and interesting) example. Let

$$\alpha:\quad 1\to A\to\Gamma\to G\to 1$$

be a group extension with A *torsion free* and *abelian*. We also identify α as an element in $H^2(G,A)$. Let l be a field. It is known that the non-zero elements of lA do not divide zero in $l\Gamma$ and that one can 'classically' localize $l\Gamma$ with respect to the set $S=lA-\{0\}$, i.e., form the ring of fractions $S^{-1}l\Gamma$, see [7] for proof of this. Now, $S^{-1}lA$ is equal to $l(A)$, the field of fractions of lA. We denote it by K. G acts on K via its action on A which, in K^*, can be thought of as the set of monomials. The inclusion map $i: A \hookrightarrow K^*$ induces $i_*(\alpha) \in H^2(G, K^*)$. Dropping the i_* from the notation we can form $K_t^\alpha G$. It is easy to check explicitly that

$$S^{-1} l\Gamma \simeq K_t^\alpha G.$$

This example shows that the construct $K_t^\alpha \Gamma$ exists in nature.

In order to make the notation $K_t^\alpha \Gamma$ more flexible and convenient we drop from it trivial things. Thus $K_t\Gamma$ stands for $K_T^\alpha \Gamma$ with $\alpha = 0$ and $K^\alpha \Gamma$ for the case where Γ acts trivially.

If Γ' is a subgroup of Γ, let $t' = t/\Gamma'$ and $\alpha' = \mathrm{res}^\Gamma_{\Gamma'}(\alpha)$. Clearly the crossed product $K_{t'}^{\alpha'} \Gamma'$ can be identified as a subring of $K_t^\alpha \Gamma$. This is used below repeatedly.

In this paper we are concerned with global dimensions of crossed products. In example (1) above $K_t^\alpha \Gamma$ is a simple artinian ring so its global dimension is 0. In example (2) gl.dim$(K\Gamma)$ is also known as the cohomological dimension of Γ over K, denoted $\mathrm{cd}_K(\Gamma)$, and depends only on char(K), see [8, p. 89]. Here we prove a general inequality (3.2 below) and deduce that always

$$(*) \qquad \mathrm{gl.dim}(K_t^\alpha \Gamma) \le \mathrm{gl.dim}(K_t \Gamma) \le \mathrm{cd}_K(\Gamma).$$

We also prove some other results, to be described in a moment. The basic idea underlying all our results is the construction of a spectral sequence. Let Γ_0 be a *normal* subgroup of Γ, $\Gamma/\Gamma_0 = G$, L the fixed field K^{Γ_0}. As in galois theory G acts on L with associated map $\bar{t}: G \to \mathrm{Aut}(L)$, i.e., $\bar{t}(\bar{\sigma})(x) = \sigma(x)$ for $x \in L$. Let $R = K_t^\alpha \Gamma$ and $R_0 = K_{t_0}^{\alpha_0} \Gamma_0$ (t_0, α_0 are the restrictions to Γ_0 of t, α). R_0 is a subring of R. If M, N are left R-modules then $\mathrm{Hom}_{R_0}(M, N)$ has a natural $L_{\bar{t}}G$ module structure, i.e., α has disappeared! Let k be the fixed field K^Γ. The group ring kG is a subring of $L_{\bar{t}}G$ and in Section 1 we show that there are natural isomorphisms

$$\mathrm{Hom}_R(M, N) \simeq \mathrm{Hom}_{kG}(k, \mathrm{Hom}_{R_0}(M, N)) \simeq \mathrm{Hom}_{L_{\bar{t}}G}(U, \mathrm{Hom}_{R_0}(M, N))$$

where $U = L_{\bar{t}}G \otimes_{kG} k$ (k is the trivial kG-module.) Thus $\mathrm{Hom}_R(-, -)$ is expressed as a composition of two 'simpler' functors and, using averaging operators, in Section 2 we show that if M is a projective R module, then $\mathrm{Hom}_{R_0}(M, N)$ is 'acyclic' in the sense that $\mathrm{Ext}^p_{kG}(k, -)$ vanishes on it for $p > 0$. Thus the conditions of a theorem of Grothendieck are satisfied and we obtain a convergent spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_{kG}(k, \mathrm{Ext}^q_{R_0}(M, N)) \Rightarrow \mathrm{Ext}^{p+q}_R(M, N).$$

In Section 3 we note this and derive the main consequences, which are the inequalities ($*$). It is easy to give examples of strict inequalities in ($*$) where the larger number is ∞; but we thought at one time that if $\mathrm{cd}_K(\Gamma) < \infty$, then both inequalities become equalities and even proved one of them (see Section 4); but gl.dim$(K_t^\alpha \Gamma) <$ $\mathrm{cd}_K \Gamma < \infty$ can happen, as shown in a letter from K.A. Brown to one of us.

Finally in Section 5 we show that gl.dim$(K_t\Gamma) = 0$ implies that Γ is a torsion group, but it can be infinite: we give a construction in which $K_t^\alpha \Gamma$ is a division ring and Γ is a (locally finite) infinite group.

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1. Module structures

We retain the notation of the introduction, so that $\Gamma_0 \lhd \Gamma$, $G = \Gamma/\Gamma_0$, $L = K^{\Gamma_0}$ etc. When working with group rings it is well known that $\mathrm{Hom}_{\Gamma_0}(-, -)$ transforms Γ-modules to G-modules. Here the situation is similar but slightly more complicated. The nice thing is that $\mathrm{Hom}_{R_0}(-, -)$ is a functor to $L_{\bar{t}}G$ modules regardless of α.

If $\sigma \in \Gamma$, its residue class in G is denoted by $\bar{\sigma}$. Let M, N be left R-modules.

1.1. Lemma. $\mathrm{Hom}_{R_0}(M, N)$ *is an* $L_{\bar{t}}G$*-module where if* $h \in \mathrm{Hom}_{R_0}(M, N)$, $\sigma \in \Gamma$ *and* $x \in L$, *then* $xu_{\bar{\sigma}}$ *acts on* h *by the rule* $xu_{\bar{\sigma}}(h(u_\sigma^{-1}m))$ *for* $m \in M$.

Proof. It is clear that multiplying elements of $\mathrm{Hom}_{R_0}(M, N)$ by elements of L is permissible. It remains to check (1) that this action is well defined, (2) that $u_{\bar{\sigma}}h \in \mathrm{Hom}_{R_0}(M, N)$ and (3) that it is an action, i.e., that

$$xu_{\bar{\sigma}}(yu_{\bar{\tau}}h) = x\sigma(y)(u_{\bar{\sigma}\bar{\tau}}h) \quad \text{where } x, y \in L;\ \sigma, \tau \in \Gamma.$$

To prove (1) let $\varrho = \sigma\mu$ where $\mu \in \Gamma_0$. Since h is R_0 linear,

$$u_\sigma u_\mu h u_\mu^{-1} h_\sigma^{-1} = u_\sigma h u_\sigma^{-1}.$$

But

$$u_\varrho h u_\varrho^{-1} = f(\sigma, \mu)^{-1} u_\sigma u_\mu h u_\mu^{-1} u_\sigma^{-1} f(\sigma, \mu) = u_\sigma h u_\sigma^{-1},$$

since u_μ cancels as does $f(\sigma, \mu)$ when it is pulled out (here we see this phenomenon for the first time).

It is immediate that $u_{\bar{\sigma}}h$ is K-linear. Now suppose $\mu \in \Gamma_0$. To prove (2) we have to show $u_{\bar{\sigma}}h(u_\mu m)) = u_\mu(u_{\bar{\sigma}}h(m))$ for $m \in M$. The computation is similar to that in (1) and can be dropped.

To prove the identity in (3) we evaluate both its sides at $m \in M$. The left hand side gives

$$\begin{aligned} xu_\sigma(yu_\tau h(u_\tau^{-1}u_\sigma^{-1}m)) &= x\sigma(y)u_\sigma u_\tau h u_\tau^{-1} u_\sigma^{-1}(m) \\ &= x\sigma(y)\{f(\sigma, \tau)u_{\sigma\tau} h u_{\sigma\tau}^{-1} f(\sigma, \tau)^{-1}\}(m). \end{aligned}$$

Pulling out $f(\sigma, \tau)^{-1}$ it comes out unchanged (suffering 2 attacks along its way, which cancel each other.) Thus we can write

$$\begin{aligned} &= x\sigma(y)u_{\sigma\tau} h u_{\sigma\tau}^{-1}(m) \\ &= x\sigma(y)u_{\bar{\sigma}\bar{\tau}} h(m). \end{aligned}$$

This completes the proof.

In fact it is clear that Hom_{R_0} defines 2 functors from the category of left R modules to the category of left $L_{\bar{\imath}}G$-modules: one contravariant $M \mapsto \mathrm{Hom}_{R_0}(M, N)$, the other covariant $N \mapsto \mathrm{Hom}_{R_0}(M, N)$. Obviously these are (finitely) additive left exact functors. Since the derived functors of, say, $M \mapsto \mathrm{Hom}_{R_0}(M, N)$ are the functors $\mathrm{Ext}^{\cdot}_{R_0}(M, N)$ we get

1.2. Lemma (in the notation of 1.1). $\mathrm{Ext}^{\cdot}_{R_0}(M, N)$ *has a natural* $L_{\bar{\imath}}G$*-module structure such that connecting homomorphisms are* $L_{\bar{\imath}}G$*-linear maps.*

From now on we shall denote the functor $M \mapsto \mathrm{Hom}_{R_0}(M, N)$ by $\Phi_N(M)$.

Let $k = K^{\Gamma}$ $(= L^G)$ be the fixed field. The group ring kG is a subring of $L_{\bar{\imath}}G$. Let $U = L_{\bar{\imath}}G \otimes_{kG} k$ (here k denotes the trivial kG-module). In what follows we sometimes treat $L_{\bar{\imath}}G$-modules as kG-modules, i.e., we only remember their kG-structure. This should be clear from the context.

1.3. Lemma. *Let* M, N *be* R*-modules. There are natural isomorphisms of functors*

$$\mathrm{Hom}_R(M, N) \simeq \mathrm{Hom}_{L_{\bar{\imath}}G}(U, \mathrm{Hom}_{R_0}(M, N)) \simeq \mathrm{Hom}_{kG}(k, \mathrm{Hom}_{R_0}(M, N)).$$

Proof. We only prove the first isomorphism; the proof of the second is similar. If $h \in \mathrm{Hom}_{L_{\bar{\imath}}G}(U, \mathrm{Hom}_{R_0}(M, N))$ we want to map it to $h(1_U)$ where $1_U = 1 \otimes 1 \in U$. So we have to show that $h(1_U) \in \mathrm{Hom}_R(M, N)$. Let $\sigma \in \Gamma$, $m \in M$. Then

$$\begin{aligned} h(1_U)(u_\sigma^{-1} m) &= u_\sigma^{-1}(u_\sigma h(1_U)(u_\sigma^{-1} m)) \\ &= u_\sigma^{-1}((u_\sigma h(1_U))(m)) \\ &= u_\sigma^{-1}(h(\bar{\sigma} \cdot 1_U)(m)) = u_\sigma^{-1}(h(1_U)(m)). \end{aligned}$$

Thus $h(1_U)$ is in $\mathrm{Hom}_R(M, N)$. It is easy to check that the map $h \mapsto h(1_U)$ is natural. To define the inverse transformation let $g \in \mathrm{Hom}_R(M, N)$. An element of $\mathrm{Hom}_{L_{\bar{\imath}}G}(U, \mathrm{Hom}_{R_0}(M, N))$ is characterised by its value at 1_U, which must be G-invariant; conversely such a G invariant element gives rise to an element of $\mathrm{Hom}_{L_{\bar{\imath}}G}(U, \mathrm{Hom}_{R_0}(M, N))$. Since g is R linear it is G invariant when considered in $\mathrm{Hom}_{R_0}(M, N)$ so we can define $\tilde{g}$ by $\tilde{g}(1_U)(m) = g(m)$ for $m \in M$. It is easily checked that this defines a natural transformation inverse to the above.

We shall need the following application of the second isomorphism of 1.3.

1.4. Lemma. *There is a natural isomorphism of* δ*-functors on* $L_{\bar{\imath}}G$*-modules*

$$\mathrm{Ext}^{\cdot}_{kG}(k, -) \simeq \mathrm{Ext}^{\cdot}_{L_{\bar{\imath}}G}(U, -).$$

This follows, of course, from the exactness of the forgetful functor mentioned before Lemma 1.3.

2. Acyclicity

We shall denote the functor $\mathrm{Hom}_{kG}(k, -)$ by Ψ. Thus if P is a kG-module, $\Psi(P)$ is simply P^G. The derived functors of Ψ are the $\mathrm{Ext}^{\cdot}_{kG}(k, -)$. If k is $\mathbb{Z}$, these are the cohomology groups $H^{\cdot}(G, -)$. Since we work only over k we will abuse the notation and denote $\mathrm{Ext}^{\cdot}_{kG}(k, -)$ by $H^{\cdot}(G, -)$. These functors are very much like the ordinary cohomology groups. For example they satisfy a 'Shapiro lemma' over k, of which we shall need a special case.

Call a kG-module *co-induced* if it is of the form $\mathrm{Hom}_k(kG, V)$ for some k-module (i.e., trivial kG-module) V; here G acts 'diagonally', see [1]. Shapiro's lemma in this case is

2.1. Lemma. *If P is a direct summand of a co-induced kG module, then it is cohomologically trivial, i.e., $H^q(G,P)=0$ for $q>0$.*

The proof is the same as in group cohomology. Recall that $\Phi_N(M)=\mathrm{Hom}_{R_0}(M,N)$.

2.2. Proposition. *If M is projective over R, then $\Phi_N(M)$ is Ψ acyclic, i.e., it is cohomologically trivial.*

Proof. By finite additivity it suffices to prove the result for M free. So suppose $M=R^I$, a free module of 'rank' $|I|$. Then $\Phi_N(M)$ is a product of $|I|$ copies of $\mathrm{Hom}_{R_0}(R,N)$. As Ψ and its derived functors commute with products, it is enough to prove that $\mathrm{Hom}_{R_0}(R,N)$ is Ψ acyclic. By 2.1, this will follow from

2.3. Proposition. $\mathrm{Hom}_{R_0}(R,N)$ *is a direct summand of a co-induced kG-module.*

To prove this we shall employ the concept of *mean value* operators.

2.4. Definition. A kG-module P admits a *mean* if there exists an additive function ('integral') $\hat{I}:\mathrm{Map}(G,P)\to P$ satisfying (i) if $c:G\to P$ is a constant function with value $a\in P$, then $\hat{I}(c)=a$; (ii) it is G-linear.

Here G acts on $\mathrm{Map}(G,P)$ diagonally, i.e., if $h:G\to P$ then, $\sigma h(\tau)=\sigma(h(\sigma^{-1}\tau))$.

2.5. Lemma. *If P admits a mean, then it is a direct summand of a co-induced module.*

Proof. There is always a kG-linear imbedding of P in $\mathrm{Hom}_k(kG,P)$ as the set of 'constant functions' viz. to $x\in P$ corresponds the constant function with value x. Identifying $\mathrm{Map}(G,P)$ and $\mathrm{Hom}_k(kG,P)$ in the obvious way, $\hat{I}$ defines a map from $\mathrm{Hom}_k(kG,P)$ to P. This map is kG-linear by 2.4(ii) and is a left inverse to the above imbedding by 2.4(i).

Thus to establish 2.2, it remains to show

2.6. Lemma. $\mathrm{Hom}_{R_0}(R,N)$ *admits a mean.*

Proof. Let $h:G\to\mathrm{Hom}_{R_0}(R,N)$. We define $\hat{I}(h)$ to be the additive function satisfying $\hat{I}(h)(xu_\sigma)=xh(\bar{\sigma})(u_\sigma)$ for $\sigma\in\Gamma$, $x\in K$. We have to check (1) $\hat{I}(h)$ is R_0 linear, and (2) (i) and (ii) of 2.4.

Proof of (1). Let $y\in L$, $\tau\in\Gamma_0$. We are to show that $\hat{I}(h)(yu_\tau xu_\sigma)=yu_\tau(\hat{I}(h)(xu_\sigma))$. Now $\hat{I}(h)(yu_\tau xu_\sigma)=\hat{I}(h)(y\tau(x)f(\tau,\sigma)u_{\tau\sigma})=y\tau(x)f(\tau,\sigma)h(\bar{\sigma})(u_{\tau\sigma})$ since $\overline{\tau\sigma}=\bar{\sigma}$. On the other hand, $yu_\tau(\hat{I}(h)(xu_\sigma))=y\tau(x)u_\tau h(\bar{\sigma})(u_\sigma)=y\tau(x)h(\bar{\sigma})(f(\tau,\sigma)u_{\tau\sigma})$ which is as desired.

Proof of (2). Obviously (i) of 2.4 holds. To prove $\hat{I}(\bar{\sigma}h)=\bar{\sigma}\hat{I}(h)$, we must check, for $x\in K$, $\tau\in\Gamma$, that $\hat{I}(\bar{\sigma}h)(xu_\tau)=(\bar{\sigma}\hat{I}(h))(xu_\tau)$. This is a confusing verification and we do it in some detail. Recall that G acts on $\mathrm{Hom}_{R_0}(R,N)$ by $\bar{\sigma}v=u_\sigma vu_\sigma^{-1}$ (composition of maps) and on $\mathrm{Map}(G,\mathrm{Hom}_{R_0}(R,N))$ similarly by $(\bar{\sigma}h)(\bar{\tau})=\bar{\sigma}(h(\sigma^{-1}\tau))$. Thus, given $h:G\to\mathrm{Hom}_{R_0}(R,N)$, $x\in K$ and $\tau\in\Gamma$

$$\begin{aligned}\bar{\sigma}\hat{I}(h)(xu_\tau)&=u_\sigma(\hat{I}(h)(u_\sigma^{-1}xu_\tau))\\&=xu_\sigma(\hat{I}(h)(f(\sigma^{-1},\sigma)^{-1}f(\sigma^{-1},\tau)u_{\sigma^{-1}\tau}))\\&=xu_\sigma f(\sigma^{-1},\sigma)^{-1}f(\sigma^{-1},\tau)h(\sigma^{-1}\tau)(u_{\sigma^{-1}\tau}).\end{aligned}$$

We have used in this computation the normalization of f mentioned in the introduction.

$$\begin{aligned}\hat{I}(\bar{\sigma}h)(xu_\tau)&=x(\bar{\sigma}h)(\tau)(u_\tau)=x(\bar{\sigma}h(\overline{\sigma^{-1}\tau}))(u_\tau)=xu_\sigma(h(\overline{\sigma^{-1}\tau})(u_\sigma^{-1}u_\tau))\\&=xu_\sigma(h(\overline{\sigma^{-1}\tau})(f(\sigma^{-1},\sigma)^{-1}\cdot f(\sigma^{-1},\tau)u_{\sigma^{-1}\tau})\\&=xu_\sigma f(\sigma^{-1},\sigma)^{-1}f(\sigma^{-1},\tau)h(\overline{\sigma^{-1}\tau})(u_{\sigma^{-1}\tau})\end{aligned}$$

and the proof of (ii) is complete.

3. The main results

In 1.3 we saw that $\mathrm{Hom}_R(M,N)=\Psi\cdot\Phi_N(M)$ and in Section 2 it is shown that Φ_N (which is a left exact contravariant functor) takes R-projectives to Ψ-acyclics. By a theorem of Grothendieck, [2, Theorem 2.4.1], there is a convergent E_2 spectral sequence with $E_2^{p,q}=R^p\Psi\cdot R^q\Phi_N(M)$ and limit $R^n(\Psi\cdot\Phi_N)(M)$. Substituting $R^p\Psi=H^p(G,-)$ etc. we get

3.1. Theorem. *There is a convergent E_2 spectral sequence*

$$H^p(G,\mathrm{Ext}^q_{R_0}(M,N))\Rightarrow\mathrm{Ext}^n_R(M,N)\quad\textit{where } M,N \textit{ are } R\textit{-modules.}$$

An immediate corollary of 3.1 and 1.4 is

3.2. Theorem. $\mathrm{gl.dim}(K_t^\alpha \Gamma) \le \mathrm{gl.dim}(K_{t_0}^{\alpha_0}\Gamma_0) + \mathrm{gl.dim}(L_{\bar t}G)$.

Proof. Say $\mathrm{gl.dim}(K_{t_0}^{\alpha_0}\Gamma_0) = r$, $\mathrm{gl.dim}(L_{\bar t}G) = s$. If $p+q=n>r+s$, then either $p>r$ or $q>s$. In both cases $E_2^{p,q}=0$ so $E_\infty^{p,q}=0$ and therefore $\mathrm{Ext}_R^n(M,N)$.

3.3. Corollary. $\mathrm{gl.dim}(K_t^\alpha \Gamma) \le \mathrm{gl.dim}(K_t \Gamma)$.

Proof. Take $\Gamma_0 = 1$ in 3.2.

3.4. Proposition. $\mathrm{gl.dim}(K_t \Gamma) \le \mathrm{gl.dim}(k\Gamma) = \mathrm{cd}_K(\Gamma)$.

Proof. Again we take $\Gamma_0 = 1$. In the spectral sequence $\mathrm{Ext}_{R_0}^q(M,N)=0$ for $q>0$, since R_0 is a field. Thus the spectral sequence collapses to isomorphisms $H^p(G, \mathrm{Hom}_K(M,N)) \simeq \mathrm{Ext}_R^p(M,N)$. Clearly our proposition follows.

The moral is, obviously, that more ('heavier') structure lowers the global dimension (by 'gravitation'). It is indeed easy to give examples showing that imposing structure can make infinite-dimensional objects into finite-dimensional ones. It requires more effort to give examples of strict inequalities with both sides finite.

3.5. Examples. Let $\Gamma = C \times G$ where C is a finite p-group, which for simplicity we can take to be cyclic, and G a group of finite cohomological dimension n. Suppose Γ acts on a field K of characteristic p via its factor C, i.e., G acts trivially. Then it is easy to see that $\mathrm{cd}_K(\Gamma)=\infty$ while $\mathrm{gl.dim}(K_t\Gamma)=n$. An example with trivial action can be obtained as follows. Let T be an infinite cyclic group, $\varphi: T \to C$ a surjection and $T' = \ker(\varphi)$. Let k be a field of characteristic p and $K = k(T')$, the field of fractions of kT'. If $S = kT' - \{0\}$, then we see that $S^{-1}k[T\times G] \simeq K^\alpha\Gamma$ where α comes from the extension $1 \to T' \to T \xrightarrow{\varphi} C \to 1$. Thus $K^\alpha\Gamma$ has global dimension n while $\mathrm{cd}_K(\Gamma) = \infty$.

4. Monotonicity

Let Γ' be a subgroup of Γ, $t' = t \mid \Gamma'$, $\alpha' = \mathrm{res}_{\Gamma'}^{\Gamma}(\alpha)$. We denote $K_{t'}^{\alpha'}\Gamma'$ by R'.

4.1. Proposition. $\mathrm{gl.dim}(R') \le \mathrm{gl.dim}(R)$.

Proof. Let $T = \{1\} \cup T'$ be a right transversal (for the cosets $\Gamma'\gamma$). The sets Γ' and $\bigcup_{\tau\in T'} \Gamma'\tau$ are stable under the action of Γ' on both sides. Thus if we write $R = R'1_R + \sum_{\tau \in T'} R'u_\tau$, the sum on the right is a bimodule over R'. Hence if N is an R'-module, it is a direct summand of $R \otimes_{R'} N$. Clearly projective R modules are projective over R'. So $\mathrm{pd}_{R'}(N) \leq \mathrm{pd}_R(R \otimes_{R'} N) \leq \mathrm{gl.dim}(R)$.

4.2. Corollary. *If* $\ker(t)$ *has finite index in* Γ *and* $\Gamma' = \ker(t)$, *then* $\mathrm{gl.dim}(R') = \mathrm{gl.dim}(R)$.

Proof. In (3.2) take $\Gamma_0 = \Gamma'$. Then $L = K$ and $\mathrm{gl.dim}(K_{\bar{t}}G) = 0$.

We now investigate what happens upon 'change of field'. Suppose F is a field containing K and Γ acts on F in a way that extends its action on K. We denote both actions by t. We already have $\alpha \in H^2(\Gamma, K^*)$ and the inclusion $j : K^* \hookrightarrow F^*$ induces $j_*(\alpha)$ and again we will abuse notation and write α for $j_*(\alpha)$. (This is a true abuse since quite often j_* is not injective.)

4.4. Proposition. *If* $\mathrm{gl.dim}(K_t^\alpha \Gamma) < \infty$, *then* $\mathrm{gl.dim}(K_t^\alpha \Gamma) \leq \mathrm{gl.dim}(F_t^\alpha \Gamma)$.

We denote $K_t^\alpha \Gamma = R$ and $F_t^\alpha \Gamma = \tilde{R}$.

Proof. Let $A = R \cdot 1_{\tilde{R}}$, and $C = \tilde{R}/A$. As a left R-module C is free. Indeed using a basis of F over K which includes 1, it is easy to see that the residue classes in C of the non 1 elements in this basis are a basis for C over R. Thus the canonical surjection $\tilde{R} \to C$ splits (over R). Let $\varphi : C \to \tilde{R}$ be such a splitting; then $\tilde{R} = A + \varphi(C)$, a direct sum. The result now follows from

4.5. Lemma. *Let* $R \subset \tilde{R}$ *be rings such that* $\tilde{R} = R \cdot 1_{\tilde{R}} + B$, *the sum being a direct sum of projective left R-modules. If* $\mathrm{gl.dim}(R) < \infty$, *it is also* $\leq \mathrm{gl.dim}(\tilde{R})$.

Proof. Suppose $\mathrm{gl.dim}(R) = n$. Let M be a (finitely generated) R-module such that $\mathrm{Ext}_R^n(M, N) \neq 0$ for some N. Note that $\mathrm{Ext}_R^n(M, -)$ is a right exact additive functor. Let $R^I \to N$ be a surjection. Then the induced map $\mathrm{Ext}_R^n(M, R)^I \to \mathrm{Ext}_R^n(M, N)$ is surjective and we see that $\mathrm{Ext}_R^n(M, R) \neq 0$. Now as in (1.4), $\mathrm{Ext}_{\tilde{R}}^n(\tilde{R} \otimes_R M, \tilde{R}) = \mathrm{Ext}_R^n(M, \tilde{R})$ and this is not 0 since it contains $\mathrm{Ext}_R^n(M, R)$.

4.6. Corollary. *Let* $k = K^\Gamma$. *Then* $\mathrm{gl.dim}(K_t\Gamma) = \mathrm{gl.dim}(K\Gamma)$ $(= \mathrm{cd}_k(\Gamma))$.

Proof. (3.4) gives $\leq$ and (4.4) gives the opposite inequality.

Problem. Let $F = K(t_1, \ldots, t_n)$, a field of rational functions, and suppose the action of Γ on K extended to F by its acting trivially on the variables. Is it true, in this case, that $\mathrm{gl.dim}(R) = \mathrm{gl.dim}(\tilde{R})$?

5. Global dimension 0

Maschke's theorem says that if G is finite and char(K) does not divide $|G|$, then KG is semisimple, i.e., has zero global dimension. By inequalities (3.3) and (3.4) (see also Introduction), if KG is semisimple, so is $K_t^\alpha G$. In this section we prove

5.1. Proposition. *If $K_t^\alpha \Gamma$ is semisimple (i.e.,* $\text{gl.dim}(K_t^\alpha \Gamma)=0$*), then Γ is a torsion group.*

Proof. By the structure theorem a non-zero-divisor in a semisimple ring is invertible. Thus the proposition follows from.

5.2. Lemma. *If $\sigma \in \Gamma$ is of infinite order, then $1-u_\sigma \in K_t^\alpha \Gamma$ is not invertible and not a zero-divisor.*

Proof (of lemma). This is well known in group rings and is proved similarily here. Suppose $(1-u_\sigma)v=0$, i.e., $v=u_\sigma v$ and, say, u_1 appears in v. Then $v=u_\sigma v=u_\sigma^2 v=\cdots$ and this implies that infinitely many powers of σ (i.e. of u_σ) appear in v. Thus $v=0$ and $1-u_\sigma$ does not divide 0. If $1-u_\sigma$ were invertible its inverse would have to be the infinite geometric series $\sum_{n=0}^\infty u_\sigma^n$ which is impossible! This 'proof' can be made respectable as follows. Let $\{\tau_i\}$ be coset representatives in Γ for the cosets Tx where T is the infinite cyclic group generated by σ. Then $K_t^\alpha \Gamma=\sum_i \Lambda u_{\tau_i}$, a direct sum, where Λ is the ring generated over K by u_σ. This shows that $(1-u_\sigma)^{-1}$, where it to exist, would have to be in Λ. The non-invertibility of $1-u_\sigma$ in Λ is proved by a degree argument and is left to the reader.

5.3. Corollary. *If Γ is a free group,* $\text{gl.dim}(K_t^\alpha \Gamma)=1$.

Proof. $\text{gl.dim}(K_t^\alpha \Gamma)>0$ by (5.1). On the other hand $\text{cd}\,\Gamma=1$ (see [8, Proposition 7] and the discussion in 1.4 there), so $\text{gl.dim}(K_t^\alpha \Gamma)\le 1$.

5.4. Example. Unlike group rings, $K_t^\alpha \Gamma$ can be semisimple for infinite groups Γ. In fact, $K_t^\alpha \Gamma$ can even be a division ring. To see this let G be an infinite locally finite group (e.g. $\mathbb{Q}/\mathbb{Z}$, $\mathrm{GL}(\mathbb{F}_p)$). We construct a group extension

$$\alpha:\quad 1 \to A \to \Gamma \xrightarrow{\pi} G \to 1$$

such that Γ is torsion free and, moreover, the group ring $k\Gamma$ over some (in fact any) field k is an integral domain.

As in example (3) of the introduction one knows, in such circumstances, that

$$S^{-1}k\Gamma \cong K_t^\alpha G$$

where:

$$S=kA-\{0\},$$

K = field of fractions of kA,

G acts on K via its action on A, and

α comes from the inclusion $A \hookrightarrow K^*$.

To see that $K_t^\alpha G$ is a division ring note that given a finite set in G it is contained in a finite subgroup H. If $\pi^{-1}H = \Gamma_H$, then

$$S^{-1}k\Gamma_H \cong K_t^\beta H \qquad (\beta = \mathrm{res}_H^G \alpha)$$

is a division algebra, being a domain finite dimensional over its center which is a field. Since G is locally finite, $\bigcup_H K_t^\beta H = K_t^\alpha G$ is a division ring. It remains to construct Γ (more precisely α) having the desired properties.

Let $\{H_j\}_{j\in J}$ be a list of all subgroups of *prime order* of G. For each j let $A_j = \mathrm{Coind}_{H_j}^G(\mathbb{Z})$. Here $\mathbb{Z}$ is the trivial $\mathbb{Z}H_j$-module and Coind is defined by

$$\mathrm{Coind}_H^G(M) = \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M) \qquad (M \text{ an } H\text{-module}).$$

This is the notation of [1, Ch. III-5] and the reader is referred to this book for details on the G-structure of $\mathrm{Coind}_H^G(M)$ and the 'Shapiro isomorphism' which we now use. It gives an isomorphism

$$s_H^G : H^2(H_j, \mathbb{Z}) \to H^2(G, A_j).$$

Since H_j is cyclic $H^2(H_j, \mathbb{Z}) \cong \mathbb{Z}/|H_j|\,\mathbb{Z}$. Let β_j be a generator and $\alpha_j = s_H^G(\beta_j)$. Finally let $A = \prod_{j\in J} A_j$, φ_j the inclusion of A_j in A (in the j-th coordinate)

$$\alpha = \prod \varphi_{j*}(\alpha_j) \quad \text{in } H^2(G, A).$$

To simplify notation we identify α_j with $\varphi_{j*}(\alpha_j)$. Thinking of α as a group extension

$$\alpha: \quad 1 \to A \to \Gamma \to G \to 1,$$

we claim Γ is torsion free. This follows from the so called 'Charlap criterion' [6] which says that Γ is torsion free $\Leftrightarrow \mathrm{res}_{H_j}^G \alpha \neq 0$ for all $j \in J$. Let us see that α_j (more precisely $\varphi_{j*}\alpha_j$) does not restrict to 0 in H_j. Now, it is not hard to see that, as $\mathbb{Z}H$-modules, if M is an H-module, $\mathrm{Coind}_H^G(M) = M \oplus M'$ such that, if p denotes the projection to M then, for every k, the composition

$$H^k(H, M) \xrightarrow{s_H^G} H^k(G, \mathrm{Coind}_H^G(M)) \xrightarrow{\mathrm{res}} H^k(H, \mathrm{Coind}_H^G(M)) \xrightarrow{p^*} H^k(H, M)$$

is the identity. In our case taking $H = H_j$, $M = \mathbb{Z}$, $k = 2$ we get

$$\mathrm{res}_{H_j}^G(\alpha_j) \neq 0, \quad \text{hence} \quad \mathrm{res}_{H_j}^G(\alpha) \neq 0.$$

This proves Γ is torsion free.

To prove $k\Gamma$ is a domain (independently of k) note that it suffices to prove that $k\Gamma'$ is a domain for every finitely generated subgroup $\Gamma' \subset \Gamma$. But Γ' is easily seen to be 'virtually abelian', i.e., its image in G is finite and its intersection with A finitely generated. Thus by results of K.A. Brown (for $\mathrm{char}(k) = 0$, see [5, p. 616]) and P. Linnell (for $\mathrm{char}(k) \neq 0$, see [4]) $k\Gamma'$ is a domain.

References

[1] K.S. Brown, Cohomology of Groups, Graduate Texts in Math. 87 (Springer, New York, 1982).
[2] A. Grothendieck, Sur quelques points d'algèbre homologique, Tohoku Math. J. 9 (1957) 119–221.
[3] G. Hochschild and J.-P. Serre, Cohomology of group extensions, Trans. Am. Math. Soc. 74 (1953) 110–134.
[4] P.A. Linnell, Zero divisors and idempotents in group rings, Math. Proc. Camb. Phil Soc. 81 (1977) 365–368.
[5] D.A. Passman, The Algebraic Structure of Group Rings (Wiley Interscience, New York, 1977).
[6] S. Rosset, Group extensions and division algebras, J. Algebra 53 (1978) 297–303.
[7] S. Rosset, A vanishing theorem for Euler characteristics, Math. Z. 185 (1984) 211–215.
[8] J.-P. Serre, Cohomologie des groupes discret, in: Prospects in Mathematics, Ann. of Math. Studies 70 (1971) 77–169.