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Crossed Products, Cohomology, and Equivariant Projective Representations

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Let k be a commutative ring and K a commutative k algebra. If G is a group acting on K via a homomorphism $t: G \to \mathrm{Aut}_k(K)$, which may (in general) have a nontrivial kernel, then the multiplicative group K^* of invertible elements of K is a G module. The elements of the "galois" cohomology group $H^2(G, K^*)$ give rise to the well known *Crossed Product Construction* (see [1]). It is defined as follows. Let $\alpha \in H^2(G, K^*)$ and let $f: G \times G \to K^*$ be a cocycle representing α, i.e., $\alpha = [f]$. The crossed product, given t and α, is a k algebra, denoted by $K_t^\alpha G$. As left K module it is $\coprod_{\sigma \in G} K u_\sigma$, while the product is defined by the rule

$$(xu_\sigma)(yu_\tau) = x\sigma(y)\, f(\sigma, \tau)\, u_{\sigma\tau} \qquad (x, y \in K; \sigma, \tau \in G).$$

It is easily verified that this is an associative k algebra, in fact it is even a K^G algebra (where K^G is the fixed ring), and—up to isomorphism of algebras—does not depend on the choice of representing cocycle.

This construction is well known in the case that G is finite and t is assumed injective. From now on this will be referred to as the "classical" case. Nonclassically, we discussed the global dimension of $K_t^\alpha G$, assuming K is a field, in our paper [1]. Crossed products are also widely used in operator algebras.

Viewing the Crossed Product Construction (henceforth the CPC) as a map from $H^2(G, K^*)$ to a certain set of algebras the question naturally arises: What is the operation on this set of algebras which will make the CPC a morphism between monoids or, even, a homomorphism of groups? Of course one can answer this question in a vacuous "tautological" way. What is being asked is a "natural" operation on some k algebras, like the tensor product over the center in the classical case, which will make the CPC into a Brauer-like, preferrably injective, homomorphism of groups (or, at least, monoids).

The answer to this question is an operation defined by Sweedler [3].

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One has to observe that the CPC carries with it more structure than just the k algebra $K_t^\alpha G$. Also available, given the materials already used (which are t and α), is a "canonical" k algebra injection of K into $K_t^\alpha G$. The compound object, which is $K_t^\alpha G$ *and* the canonical map $K \to K_t^\alpha G$, is called a *K/k algebra*. In Section 2 we show that the CPC is a homomorphism from $H^2(G, K^*)$ to the monoid of all isomorphism classes of K/k algebras (with the Sweedler multiplication) and that, under some hypotheses, it is also *injective*.

In our attempts to fathom the motive behind the Sweedler operation we found a characterization of it. This characterization says that, in a certain sense Sweedler's definition is the only one possible. This is described in Section 3. The formulation that is thereby obtained in amenable to generalization in the non-commutative direction. If R is a ring and I is a right ideal in R the "idealizer" of I is the subring $\{x \in R : xI \subset I\}$. We denote it by $\mathrm{Id}(I)$. It is the unique maximal subring that contains I as a 2-sided ideal. Sweedler's construction is describable as follows. Let A, B be K/k algebras. Let I be the kernel of the canonical projection $A \otimes_k B \to A \otimes_K B$ (A, B taken as left K modules). It is the right ideal of the ring $A \otimes_k B$ generated by the set $\{x \otimes 1 - 1 \otimes x : x \in K\}$. Then the Sweedler product $A \times_K B$ equals $\mathrm{Id}(I)/I$. In Section 4 we take a small step towards generalizing Sweedler's construction by showing that if the injectivity assumption on $t: G \to \mathrm{Aut}_k(K)$ is *dropped* it is still possible, for *some* α and β, to find a right ideal J in $K_t^\alpha G \otimes_k K_t^\beta G$ such that $\mathrm{Id}(J)/J = K_t^{\alpha\beta} G$. It seems to us that more work remains to be done in this direction.

Finally in Section 5 we follow a different direction altogether and exhibit an analogue, in the present general context, of the classical descent theory. We define equivariant projective representations of G on KG and show that up to a certain equivalence relation they are classified by $H^1(G, P\,\mathrm{Aut}_K(KG))$ where $P\,\mathrm{Aut}$ denotes "automorphisms up to proportionality in K^*." We then show that a certain subset of this set, which is an H^1 with coefficients in an abelian subgroup of $P\,\mathrm{Aut}_K(KG)$, is naturally a group which is mapped, by a connecting homomorphism, isomorphically onto $H^2(G, K^*)$. Finally we show that this subgroup of $H^1(G, P\,\mathrm{Aut}_K(KG))$ classifies the "regular" (we also use the adjective "diagonal" below) equivariant projective representations.

1. *K/k* Algebras

As above let k be a commutative ring and K a commutative k algebra. A *K/k algebra* is a pair (A, i) where A is a k-algebra and $i: K \to A$ is a homomorphism of k-algebras (sending 1_K to 1_A). We will usually abuse the notation and refer to the K/k algebra as A, if i is clear. A morphism of K/k algebras A, B is an algebra morphism $A \to B$ such that the diagram

$$\begin{array}{ccc} & K & \\ \swarrow & & \searrow \\ A & \longrightarrow & B \end{array}$$

commutes.

If G acts on K via $t\colon G \to \mathrm{Aut}_k(K)$ and $\alpha \in H^2(G, K^*)$ let $f\colon G \times G \to K^*$ be a cocycle representing α. Let R be the crossed product obtained using f. The unit of R, 1_R, is $f(1,1)^{-1}u_1$ and there is an embedding $K \hookrightarrow R$ defined by $x \to x \cdot 1_R$. If R' is the crossed product obtained using another cocycle g (equivalent of f) let $\lambda\colon G \to K^*$ be a 1 cochain such that $f = g \cdot d\lambda$. By definition $R = \coprod_G Ku_\sigma$, Ku_σ with $u_\sigma u_\tau = f(\sigma, \tau)u_{\sigma\tau}$ and $R' = \coprod_G Kv_\sigma$ with $v_\sigma v_\tau = g(\sigma, \tau)v_{\sigma\tau}$. The K-linear map from R to R' defined by $u_\sigma \to \lambda(\sigma)v_\sigma$ is an isomorphism of algebras sending 1_R to $1_{R'}$ and, being K linear, commutes with the embeddings of K in R and R'. Thus we have proved the following.

(1.1) Proposition. *Given $t\colon G \to \mathrm{Aut}_k(K)$ the Crossed Product Construction ($=$CPC) is a map from $H^2(G, K^*)$ to the set of isomorphism classes of K/k-algebras.*

We denote the K/k algebra obtained from α by $K_t^\alpha G$. It is often possible to prove stronger results if one assumes that t is injective. For example

(1.2) Lemma. *Assume t injective and K a domain. Then* (1) *the elements of $K_t^\alpha G$ that commute with K elementwise are precisely the elements of K.* (2) *The elements of $K_t^\alpha G$ normalizing K are of the form $xu_\sigma (x \in K, \sigma \in G)$.*

Proof. If $\sum xa_\sigma u_\sigma = (\sum a_\sigma u_\sigma)x$ for every $x \in K$ we must show that $a_\sigma = 0$ if $\sigma \neq 1$. But $(\sum a_\sigma u_\sigma)x = (\sum a_\sigma \sigma(x))u_\sigma$. If $\sigma \neq 1$ let x be such that $\sigma(x) \neq x$. Then the equality $a_\sigma x = a_\sigma \sigma(x)$ implies $a_\sigma = 0$. Similarly if $\sum xa_\sigma u_\sigma = (\sum a_\sigma u_\sigma) f(x)$ for some endomorphism $f\colon K \to K$ then the equality $\sum a_\sigma(x - \sigma f(x))u_\sigma = 0$ implies $f(x) = \sigma^{-1}(x)$ for every σ such that $a_\sigma \neq 0$. This implies that only one a_σ can be non-zero.

Remark. The assumption that K be a domain can be weakened to "given $\sigma \in G$ there exists an element x such that $x - \sigma(x)$ is not a zero divisor." This technical condition may be useful in analytic contexts where K can be a "large" ring of continuous functions. Such rings are hardly ever domains.

2. Multiplication of K/k Algebras

Let A, B be K/k algebras. The k algebra $A \otimes_k B$ has a natural maps from $K: x \to x \otimes 1$ and $x \to 1 \otimes x$. There is a canonical surjection

$$\pi: A \otimes_k B \to A \otimes_K B,$$

where the tensor product over K is with respect to the left K module structures of A and B. $A \otimes_K B$ is not an algebra in general. But Sweedler [3] has observed that $A \otimes_K B$ has a "canonical" submodule which is an algebra under the naive, coordinatewise, multiplication..

The Sweedler submodule is

$$A \times_K B = \left\{ \sum a_i \otimes b_i \in A \otimes_K B\text{: for every } x \in K, \sum a_i \otimes b_i x = \sum a_i x \otimes b_i \right\}.$$

The product structure is defined by the rule $(\sum_i a_i \otimes b_i)(\sum_j a'_j \otimes b'_j) = \sum_{i,j} a_i a'_j \otimes b_i b'_j$.

(2.1) Lemma. *With the above multiplication and with the map* $K \to A \times_K B$ *by* $x \to x \otimes 1 = 1 \otimes x$ $A \times_K B$ *is a* K/k *algebra.*

The proof is straightforward but demands a genuine understanding of tensor products. (see [3, Proposition (3.1)]). We shall refer to $A \times_K B$ as the Sweedler product of A and B (over K).

We now show that the Sweedler product $\times_K$ answers the question raised in the introduction.

(2.2) Theorem. *Assume that* $t: G \to \mathrm{Aut}(K/k)$ *is injective. If* $\alpha, \beta \in H^2(G, K^*)$ *then* $K_t^\alpha G \times_K K_t^\beta G \approx K_t^{\alpha\beta} G$ *as* K/k *algebras.*

Proof. We start by identifying the elements of $K_t^\alpha G \times_K K_t^\beta G$ inside the tensor product $K_t^\alpha G \otimes_K K_t^\beta G$. Suppose that $K_t^\alpha G = \amalg K u_\sigma$ with $u_\sigma u_\tau = f(\sigma, \tau) u_{\sigma\tau}$ and $K_t^\beta G = \amalg K v_\sigma$ with $v_\sigma v_\tau = g(\sigma, \tau) v_{\sigma\tau}$, f, g cocycles representing $\alpha, \beta \in H^2(G, K^*)$, respectively. If $\sum_{\sigma,\tau} a_{\sigma,\tau} u_\sigma \otimes v_\tau$ $(a_{\sigma,\tau} \in K)$ is to be in $K_t^\alpha G \times_K K_t^\beta G$ it must satisfy $\sum a_{\sigma,\tau} u_\sigma x \otimes v_\tau = \sum a_{\sigma,\tau} u_\sigma \otimes v_\tau x$ for every $x \in K$.

Remembering that $u_\sigma x = \sigma(x) u_\sigma$ and $v_\sigma x = \sigma(x) v_\sigma$ for $\sigma \in G$, we see that

$$\sum a_{\sigma,\tau} u_\sigma x \otimes v_\tau = \sum a_{\sigma,\tau} \sigma(x)\, u_\sigma \otimes v_\tau = \sum a_{\sigma,\tau} u_\sigma \otimes \sigma(x) v_\tau$$

(since $\otimes_K$ is the tensor product of the two left K structures)

$$= \sum a_{\sigma,\tau} u_\sigma \otimes v_\tau \tau^{-1} \sigma(x).$$

Thus we have $\sum a_{\sigma,\tau} u_\sigma \otimes v_\tau(\tau^{-1}\sigma(x)-x)=0$ or, moving scalars to the left

$$\sum_{\sigma,\tau} a_{\sigma,\tau}(\sigma(x)-\tau(x))\, u_\sigma \otimes v_\tau = 0$$

for every $x \in K$. Since K is assumed to be a domain and t injective, we see that $a_{\sigma,\tau}=0$ if $\sigma \neq \tau$. This shows that $K_t^\alpha G \times_K K_t^\beta G = \{\sum_\sigma a_\sigma u_\sigma \otimes v_\sigma\}$; i.e., it is a kind of "diagonal." How are the elements $u_\sigma \otimes v_\sigma$ and $u_r \otimes v_r$ multiplied?

One sees easily that

$$(u_\sigma \otimes v_\sigma)(u_\tau \otimes v_\tau) = f(\sigma, \tau)\, g(\sigma, \tau)\, u_{\sigma\tau} \otimes v_{\sigma\tau}.$$

Thus if $K_t^{\alpha\beta} G = \amalg K w_\sigma$ with $w_\sigma w_\tau = f(\sigma,\tau)\, g(\sigma,\tau)\, w_{\sigma\tau}$ it is plain that the map

$$K_t^\alpha G \times_K K_t^\beta G \to K_t^{\alpha\beta} G$$

sending $\sum a_\sigma u_\sigma \otimes v_\sigma$ to $\sum a_\sigma w_\sigma$ is an isomorphism of K/k algebras. This ends the proof.

Remark. As before the condition that K be a domain can be weakened to "given $\sigma \neq 1 (\sigma \in G)$ there exists $x \in K$ such that $\sigma(x)-x$ is not a zero divisor in K."

The theorem just proved can be viewed as saying that the "CPC" map CPC: $H^2(G, K^*) \to$ {classes of K/k algebras (up to isomorphisms) with the Sweedler product} is a homomorphism. To narrow down its range is now a matter of choice. The following has been suggested by D. Zelinsky.

Let $\mathscr{C}$ be the set of (isomorphism classes) of K/k algebras A enjoying the following property: ($\mathscr{C}$) A is a direct sum of K bimodules $\{U_\sigma\}_{\sigma \in G}$, each U_σ being a free left K-module of rank 1 and its right structure is related to its left structure by

$$ux = \sigma(x)u \qquad (u \in U_\sigma,\, x \in K).$$

It is easy to see that, in A, $U_\sigma U_\tau \subset U_{\sigma\tau}$ if K is assumed to be an integral domain. If K is a field it turns out that $U_\sigma U_\tau = U_{\sigma\tau}$ and $\mathscr{C}$ is exactly the image of the CPC.

We now turn to the question of *injectivity.*

(2.3) THEOREM. *If K is a domain and $t\colon G \to \mathrm{Aut}_k(K)$ is injective then $CPC\colon H^2(G, K^*) \to \mathscr{C}$ is injective.*

Proof. Let $\alpha, \beta \in H^2(G, K^*)$ and assume $h\colon K_t^\alpha G \simeq K_t^\beta G$ as K/k algebras. We must prove $\alpha = \beta$. Write $K_t^\alpha G = \amalg K u_\sigma$, with $u_\sigma u_\tau = f(\sigma,\tau) u_{\sigma\tau}$ where f

represents α, and $K_t^\beta G = \amalg Kv_\sigma$ with $v_\sigma v_\tau = g(\sigma, \tau)\, v_{\sigma\tau}$, g represents β. We prove that f is cohomologous to g. Let $h(u_\sigma) = z_\sigma$. If $x \in K$ $h(xu_\sigma) = h(u_\sigma(\sigma^{-1}(x))) = h(u_\sigma)\,\sigma^{-1}(x) = xh(u_\sigma)$. Thus z_σ *normalizes* K and by (1.2) z_σ is proportional to v_σ. Say $z_\sigma = \lambda_\sigma v_\sigma$ where $\lambda_\sigma \in K$. Clearly $\lambda_\sigma \in K^*$ since z_σ and v_σ are invertible.

As h is a homomorphism

$$h(u_\sigma u_\tau) = h(u_\sigma)\, h(u_\tau) = \lambda_\sigma v_\sigma \lambda_\tau v_\tau = \lambda_\sigma \sigma(\lambda_\tau)\, g(\sigma, \tau) v_{\sigma\tau}.$$

But it also equals $h(f(\sigma, \tau) u_{\sigma\tau}) = f(\sigma, \tau)\, \lambda_{\sigma\tau} v_{\sigma\tau}$. Thus $f(\sigma, \tau) = \lambda_{\sigma\tau}^{-1} \lambda_\sigma \sigma(\lambda_\tau)$ $g(\sigma, \tau)$ proving that f is equivalent to g.

3. A Characterization of the Sweedler Product

Let A, B be K/k algebras (where k, K are as above). Then $A \otimes_k B$ is, in a natural way, a $K \otimes_k K/k$ algebra. If C is a $K \otimes_k K/k$ algebra, let $\rho: C \to A \otimes_k B$ be a $K \otimes_k K/k$ algebra map. Let $\pi: A \otimes_k B \to A \otimes_K B$ ($\otimes_K$ of left K-modules) be the natural projection. If one endows $A \otimes_k B$ with a K structure via either of the two natural maps $K \to K \otimes_k K$ ($x \to x \otimes 1$, $x \to 1 \otimes x$), then π is a K-module morphism.

Let $\varphi = \pi \circ \rho$. We want to *define* multiplication in the image of φ by the formula $\varphi(c)\, \varphi(c') = \varphi(cc')$. When is it possible? Obviously a necessary and sufficient condition is that $\ker(\varphi)$ is a 2-sided ideal of C. Let us call this product (*). How does the (*) product look in $A \otimes_K B$?

(3.1) Theorem (with the above notation). $\ker(\varphi)$ *is a* 2-*sided ideal if and only if* image$(\varphi) \subset A \times_K B$. *If this is the case then the* (*) *product defined above, in* image(φ), *is the restriction to* image(φ) *of the product in* $A \times_K B$ *defined in Section* 2 (*i.e., Sweedler's*).

In other words if $\varphi(c) = \sum a_i \otimes b_i$ and $\sum a_i x \otimes b_i = \sum a_i \otimes b_i x$, $\varphi(c') = \sum a'_j \otimes b'_j$ and $\sum a'_j x \otimes b'_j = \sum a'_j \otimes b'_j x$ for all $x \in K$ then $\varphi(cc') = \sum a_i a'_j \otimes b_i b'_j$.

Remark. If $x, y \in K$ we denote the image of $x \otimes y$ in C by $x \otimes y$. This is a convenient abuse of notation and should not cause confusion.

Proof. Suppose $\ker(\varphi)$ is a 2-sided ideal. If $c \in C$ let $\rho(c) = \sum a_i \otimes_k b_i$ then $\rho(c(1 \otimes x)) = \rho(c)\, \rho(1 \otimes x) = \sum a_i \otimes_k b_i x$ for $x \in K$, while

$$\rho(c(x \otimes 1)) = \rho(c)\, \rho(x \otimes 1) = \sum a_i x \otimes_k b_i.$$

Thus $\varphi(c(1 \otimes x)) = \sum a_i \otimes_K b_i x$, $\varphi(c \cdot (x \otimes 1)) = \sum a_i x \otimes_K b_i$. It remains to show that they are equal. Now φ is multiplicative, by definition, relative to the $(*)$ product so $\varphi(c(1\otimes x))=\varphi(c)\,\varphi(1\otimes x)$ and we claim that $\varphi(1\otimes x)=\varphi(x\otimes 1)$. This follows easily from the assumption that ρ is a $K\otimes K/k$ algebra morphism. Thus $\sum a_i x\otimes b_i=\sum a_i\otimes b_i x$ and $\varphi(c)\in A\times_K B$.

Conversely suppose $\varphi(C)\subset A\times_K B$. Then if $\varphi(c)=0$ and $c'\in C$ we must prove $\varphi(cc')=\varphi(c'c)=0$. We prove first, that $\varphi(cc')=0$. Write $\varphi(c)=\sum a_i\otimes b_i$ in $A\otimes_K B$. We know that $\sum a_i\otimes_k b_i$ is an element of the form $\sum(y_s\otimes 1-1\otimes y_s)(u_s\otimes v_s)$ with $y_s\in K$. We call such elements (which make up $\ker(\pi)$) *null elements*. Then $\rho(c)\,\rho(c')$ is clearly also null. Thus $\varphi(cc')=\pi(\rho(cc'))=\pi(\rho(c)\,\rho(c'))=0$.

To show that $\varphi(c'c)=0$ is somewhat more complicated. Write $\varphi(c')=\sum_j c_j\otimes_K d_j$, satisfying $(\ddagger)\sum c_j x\otimes d_j=\sum c_j\otimes d_j x$ for $x\in K$. Then

$$\rho(c')=\sum_j c_j\otimes_k d_j+\sum_l(z_l\otimes 1-1\otimes z_l)(f_l\otimes g_l)\qquad(z_l\in K, f_l\in A, g_l\in B).$$

It follows that $\rho(c')\,\rho(c)=\sum_{j,s}(c_j\otimes d_j)(y_s\otimes 1-1\otimes y_s)(u_s\otimes v_s)+\text{null}$ elements. It remains to show that the sum on the right is a null element too. But this sum can be written as $\sum_s(\sum_j c_j y_s\otimes d_j-c_j\otimes d_j y_s)(u_s\otimes v_s)$ and the inner sum is a null element because of $(\ddagger)$. Whence so is the whole sum. This proves that $\varphi(c'c)=0$.

To prove the last statement of the theorem note that we showed above that if $\rho(c)=\sum a_i\otimes_k b_i$, $\rho(c')=\sum a_j\otimes_k b_j$ in $A\otimes_k B$ then since φ satisfies the assumptions (that $\ker(\varphi)$ is a 2-sided ideal and ρ is a $K\otimes K/k$ algebras map) the image $\varphi(c)$ and $\varphi(c')$ satisfy

$$\sum a_i x\otimes_K b_i=\sum a_i\otimes_K b_i x$$

$$\sum a_j' x\otimes_K b_j'=\sum a_j'\otimes_K b_j' x,\qquad \text{all } x\in K.$$

Now, according to the $(*)$ product $\varphi(c)\,\varphi(c')=\varphi(cc')$, so it remains to show that

$$\varphi(cc')=\sum a_i a_j'\otimes_K b_i b_j'.$$

But

$$\varphi(cc')=\pi(\rho(cc'))=\pi(\rho(c)\,\rho(c'))=\pi\left(\sum_{i,j}a_i a_j'\otimes_k b_i b_j'\right)=\sum_{i,j}a_i a_j'\otimes_K b_i b_j'$$

and the proof is complete.

4. A Generalization of Sweedler's Product

Suppose that G acts on the commutative k-algebra K but not faithfully, i.e., the morphism $t: G \to \mathrm{Aut}_k(K)$ has a kernel, H, not $\{1\}$. Then the conclusion of Theorem (2.2) does not hold. We show now that in some cases it is possible to modify the construction $\times_K$ to obtain the desired result. In this section we assume that K is a domain.

As in the introduction we will define the new operation $\times_K$ as $\mathrm{Id}(J)/J$ for an appropriate right ideal J in $K_t^\alpha G \otimes_k K_t^\beta G$. For our procedure to work we need to make a rather restrictive

(4.1) Assumption. Let $\inf: H^2(G/H, K^*) \to H^2(G, K^*)$ be the inflation map. Then either α or β is in the image of inf.

Say $\alpha \in \mathrm{Im}(\inf)$.

Let $K_t^\alpha G = \coprod_{\sigma \in G} K u_\sigma$ with $u_\sigma u_\tau = f(\sigma, \tau) u_{\sigma\tau}$ and $K_t^\beta G = \coprod_{\sigma \in G} K v_\sigma$ with $v_\sigma v_\tau = g(\sigma, \tau) v_{\sigma\tau}$. We assume f and g are normalized and, since f represents an "inflated" element, that f is moreover normalized to satisfy $f(\sigma, \tau) = 1$ if σ or τ are in H.

Let $J \subset K_t^\alpha G \otimes K_t^\beta G$ be the right ideal generated by the set

$$\{x \otimes 1 - 1 \otimes x : x \in K\} \cup \{(u_\sigma - 1) \otimes 1 : \sigma \in H\}.$$

Let $B = \mathrm{Id}(J)$.

(4.2) Theorem. *With the above notation, and assuming* (4.1), $B/J = K_t^{\alpha\beta} G$.

Proof. The main part of the proof is to identify B. To do that we can use the following principles.

(T_1) If $b \in B$ and $x \in K$ then $(x \otimes 1)b$ and $(1 \otimes x)b$ differ by an element of J. Thus elements of K can be moved across the tensor sign (on the left).

(T_2) Similarly if $b \in B$ and $\sigma \in H$ then $(u_\sigma \otimes 1)b$ and b differ by an element of J. Thus u_σ, when multiplying on the left, can be moved across the tensor sign becoming 1 in the process.

We start by identifying some elements of B and then show that these, together with J, make up all of B.

If $D = \{(\sigma, \tau) \in G \times G : \bar\sigma = \bar\tau \text{ in } G/H\}$ we claim that finite sums of the type

$$\sum_{(\sigma, \tau) \in D} a_{\sigma,\tau} u_\sigma \otimes v_\tau \qquad (a_{\sigma,\tau} \in K)$$

are in B. Indeed, if $x \in K$, then

$$\left(\sum_{(\sigma,\tau)\in D} a_{\sigma,\tau} u_\sigma \otimes v_\tau\right)(x\otimes 1 - 1\otimes x)$$
$$= \sum_{(\sigma,\tau)\in D} a_{\sigma,\tau}(u_\sigma x \otimes v_\tau - u_\sigma \otimes v_\tau x)$$
$$= \sum_{(\sigma,\tau)\in D} a_{\sigma,\tau}(\sigma(x)\, u_\sigma \otimes v_\tau - u_\sigma \otimes \tau(x) v_\tau)$$

and by (T_1) this is congruent modulo J to

$$\sum_{(\sigma,\tau)\in D} a_{\sigma,\tau}(\sigma(x) - \tau(x))\, u_\sigma \otimes v_\tau = 0.$$

Similarly, using (T_2), it is seen that if $h \in H$

$$\left(\sum_{(\sigma,\tau)\in D} a_{\sigma,\tau} u_\sigma \otimes v_\tau\right)(u_h - 1)\otimes 1$$

is in J.

In order to prove the converse we first need to establish some terminology and notation. Let $\{\sigma_i\}$ be a fixed set of representatives for the cosets of H in G. (It can be assumed that $\sigma_1 = 1$.) Every element of the type

$$\sum_{(\sigma,\tau)\in D} a_{\sigma,\tau} u_\sigma \otimes v_\tau$$

(which we have just shown to be in B) can be represented, modulo J, by a sum

$$\sum_{(\sigma_i,\tau)\in D} a_{i,\tau} u_i \otimes v_\tau \qquad (u_i = u_{\sigma_i};\ a_{i,\tau} \in K), \tag{*}$$

where the pairs (σ_i, τ) are distinct. This is seen by using transformations of type T_1 and T_2. We denote by B_0 the left K (where $K = K\otimes 1 \subset K\otimes K$) module generated by elements of type (*).

In these terms we are to prove

(4.3) PROPOSITION. (a) $B = B_0 + J$. *Moreover* (b) $B_0 \cap J = \{0\}$. *In other words B is a direct sum of B_0 and J.*

The second part, (b), will be needed to establish the isomorphism of B/J and $K_t^{\alpha\beta} G$.

Proof. We will need to use the existence of a K-linear function

$$\eta : K_t^\alpha G \otimes_k K_t^\beta G \to K_t^\alpha G \otimes_k K_t^\beta G$$

which satisfies for $x, y \in K$

$$\eta(xu_\sigma \otimes yv_\tau) = xyu_{r(\sigma)} \otimes v_\tau,$$

where $r(\sigma) \in \{\sigma_i\}$ represents σ, i.e., $\sigma H = r(\sigma) H$. It is easy to see that such a function exists and is unique.

It is obvious that $\eta \mid B_0$ is the identity of B_0. We claim that η is zero on J. Indeed every element of J is a sum of elements of type $(x \otimes 1 - 1 \otimes x)(au_\sigma \otimes bv_\tau)$, $((u_h - 1) \otimes 1)(yu_\tau \otimes zv_\rho)$, where $x, y, z, a, b \in K$, $h \in H$, and $\sigma, \tau, \rho \in G$. So we only need to check that η vanishes on such elements, and this is a simple computation. But note that the proof that

$$\eta(((u_h - 1) \otimes 1)(yu_r \otimes zv_\rho)) = 0$$

uses the fact that α is an inflation, i.e., that if $h \in H$, $u_h u_\sigma = u_{h\sigma}$ for arbitrary $\sigma \in G$.

Part (b) follows easily now, since, if $b \in B_0 \cap J$ then

$$b = \eta(b) = 0.$$

We now prove (a). Let $\sum_{\sigma, \tau} x_\sigma u_\sigma \otimes y_\tau v_\tau \in B$. We are to show that, modulo J, it is in B_0. Using T_1 and T_2 it can be transformed to an element of the type

$$\sum_{i, \tau} a_{i, \tau} u_i \otimes v_\tau \qquad (u_i = u_{\sigma_i}),$$

where $a_{i, \tau} \in K$ and the pairs (σ_i, τ) are all distinct. Such a sum can be split to its "diagonal" and "non-diagonal" parts:

$$\sum_{i, \tau} a_{i,\tau} u_i \otimes v_\tau = \sum_{(\sigma_i, \tau) \in D} + \sum_{(\sigma_i, \tau) \notin D}.$$

The diagonal part is the first sum on the right and the non-diagonal part is the second sum. Since the diagonal part is already known to be in B_0, we can concentrate on the non-diagonal part. So we assume that we are given a purely non-diagonal element b in B. We shall prove it is necessarily zero. Write

$$b = \sum_{i, \tau} a_{i,\tau} u_i \otimes v_\tau.$$

We collect together those pairs (i, τ) with fixed i and coset τH:

$$b_{i, r(\tau)} = \sum_{\tau H = r(\tau) H} a_{i,\tau} u_i \otimes v_\tau.$$

Then $b=\sum_i \sum_{r(\tau)} b_{i,r(\tau)}$. We want to prove $b_{i,r(\tau)}=0$ for each i and $r(\tau)\in\{\sigma_i\}$.

As $b\in B$, we know that, for $x\in K$, $b(x\otimes 1-1\otimes x)\in J$. Now, $b_{i,r(\tau)}(x\otimes 1-1\otimes x)=(\sigma_i(x)-\tau(x))\,b_{i,r(\tau)}$ and this is non-zero (for some x) if $b_{i,r(\tau)}\neq 0$ since b is purely non-diagonal and K a domain.

We fix i and $r(\tau)$. We will show actually that there exist $\lambda\neq 0$, $\lambda\in K$, such that $\lambda b_{i,r(\tau)}\in J$. As $\eta(\lambda b_{i,r(\tau)})=\lambda b_{i,r(\tau)}$ this would prove that $\lambda b_{i,r(\tau)}$, and hence $b_{i,r(\tau)}$ vanish.

Let $x_0\in K$ be such that $\sigma_i(x_0)\neq\tau(x_0)$ for the pair $(i, r(\tau))$ fixed above. We know that

$$b(x_0\otimes 1-1\otimes x_0)\equiv\sum_j\sum_{r(\tau)}(\sigma_j(x_0)-\tau(x_0))\,b_{j,r(\tau)}=\bar{b}\in J.$$

Here $\equiv$ denotes congruence modulo J.

For each j let

$$c_j=\sum_{r(\tau)}(\sigma_j(x_0)-\tau(x_0))\,b_{j,r(\tau)}.$$

If $j\neq i$ (recall that i was fixed above) let $y_j\in K$ be such that $\sigma_j(y_j)\neq\sigma_i(y_j)$. Note that

$$(\sigma_j(y_j)\otimes 1)\,\bar{b}-\bar{b}(y_j\otimes 1)$$

is a sum of left K multiples of c_j's but that (i) c_j cancels out, (ii) c_i does appear, multiplied by a non-zero factor (from K). If we carry out this process for each j $(j\neq i)$ the end result is a non-zero multiple of c_i. Thus we proved there exists $\lambda\neq 0$, $\lambda\in K$, such that $\lambda c_i\in J$.

It should be noticed that in the computation above we used implicitly (to show that $(y_j\otimes 1)\bar{b}\in J$) the fact, established previously, that $K\otimes_k K\subset B$.

A similar computation, but with $1\otimes y$ instead of $y\otimes 1$ and utilizing T_1, proves now that there exists $\mu\neq 0$, $\mu\in K$, such that $\mu b_{i,r(\tau)}\in J$. As observed above, this completes the proof of the proposition.

Continuation of the proof of (4.2). We describe a homomorphism $\varphi\colon B\to K_t^{\alpha\beta}G$, by defining $\varphi(J)=0$ while if $b=\sum_{(i,\tau)\in D} a_{i,\tau}u_i\otimes v_\tau\in B_0$ then we define

$$\varphi(b)=\sum a_{i,\tau}w_\tau.$$

Here $K_t^{\alpha\beta}G$ is taken as $\coprod_{\tau\in G} Kw_\tau$ with $w_\sigma w_\tau=f(\sigma,\tau)\,g(\sigma,\tau)\,w_{\sigma\tau}$. Clearly $\varphi\,|\,B_0\colon B_0\to K_t^{\alpha\beta}G$ is bijective and K-linear. It remains to prove φ is multiplicative.

It suffices to prove that if $b_1, b_2 \in B_0$ then $\varphi(b_1 b_2) = \varphi(b_1)\,\varphi(b_2)$. Since elements of B_0 are diagonal, we can write

$$b_1 = \sum_\tau a_\tau^{(1)} u_{r(\tau)} \otimes v_\tau, \qquad b_2 = \sum_\sigma a_\sigma^{(2)} u_{r(\tau)} \otimes v_\sigma,$$

where $a_\tau^{(1)}, a_\sigma^{(2)} \in K$ and we recall that $r(\tau) \in \{\sigma_i\}$ is the representative of the coset τH. Clearly

$$\varphi(b_1)\,\varphi(b_2) = \sum_\tau a_\tau^{(1)} w_\tau \cdot \sum_\sigma a_\sigma^{(2)} w_\sigma = \sum_{\tau,\sigma} a_\tau^{(1)} \tau(a_\sigma^{(2)})\, f(\tau, \sigma)\, g(\tau, \sigma)\, w_{\tau\sigma}.$$

On the other hand

$$\begin{aligned} b_1 b_2 &= \sum_{\tau,\sigma} a_\tau^{(1)} \tau(a_\sigma^{(2)})\, f(r(\tau), r(\sigma))\, u_{r(\tau)\cdot r(\sigma)} \otimes g(\tau, \sigma)\, v_{\tau\sigma} \\ &\equiv \sum_{\tau,\sigma} a_\tau^{(1)} \tau(a_\sigma^{(2)})\, f(\tau, \sigma)\, g(\tau, \sigma)\, u_{r(\tau\sigma)} \otimes v_{\tau\sigma} \qquad (\mathrm{mod}\ J). \end{aligned}$$

The last congruence follows from the fact that α is an inflation, a T_2 transformation (to get rid of a u_h) and a T_1 transformation (to move $g(\tau, \sigma)$ across the tensor sign).

It is now easily seen that $\varphi(b_1 b_2) = \varphi(b_1)\,\varphi(b_2)$.

This completes the proof of (4.2).

Finally, to end this section, we discuss briefly the generalization of Theorem (3.1) to the context of working with $\mathrm{Id}(I)/I$ (as described in the introduction). Let A be a k-algebra. It is not assumed that A is a K/k algebra.

Let I be a right ideal in A. We denote by π the canonical projection $A \to A/I$.

(4.4) THEOREM. *Let C be a k-algebra and $\rho: C \to A$ a k-algebra homomorphism whose image contains I. Let $\varphi = \pi \circ \rho: C \to A \otimes_k A/I$. Then* $\ker(\varphi)$ *is a 2-sided ideal if, and only if,* $\mathrm{image}(\varphi) \subseteq \mathrm{Id}(I)$. *If this is the case then φ is an algebra homomorphism into* $\mathrm{Id}(I)/I$.

The proof is left to the reader. It is much the same as the proof of (3.1), and is in fact simpler.

5. EQUIVARIANT PROJECTIVE REPRESENTATIONS

As before we fix an action $t: G \to \mathrm{Aut}_k(K)$ of G on K (over k). We also fix, for the duration of this section, a homomorphism $u: \sigma \to u_\sigma$ from G to the group of *k-linear* automorphisms of the K module KG: u_σ acts on $x\tau$ ($x \in K$, $\tau \in G$) by

$$u_\sigma(x\tau) = \sigma(x)\sigma\tau.$$

This homomorphism is the *equivariant regular representation* of G (given t). In general a $(K/k, \sigma)$ automorphism of KG is a k-linear automorphism $\varphi: KG \to KG$ which satisfies

$$\varphi(xa) = \sigma(x)\,\varphi(a), \qquad x \in K, a \in KG.$$

The set of such automorphisms will be denoted by $\mathrm{Aut}_{K/k,\sigma}(KG)$. The union, over G, of these sets is a group $\mathrm{Aut}_{K/k}(KG)$. For example, the elements u_σ mentioned above lie in $\mathrm{Aut}_{K/k,\sigma}(KG)$.

It is clear that $\mathrm{Aut}_K(KG) = \mathrm{Aut}_{K/k,1}(KG)$ is a normal subgroup of $\mathrm{Aut}_{K/k}(KG)$. Two of its subgroups are important to us. One is K^*, the group of invertible elements of K considered as operators on KG (by left multiplication). The other is the group of "diagonal" K-linear automorphisms of KG, denoted by $D\,\mathrm{Aut}_K(KG)$, and defined by saying that $\varphi \in D\,\mathrm{Aut}_K(KG)$ if relative to the "canonical" basis of KG (i.e., G) its matrix is diagonal. It is easily seen that K^* is *normal* (but not central!) in $\mathrm{Aut}_{K/k}(KG)$. The factor groups obtained by dividing out K^* will be designated by prefixing a P. Thus

$$\mathrm{Aut}_{K/k}(KG)/K^* = P\,\mathrm{Aut}_{K/k}(KG)$$

$$\mathrm{Aut}_K(KG)/K^* = P\,\mathrm{Aut}_K(KG)$$

$$D\,\mathrm{Aut}_K(KG)/K^* = PD\,\mathrm{Aut}_K(KG), \qquad \text{etc.}$$

An *equivariant projective representation* (EPR for short) is a map

$$v: G \to \mathrm{Aut}_{K/k}(KG)$$

such that

(i) $v(\sigma) \in \mathrm{Aut}_{K/k,\sigma}(KG)$ for every $\sigma \in G$, and

(ii) the composition $G \to^{v} \mathrm{Aut}_{K/k}(KG) \to P\mathrm{Aut}_{K/k}(KG)$ is a homomorphism.

It is denoted by $\hat{v}$.

An equivariant projective representation is called *regular* if it satisfies, in addition to (i) and (ii),

(iii) $v(\sigma)(x\tau) = \sigma(x)\,p(\sigma, \tau)\sigma\tau$ with $p(\sigma, \tau) \in K^*$.

If $v: G \to \mathrm{Aut}_{K/k}(KG)$ is an equivariant projective representation then since $\hat{v}$ is a homomorphism $v(\sigma\tau)$ and $v(\sigma)\cdot v(\tau)$ are proportional, with factor of proportionality $f(\sigma, \tau) \in K^*$:

$$v(\sigma)\, v(\tau) = f(\sigma, \tau)\, v(\sigma\tau).$$

The associativity $(v(\sigma)\, v(\tau))\, v(\rho) = v(\sigma)(v(\tau)\, v(\rho))$ means that $f: G \times G \to K^*$ is a cocycle. We call f the "associated cocycle" of v.

We now come to the important notion of *equivalence* of equivariant projective representations. Two equivariant projective representations $v, w: G \to \mathrm{Aut}_{K/k}(KG)$ will be considered equivalent if there is a K-linear automorphism φ of KG and elements $\lambda_\sigma \in K^*$ such that for every $\sigma \in G$, $z \in KG$

$$\varphi(v(\sigma)z) = \lambda_\sigma w(\sigma)(\varphi(z)).$$

This can be rephrased as follows. $P\,\mathrm{Aut}_K(KG)$ acts on $P\,\mathrm{Aut}_{K/k}(KG)$ by "conjugation." If $\varphi \in \mathrm{Aut}_K(KG)$, let $\hat{\varphi}$ be its image in $P\,\mathrm{Aut}_K(KG)$. To say that v is equivalent to w is the same as saying that $\hat{\varphi}\hat{v}\hat{\varphi}^{-1} = \hat{w}$; i.e., $\hat{\varphi}\hat{v}(\sigma)\hat{\varphi}^{-1} = \hat{w}(\sigma)$ for all $\sigma \in G$. It is easily seen that the relation just defined is indeed an equivalence relation. We define an action of G on $\mathrm{Aut}_{K/k}(KG)$ and its subgroup $\mathrm{Aut}_K(KG)$, also by "conjugation": If $\psi \in \mathrm{Aut}_{K/k,\tau}(KG)$ and $\sigma \in G$ then $\sigma\psi = u_\sigma \cdot \psi \cdot u_\sigma^{-1}$. It is immediately seen that $\sigma\psi \in \mathrm{Aut}_{K/k,\sigma\tau\sigma^{-1}}(KG)$. This action of G on $\mathrm{Aut}_{K/k}(KG)$ also normalizes $D\,\mathrm{Aut}_K(KG)$ and K^* and thus defines an action of G on $P\,\mathrm{Aut}_{K/k}(KG)$, $P\,\mathrm{Aut}_K(KG)$, and $PD\,\mathrm{Aut}_K(KG)$.

If $v: G \to \mathrm{Aut}_{K/k}(KG)$ us an equivariant projective representation then, for each $\sigma \in G$, $v(\sigma)u_\sigma^{-1} = \varphi(\sigma) \in \mathrm{Aut}_K(KG)$.

(5.1) LEMMA. *$\hat{\varphi}: G \to P\,\mathrm{Aut}_K(KG)$ is a 1-cocycle.*

This simply means that for $\sigma, \tau \in G$, $\hat{\varphi}(\sigma\tau) = \hat{\varphi}(\sigma)\,\sigma(\hat{\varphi}(\tau))$; we refer the reader to [2, Appendix, p. 123] for general information in non-commutative cohomology.

Proof.

$$\begin{aligned}\varphi(\sigma\tau) &= v(\sigma\tau)u_{\sigma\tau}^{-1} = f^{-1}(\sigma, \tau)\, v(\sigma)\, v(\tau)\, u_\tau^{-1} u_\sigma^{-1}\\ &= f^{-1}(\sigma, \tau)\, \varphi(\sigma)\, u_\sigma \varphi(\tau)\, u_\tau u_\tau^{-1} u_\sigma^{-1}\\ &= f^{-1}(\sigma, \tau)\, \varphi(\sigma)\, \sigma(\varphi(\tau))\end{aligned}$$

so $\hat{\varphi}(\sigma\tau) = \hat{\varphi}(\sigma)\, \sigma(\hat{\varphi}(\tau))$, as required.

This defines a map from the set of equivariant projective representations of G in KG (with fixed action, t, of G on K) into the *set* $H^1(G, P\,\mathrm{Aut}_K(KG))$ by $v \to [\hat{\varphi}] \in H^1(G, P\,\mathrm{Aut}_K(KG))$. If w is an equivariant projective representation equivalent to v, write $w(\sigma) = \psi(\sigma)u_\sigma$, where $\psi(\sigma) \in \mathrm{Aut}_K(KG)$. Let $\eta \in \mathrm{Aut}_K(KG)$ be an automorphism implementing the equivalence $v \sim w$. Thus for every $\sigma \in G$

$$\eta\varphi(\sigma)u_\sigma = \lambda_\sigma \psi(\sigma)\, u_\sigma \eta \qquad (\lambda_\sigma \in K^*)$$

which implies $\eta\varphi(\sigma) = \lambda_\sigma \psi(\sigma)\,\sigma(\eta)$ or $\hat{\eta}\hat{\varphi}(\sigma)\,\sigma(\hat{\eta})^{-1} = \hat{\psi}(\sigma)$. Thus we see that the map $v \to [\hat{\varphi}]$ is constant on equivalence classes.

Let $E_{K/k}(G)$ denote the set of equivalence classes of equivariant projective representations of G in KG. We denote the class of v by $[v]$. The above discussion proves that we have defined a map

$$d\colon E_{K/k}(G) \to H^1(G, P\,\mathrm{Aut}_K(KG)).$$

(5.2) PROPOSITION. *The map d is bijective.*

Proof. We define an inverse to d. Given a class in $H^1(G, P\,\mathrm{Aut}_K(KG))$ choose a representative cocycle, i.e., a map $\theta\colon G \to P\,\mathrm{Aut}_K(KG)$. Choose $\varphi\colon G \to \mathrm{Aut}_K(KG)$ lifting θ, i.e., such that $\hat{\varphi} = \theta$. Let $v\colon G \to \mathrm{Aut}_{K/k}(KG)$ by $v(\sigma) = \varphi(\sigma)u_\sigma$. It is easily seen (since θ is a cocycle) that v is an equivariant projective representation. It remains to prove that the correspondence $[\theta]$ goes to $[v]$ just described is well defined and that it is inverse to d. This is routine and is left to the reader.

The short exact sequence of groups with G action

$$1 \to K^* \to \mathrm{Aut}_K(KG) \to P\,\mathrm{Aut}_K(KG) \to 1$$

gives rise to a connecting homomorphism $\delta\colon H^1(G, P\,\mathrm{Aut}_K(KG)) \to H^2(G, K^*)$. How does the map $\delta \circ d$ look? It is immediately checked that given an equivariant projective representation $v\colon G \to \mathrm{Aut}_{K/k}(KG)$ then $\delta(d[v]))$ is represented by the cocycle $f(\sigma, \tau)$ which is defined by $v(\sigma)\,v(\tau) = f(\sigma, \tau)\,v(\sigma\tau)$, $f(\sigma, \tau) \in K^*$.

(5.3) LEMMA. $\delta\colon H^1(G, P\,\mathrm{Aut}_K(KG)) \to H^2(G, K^*)$ *is onto.*

Proof. Let $f\colon G \times G \to K^*$ be a 2-cocycle. We exhibit an equivariant projective representation v such that $\delta(d([v])) = [f]$. Define $v\colon G \to \mathrm{Aut}_K(KG)$ by $v(\sigma)(\sum_\tau a_\tau \tau) = \sum_\tau \sigma(a_\tau)\, f(\sigma, \tau)\sigma\tau$. This is easily seen to be a *regular* equivariant projective representation satisfying the requirements.

There is an *abelian* subgroup of $P\,\mathrm{Aut}_K(KG)$, $PD\,\mathrm{Aut}_K(KG)$, which we defined above. The inclusion $PD\,\mathrm{Aut}_K(KG) \subsetneq P\,\mathrm{Aut}_K(KG)$ induces a map in cohomology

$$e\colon H^1(G, PD\,\mathrm{Aut}_K(KG)) \to H^1(G, P\,\mathrm{Aut}_K(KG))$$

and the short exact sequence

$$1 \to K^* \to D\,\mathrm{Aut}_K(KG) \to PD\,\mathrm{Aut}_K(KG) \to 1 \qquad (*)$$

induces a connecting *homomorphism* (since $PD\,\mathrm{Aut}_K(KG)$ is abelian)

$$\delta_1 : H^1(G, PD\,\mathrm{Aut}_K(KG)) \to H^2(G, K^*).$$

Clearly the diagram

$$\begin{array}{ccc} H^1(G, PD\,\mathrm{Aut}_K(KG)) & & \\ & \searrow^{\delta_1} & \\ {\scriptstyle e}\downarrow & & H^2(G, K^*) \\ & \nearrow_{\delta} & \\ H^1(G, P\,\mathrm{Aut}_K(KG)) & & \end{array}$$

commutes (this follows from the definition of connecting maps).

(5.4) PROPOSITION. *δ_1 is an isomorphism.*

COROLLARY. *e is injective.*

Note also that the proposition gives another proof for (5.3).

The proposition will follow from an application of "Shapiro's lemma" which we now recall. If M is a G module let M_0 be M as an abelian group but with a trivial G-action. Let $\mathrm{Coind}(M) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G, M)$ with diagonal action of G; i.e., if $h: G \to M$ and $\sigma, \tau \in G$ then $(\sigma h)(\tau) = \sigma(h(\sigma^{-1}\tau))$.

(5.5) LEMMA. $\mathrm{Coind}(M)$ *is cohomologically trivial, i.e.,* $H^i(G, \mathrm{Coind}(M)) = 0$ *for* $i > 0$.

Proof. This is well known for M_0 (see [2, p. 112]), so it suffices to prove that $\mathrm{Coind}(M) \cong \mathrm{Coind}(M_0)$ as G-modules. The isomorphism is similar to the isomorphism described in [2] (see the remark on p. 118): if $h: G \to M$ let $h_0: G \to M_0$ be defined by $h_0(\sigma) = \sigma^{-1}h(\sigma)$ (the latter considered in M_0). This proves the lemma.

(5.6) LEMMA. $D\,\mathrm{Aut}_K(KG) \cong \mathrm{Map}(G, K^*) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G, K^*)$.

Here, too, G acts on $\mathrm{Map}(G, K^*)$ diagonally as above.

Proof. The second isomorphism is easy and given by restriction. To prove the first isomorphism let ρ be a K-linear "diagonal" automorphism of KG. Then $\rho(\tau) = a(\tau)\tau$ with $a(\tau) \in K^*$. To ρ we correspond the map

$h_\rho: \tau \to a(\tau)$. It is clearly bijective, and we must show that $h_{\sigma(\rho)} = \sigma(h_\rho)$. Now $\sigma(\rho) = u_\sigma \circ \rho \circ u_\sigma^{-1}$ so $h_{\sigma(\rho)}(\tau) = \sigma(a(\sigma^{-1}\tau))$ and $\sigma(h_\rho)(\tau) = \sigma(h_\rho(\sigma^{-1}\tau)) = \sigma(a(\sigma^{-1}\tau))$, as required.

Proof of (5.4). According to (5.5) and (5.6), $D \operatorname{Aut}_K(KG)$ is cohomologically trivial. Thus the long cohomology exact sequence of (*) says that for $i > 0$ the connecting homomorphisms

$$H^i(G, PD \operatorname{Aut}_K(KG)) \to H^{i+1}(G, K^*)$$

are isomorphisms. The case $i = 1$ is (5.4).

We wish, finally, to explain what the elements of $H^1(G, PD \operatorname{Aut}_K(KG))$ look like when considered as elements of $E_{K/k}(G)$ (which is identified with $H^1(G, P \operatorname{Aut}_{K/k}(KG))$ via d). Recall that we defined above the notion of a *regular* equivariant projective representation v to be one that is *diagonal* relative to the canonical basis of KG, which is G. If, say, $v(\sigma)\tau = p(\sigma, \tau)(\sigma\tau)$ it does *not* follow that p is a 2-cocycle. If p *is* the 2-cocycle corresponding to v, i.e., such that $v(\sigma)\, v(\tau) = p(\sigma, \tau)\, v(\sigma\tau)$ for all $\sigma, \tau \in G$, then we call v *standard.* Of course to every regular representation corresponds a standard representation in an obvious way.

(5.7) Lemma. *A regular equivariant projective representation and its corresponding standard equivariant projective representation are equivalent.*

Proof. Let $v: G \to \operatorname{Aut}_{K/k}(KG)$ be the regular equivariant projective representation, so that $v(\sigma)\tau = p(\sigma, \tau)\sigma\tau$ and let $f: G \times G \to K^*$ be the cocycle associated with v, so that

$$v(\sigma)\, v(\tau) = f(\sigma, \tau)\, v(\sigma\tau).$$

The corresponding standard equivariant projective representation is the map $w: G \to \operatorname{Aut}_{K/k}(KG)$ defined by $w(\sigma)(x\tau) = \sigma(x)\, f(\sigma, \tau)\sigma\tau$ $(x \in K;\ \sigma, \tau \in G)$. Equivalence of v and w is given by a K-linear automorphism φ of KG satisfying $\varphi(v(\sigma)z) = \lambda_\sigma w(\sigma)\, \varphi(z)$ for some $\lambda_\sigma \in K^*$. If we try to define a *diagonal* such automorphism $\varphi(\tau) = c_\tau \tau$ with $c_\tau \in K^*$ we see that the "constant" c_τ must satisfy (taking $z = 1 \in KG$)

$$p(\sigma, 1)c_\sigma = \lambda_\sigma c_1\, f(\sigma, 1).$$

Thus we may try if the choice $c_\sigma = f(\sigma, 1)/p(\sigma, 1)$ does the job. Note that $f(1, 1) = p(1, 1)$ will follow from the computation below so that $c_1 = 1$. We need to check if

$$\varphi(v(\sigma)\tau) = \varphi(\, p(\sigma, \tau)\sigma\tau) = p(\sigma, \tau)\, f(\sigma\tau, 1)\, p(\sigma\tau, 1)^{-1}\sigma\tau$$

(where $\sigma, \tau \in G$) is equal to

$$w(\sigma)\varphi(\tau) = w(\sigma)(f(\tau, 1)\, p(\tau, 1)^{-1}\tau) = \sigma(f(\tau, 1)\, p(\tau, 1)^{-1})\, f(\sigma, \tau)\sigma\tau.$$

Thus we need to compare $f(\sigma\tau, 1)\, p(\sigma, \tau)\, p(\sigma\tau, 1)^{-1}$ with $f(\sigma, \tau)\, \sigma(f(\tau, 1))\, \sigma(p(\tau, 1)^{-1})$. Collecting f's and p's together we ask if the equality

$$f(\sigma, \tau)\, \sigma(f(\tau, 1))\, f(\sigma\tau, 1)^{-1} = p(\sigma, \tau)\, \sigma(p(\tau, 1))\, p(\sigma\tau, 1)^{-1}$$

holds. As f is a cocycle we see that the relation $df(\sigma, \tau, 1) = 1$ implies that the left hand side equals $f(\sigma, \tau)$. Now the associativity $v(\sigma)(v(\tau)1) = (v(\sigma)\, v(\tau))(1)$ implies exactly

$$\sigma(p(\tau, 1))\, p(\sigma, \tau) = f(\sigma, \tau)\, p(\sigma\tau, 1)$$

which is the required equality.

Now if $f: G \times G \to K^*$ is a 2-cocycle, it is easy to see that if v is the standard equivariant projective representation associated with f (i.e., $v(\sigma)\tau = f(\sigma, \tau)\sigma\tau$) the image of $[v]$ under $\delta \circ d$ is $[f]$. Combined with (5.7) we get

(5.8) PROPOSITION. *$d^{-1}H^1(G, PD\,\mathrm{Aut}_K(KG))$ is the set of equivalence classes of regular equivariant projective representations.*

REFERENCES

1. E. ALJADEFF AND S. ROSSET, Global dimensions of crossed products, *J. Pure Appl. Algebra* **40** (1986), 103–113.
2. J. P. SERRE, "Local Fields," Springer-Verlag, New York/Berlin, 1979.
3. M. E. SWEEDLER, Groups of simple algebras, *Publ. Math. IHES* **44** (1974), 79–190.