Graded identities of matrix algebras and the universal graded algebra

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1   Introduction.

In the last decade, group gradings and graded identities of finite dimensional central simple algebras have been an active area of research. We refer the reader to Bahturin, et al [6] and [7]. There are two basic kinds of group grading, elementary and fine. It was proved by Bahturin and Zaicev [7] that any group grading of $M_n(\mathbb{C})$ is given by a certain composition of an elementary grading and a fine grading. In this paper we are concerned with fine gradings on $M_n(\mathbb{C})$ and their corresponding graded identities.

Let R be a simple algebra, finite dimensional over its center k and G a finite group. We say that R is fine graded by G if $R \cong \oplus_{g \in G} R_g$ is a grading and $\dim_k(R_g) \leq 1$. Thus any component is either 0 or isomorphic to k as a k–vector space. It is easy to show that $\text{Supp}(R)$, the subset of elements of G for which R_g is not 0, is a subgroup of G. Moreover

R is strongly graded by $\operatorname{Supp}(R)$, namely $R_g R_h = R_{gh}$ for every $g, h \in \operatorname{Supp}(R)$. Since every group Γ containing $\operatorname{Supp}(R)$ provides a fine grading of R (just by putting $R_g = 0$ for g outside $\operatorname{Supp}(R)$) we will restrict our attention to the case where $G = \operatorname{Supp}(R)$. In this case it has been shown (see Bahturin and Zaicev [7]) that the existence of such a grading is equivalent to R being isomorphic to a twisted group algebra $k^c G$ where $[c]$ is an element in $H^2(G, k^\times)$ and G acts trivially on $k^\times$.

A group G is said to be of central type if it admits a cocycle c, $[c] \in H^2(G, \mathbb{C}^\times)$, for which the twisted group algebra $\mathbb{C}^c G$ is central simple over $\mathbb{C}$. By definition such a cocycle c is called nondegenerate. As we have just seen fine gradings arise from such groups and cocycles. Groups of central type appear in the theory of projective representations of finite groups and in the classification of finite dimensional Hopf algebras. Using the classification of finite simple groups, Howlett and Isaacs [12] proved that any group of central type is solvable.

Given a (fine) G–grading on $M_n(\mathbb{C})$ we consider the set of graded identities: let k be a subfield of $\mathbb{C}$ and let $\Omega = \{x_{ig} : i \in \mathbb{N}, g \in G\}$ be a set of indeterminates. Let $\Sigma(k) = k\langle \Omega \rangle$ be the noncommutative free algebra generated by Ω over k. A polynomial $p(x_{ig}) \in \Sigma(k)$ is a graded identity of $M_n(\mathbb{C})$ if the polynomial vanishes upon every substitution of the indeterminates x_{ig} by elements of degree g in $M_n(\mathbb{C})$. It is clear that the set of graded identities is an ideal of $\Sigma(k)$. Furthermore, it is a graded T–ideal. Recall that an ideal (in $\Sigma(k)$) is a graded T–ideal if it is closed under all G–graded endomorphisms of $\Sigma(k)$. In section two we show that all graded identities are already defined over a certain finite cyclotomic field extension $\mathbb{Q}(\mu)$ of $\mathbb{Q}$. Moreover $\mathbb{Q}(\mu)$ is minimal and unique with that property. We refer to $\mathbb{Q}(\mu)$ as the field of definition of the graded identities. We consider the free algebra $\Sigma(\mathbb{Q}(\mu))$ and denote by $T(\mathbb{Q}(\mu))$ the T–ideal of graded identities of $M_n(\mathbb{C})$ in $\Sigma(\mathbb{Q}(\mu))$. We introduce a set of special graded identities which we call "elementary". We show that a certain finite subset of the set of elementary identities generate $T(\mathbb{Q}(\mu))$. In particular $T(\mathbb{Q}(\mu))$ is finitely generated as a T–graded ideal.

In section three we examine the algebra $U_G = \Sigma(\mathbb{Q}(\mu))/T(\mathbb{Q}(\mu))$, which we call the universal G–graded algebra. First we introduce another algebra analogous to the ring of generic matrices in the classical theory. We show this algebra is isomorphic to U_G and thus are able to prove that the center $Z = Z(U_G)$ is a domain and that if F denotes the field of fractions of Z, then the algebra $Q(U_G) = F \otimes_Z U_G$ is an F–central simple algebra of dimension equal to the order of G. In particular U_G is a prime ring. Moreover we show there is a certain multiplicatively closed subset M in Z such that the central localization $M^{-1} U_G$ is an Azumaya algebra over its center $S = M^{-1} Z$. The simple images of this Azumaya algebra are the graded forms of $M_n(\mathbb{C})$, that is the G–graded central simple L–algebras B such that $B \otimes_L \mathbb{C}$ is isomorphic as a graded algebra to $M_n(\mathbb{C})$, where L varies over all subfields of $\mathbb{C}$. We also give quite explicit determinations of S and F.

Note that these algebras depend on the given grading on $M_n(\mathbb{C})$ and so we should write $U_{G,c}$, $M^{-1}U_{G,c}$ and $Q(U_{G,c})$, where c is the given nondegenerate two-cocycle. We will omit the c except in those cases where we need to emphasize the particular grading.

Unlike the case of classical polynomial identities, the central simple algebra $Q(U_G)$ is not necessarily a division algebra. Based on earlier work of the authors [2] we in fact show that $Q(U_{G,c})$ is a division algebra if and only if the group G belongs to a certain explicit list Λ of groups. This is independent of the cocycle c, a fact we will return to below. The list Λ consists of a very special family of nilpotent groups. Roughly speaking these are the nilpotent groups for which each Sylow-p subgroup is the direct product of an abelian group of the form $A \times A$ (called of symmetric type) and possibly a unique nonabelian group of the form $C_{p^n} \ltimes C_{p^n}$. For $p = 2$ an extra family of non abelian groups can occur, namely $C_2 \times C_{2^{n-1}} \ltimes C_{2^n}$. The precise definition of Λ is given in the last section.

The main result of the final section is that for every group G on the list Λ the automorphism group of G acts transitively on the cohomology classes represented by nondegenerate two-cocycles. It follows that the algebras $Q(U_{G,c})$ are all isomorphic for a fixed G (but not graded isomorphic).

For a group G of central type let $\mathrm{ind}(G)$ denote the maximum over all nondegenerate cocycles c of the indices of the simple algebras $Q(U_{G,c})$. We have just seen that if G is not on the list, then $\mathrm{ind}(G)$ is strictly less than the order of G. In a forthcoming paper, Aljadeff and Natapov [4], it is shown that in fact the groups on the list are the only groups responsible for the index. The precise result is stated in the last section (Theorem 21). It follows from this theorem that if G is a p-group of central type then $\mathrm{ind}(G) \leq \max(\mathrm{ord}(H)^{1/2})$ where the maximum is taken over all p-groups H on the list that are sub-quotients of G.

If $Q(U_G)$ is not a division algebra, there are graded identities over the field of definition that are the product of nonidentities. We present an explicit example of a grading on $M_6(\mathbb{C})$ (for the group $G = S_3 \ltimes C_6$) for which there is a graded identity over the field of definition that is the cube of a nonidentity.

Even in the case where $Q(U_G)$ is a division algebra it is possible that this universal algebra does not remain a division algebra under extension of the coefficient field. In other words the algebra $Q(U_G) \otimes_{\mathbb{Q}(\mu)} \mathbb{C}$ may not be a division algebra. This means that there can be gradings on matrices for which the algebra $Q(U_G)$ is a division algebra and yet there are graded identities over $\mathbb{C}$ that are the products of nonidentities. In the last section we compute explicitly the index of this extended algebra $Q(U_G) \otimes_{\mathbb{Q}(\mu)} \mathbb{C}$ for groups G on the list Λ.

2   Graded Identities.

Let $M_n(\mathbb{C})$ have a fine grading by the finite group G. In this section we investigate the T–ideal of G–graded identities on $M_n(\mathbb{C})$. As we have seen in the introduction the existence of such a grading implies that G is of central type and that there is a nondegenerate cocycle c such that $M_n(\mathbb{C})$ is isomorphic to the twisted group algebra $\mathbb{C}^c G$. We want to make this more precise. Let $A = M_n(\mathbb{C})$. The G–grading induces a decomposition $A \cong \oplus A_g$ where each homogeneous component is 1-dimensional over $\mathbb{C}$. Every homogeneous component A_g is spanned by an invertible element u_g (which we also fix from now on) and hence any element in A_g is given by $\lambda_g u_g$ where $\lambda_g \in \mathbb{C}$. The multiplication is given by $u_g u_h = c(g,h) u_{gh}$ where $c(g,h) \in \mathbb{C}^\times$ is a two-cocycle. We let $[c] \in H^2(G, \mathbb{C}^\times)$ be the corresponding cohomology class. In this way we identify A (as a G-graded algebra) with the twisted group algebra $\mathbb{C}^c G$. Replacing c by a cohomologous cocycle c' produces a twisted group algebra $\mathbb{C}^{c'} G$ that is isomorphic as a G–graded algebra to A.

We want to obtain information about the T–ideal of identities of $M_n(\mathbb{C})$. We begin with a special family of identities. As in the introduction we let $\Sigma(\mathbb{C}) = \mathbb{C}\langle \Omega \rangle$ denote the free algebra generated by Ω over $\mathbb{C}$, where $\Omega = \{x_{ig} : i \in \mathbb{N}, g \in G\}$.

Let $Z_1 = x_{r_1 g_1} x_{r_2 g_2} \cdots x_{r_k g_k}$ and $Z_2 = x_{s_1 h_1} x_{s_2 h_2} \cdots x_{s_l h_l}$ be two monomials in $\Sigma(\mathbb{C})$. We say Z_1 and Z_2 are *congruent* if the following three properties are satisfied:

a) $g_1 \cdots g_k = h_1 \cdots h_k$

b) $k = l$ (equal length)

c) there exists an element $\pi \in Sym(k)$ such that for $1 \leq i \leq k$, $s_i = r_{\pi(i)}$ and $h_i = g_{\pi(i)}$.

We say Z_1 and Z_2 are *weakly congruent* if they satisfy condition (a). Clearly congruence and weakly congruence are equivalence relations.

Let $c(g_1, g_2, \ldots, g_k)$ be the element in $\mathbb{C}^\times$ determined by $u_{g_1} u_{g_2} \cdots u_{g_k} = c(g_1, \ldots, g_k) u_{g_1 \cdots g_k}$ in $\mathbb{C}^c G$. For any two congruent monomials $Z_1 = x_{r_1 g_1} x_{r_2 g_2} \cdots x_{r_k g_k}$ and $Z_2 = x_{r_{\pi(1)} g_{\pi(1)}} x_{r_{\pi(2)} g_{\pi(2)}} \cdots x_{r_{\pi(k)} g_{\pi(k)}}$ consider the binomial

$$B = B(Z_1, Z_2) = Z_1 - \frac{c(g_1, \ldots, g_k)}{c\left(g_{\pi(1)}, \ldots, g_{\pi(k)}\right)} Z_2$$

It is easy to see that B is a graded identity. We will call it an *elementary* identity. We refer to $c(B) = \frac{c(g_1, \ldots, g_k)}{c(g_{\pi(1)}, \ldots, g_{\pi(k)})}$ in $\mathbb{C}^\times$ as the coefficient of the elementary identity B.

This element $c(B)$ can be understood homologically. Let $F = F(\Omega)$ be the free group generated by $\Omega = \{x_{ig} : i \in \mathbb{N}, g \in G\}$. Let $\nu : F \longrightarrow G$ ($x_{ig} \longmapsto g$) be the natural map and $R = \ker(\nu)$. If $\pi \in Sym(k)$ and $Z_1 = x_{r_1 g_1} x_{r_2 g_2} \cdots x_{r_k g_k}$, $Z_2 = x_{r_{\pi(1)} g_{\pi(1)}} x_{r_{\pi(2)} g_{\pi(2)}} \cdots x_{r_{\pi(k)} g_{\pi(k)}}$ are congruent, then the element $z = Z_1 (Z_2)^{-1}$ lies in $[F, F] \cap$

R. By the Hopf formula the Schur multiplier $M(G)$ of G is given by $M(G) = [F, F] \cap R/[F, R]$ and so we may consider the element $\overline{z} \in M(G)$. Moreover the Universal Coefficients Theorem provides an isomorphism ϕ from $H^2(G, \mathbb{C}^\times)$ to $\mathrm{Hom}(M(G), \mathbb{C}^\times)$,

PROPOSITION 1. *Let the notation be as given above.*

(a) *The element $c(B)$ is independent of the representing cocycle c of $[c]$.*

(b) *If $z = Z_1(Z_2)^{-1}$ is as given above then $c(B) = \phi([c])(\overline{z})$.*

(c) *Each $c(B)$ is a root of unity, in fact $c(B)^{\mathrm{ord}(G)} = 1$.*

(d) *If J denotes the set of all elementary identities then the set $\mu = \{c(B) \in \mathbb{C}^\times : B \in J\}$ is finite.*

Proof. Statement (a) is clear. For the proof of (b) we consider the central extension

$$\{1\} \to R/[F, R] \to F/[F, R] \to G \to \{1\}$$

induced by the map ν and let

$$\{1\} \to \mathbb{C}^\times \to \Gamma \to G \to \{1\}$$

be the central extension that corresponds to the cocycle c (i.e. $\{u_g\}_{g \in G}$ is a set of representatives in Γ and $u_g u_h = c(g, h) u_{gh}$). It is easy to see that the map $\gamma : F/[F, R] \to \Gamma$ given by $\gamma(x_{ig}) = u_g$ induces a map of extensions and it is well known its restriction $\gamma : [F, F] \cap R/[F, R] = M(G) \to \mathbb{C}^\times$ is independent of the presentation and corresponds to $[c]$ by means of the Universal Coefficient Theorem. That is $\phi([c]) = \gamma$. It follows that $\phi([c])(\overline{z}) = \gamma(\overline{z}) = u_{g_1} \cdots u_{g_r} (u_{g_{\pi(1)}} \cdots u_{g_{\pi(r)}})^{-1} = \frac{c(g_1, \ldots, g_k)}{c(g_{\pi(1)}, \ldots, g_{\pi(k)})} = c(B)$ as desired.

Statements (c) and (d) are direct consequences of (b). This completes the proof of the proposition. ∎

From the Proposition it follows that the set J is defined over $Q(\mu)$. Our next task is to show that the the T–ideal of graded identities $T(\mathbb{C})$ is spanned by J and that $Q(\mu)$ is the field of definition for the graded identities. We will need the following terminology:

Let k be a subfield of $\mathbb{C}$ and let $p(x_{r_i g_i}) = \sum \lambda_{r_1 g_1, r_2 g_2, \ldots, r_k g_k} x_{r_1 g_1} x_{r_2 g_2} \cdots x_{r_k g_k}$ be a polynomial in $\Sigma(k)$ ($\subseteq \Sigma(\mathbb{C})$). We say $p(x_{r_i g_i})$ is *reduced* if the monomials $x_{r_1 g_1} x_{r_2 g_2} \cdots x_{r_k g_k}$ are all different. If $p(x_{r_i g_i})$ is reduced we say it is weakly homogeneous if its monomials are all weakly congruent and we say $p(x_{r_i g_i})$ is homogeneous is if its monomials are all congruent. It is clear that any reduced polynomial can be written uniquely as the sum of its weakly homogeneous components and that each weakly homogeneous component can be written uniquely as the sum of its homogeneous components.

PROPOSITION 2. *If $p(x_{r_i g_i})$ is a graded identity then its homogeneous components are graded identities as well. Moreover, any graded identity is a linear combination of elementary identities.*

Proof. Let us show first that its weakly homogeneous components are graded identities. Indeed, the replacement of $x_{r_i g_i}$ by $u_{g_i} \in \mathbb{C}^c G$ maps weakly homogeneous components of $p(x_{r_i g_i})$ to different homogeneous components (in the graded decomposition of $\mathbb{C}^c G$). Since these are linearly independent, they all must be 0. Next, assume that we have at least two homogeneous components in a weakly homogeneous component which we denote by Π. Then there exists an indeterminate $y = x_{r_j g_j}$ which appears with different multiplicities in (at least) two different monomials. Let $s \geq 1$ be the maximal multiplicity. We decompose Π into (at most) $s + 1$ components $U_s, \ldots, U_0$ where U_i consists of the monomials of Π that contain y with multiplicity i. Of course, by induction, the result will follow if we show that U_s is a graded identity. Suppose not. Then there is a substitution which does not annihilate U_s. Note that since Π is weakly homogeneous the image of all components are multiples of u_g in $\mathbb{C}^c G$ and hence there is an evaluation that maps U_i to $\lambda_i u_g$ with λ_s not zero. We may multiply the evaluation for y by a central indeterminate z. Then we get a non zero polynomial in z, whose coefficients are λ_i and for any evaluation of z we get zero. This is of course impossible in a field of characteristic zero.

To complete the proof of the proposition it suffices to show that every homogeneous identity is a linear combination of elementary identities. But this is clear since any two monomials which are congruent determine an elementary identity (and monomials are not identities). ∎

PROPOSITION 3. *If L is a subfield of $\mathbb{C}$ such that $\mathbb{Q}(\mu) \subseteq L \subseteq \mathbb{C}$ then the set J spans $T(L)$ over L. Conversely, if $L \subseteq \mathbb{C}$ is a field of definition for $T(\mathbb{C})$, that is $T(\mathbb{C}) = T(L) \otimes_L \mathbb{C}$, then L contains $\mathbb{Q}(\mu)$.*

Proof. Fix a monomial $Z = x_{r_1 g_1} x_{r_2 g_2} \cdots x_{r_k g_k}$. Let $Z = Z_1, Z_2 \ldots, Z_d$ be the distinct monomials that are congruent to Z and let W be the d-dimensional $\mathbb{C}$–vector space spanned by these monomials. Each Z_i, for $1 \leq i < d$ determines an elementary identity $Z_i - c_i Z_d$ and these identities form a basis for the subspace Y of graded identities in W. In particular the dimension of Y over $\mathbb{C}$ is $d - 1$. Furthermore if $\gamma = \{c_1, \ldots, c_{d-1}\}$ then clearly Y is defined over $\mathbb{Q}(\gamma)$. But $\gamma \subseteq \mu$ and so it follows that $T(\mathbb{C})$ is defined over $\mathbb{Q}(\mu)$ and hence over any field L that contains $\mathbb{Q}(\mu)$.

For the converse let W_L denote the L–span of the monomials $Z_1, \ldots, Z_d$ and let $Y_L = Y \cap W_L$. We claim that if L is a field of definition for Y (that is, $Y = Y_L \otimes_L \mathbb{C}$), then $L \supseteq \mathbb{Q}(\gamma)$. Because Z is arbitrary, it will follow from the claim that L contains μ. To

prove the claim, let $f_1, \ldots, f_{d-1}$ be a basis of Y_L over L. Express $f_1, \ldots, f_{d-1}$ using the monomials Z_i. The coefficient matrix is a $(d-1) \times d$ matrix over L. This matrix may be row reduced to the normal form (I_{d-1}, C') where C' is a $(d-1) \times 1$ matrix over L. The column vector C' is uniquely determined. In other words there are unique scalars a_i in L, for $1 \le i < d$, such that the vectors $Z_i - a_i Z_d$ form a basis for Y_L over L. On the other hand the identities $Z_i - c_i Z_d$ form a basis of $Y_{L(\gamma)}$ over $L(\gamma)$. It follows that $a_i = c_i \in L$ for all i. ∎

We now show that the T–ideal of graded identities is finitely generated. Let $n = \mathrm{ord}(G)$. Consider the set $V = \{x_{ig} : 1 \le i \le n \ , g \in G\}$ of indeterminates and let E be the set of elementary identities of length $\le n$ (that is where the monomials are of length $\le n$) and such that its indeterminates are elements of V. Clearly E is a finite set.

THEOREM 4. *The ideal of graded identities is generated as a T–ideal by E. In particular the ideal is finitely generated as a T–ideal.*

Proof. Denote by I' the T–ideal generated by E. We will show that I' contains all graded identities. Clearly it is enough to show that I' contains all elementary identities. Let $B = x_{r_1 g_1} x_{r_2 g_2} \cdots x_{r_k g_k} - c(B) x_{r_{\pi(1)} g_{\pi(1)}} x_{r_{\pi(2)} g_{\pi(2)}} \cdots x_{r_{\pi(k)} g_{\pi(k)}}$ be an elementary identity of length k. Clearly, if $k \le n$, then B is in I'. We assume therefore that $k > n$ and proceed by induction on k. The first observation is that if there exist i, t, with $1 \le i, t \le k-1$ $i = \pi(t)$ and $i+1 = \pi(t+1)$ then we can reduce the length of the word: We let $g = g_i g_{i+1}$ and replace $x_{r_i g_i} x_{r_{i+1} g_{i+1}}$ by x_{rg}, where $1 \le r \le n$ and x_{rg} does not appear in B. The resulting identity (one has to check that the coefficient is right) has shorter length and so we may assume this identity is in I'. But we can obtain the longer one from the shorter one by the substitution of $x_{r_i g_i} x_{r_{i+1} g_{i+1}}$ for x_{rg}, so we are done in this case.

Next, note that since $k > n$, the pigeonhole principle applied to the expressions $g_1 \cdots g_s$, $s = 1, \ldots, k$ shows that there are integers $1 \le i < j \le n$ such that $g_i g_{i+1} \cdots g_j = 1$. Observe also, that if γ is a cyclic permutation of the numbers $i, i+1, \ldots, j$ then we have $g_{\gamma(i)} \cdots g_{\gamma(j)} = 1$. That means that if we replace the string $x_{r_i g_i} x_{r_{i+1} g_{i+1}} \cdots x_{r_j g_j}$ by a cyclic permutation of the variables in the first monomial and leave the second monomial alone, we do not change our identity modulo I'. Also for all $g \in G$, we have $g g_i g_{i+1} \cdots g_j = g_i g_{i+1} \cdots g_j g$. This means that modulo I' we can move the string $x_{r_i g_i} x_{r_{i+1} g_{i+1}} \cdots x_{r_j g_j}$ anywhere in the monomial. So, combining these two things we can cyclically move the substring and move it anywhere in the first monomial. But by doing these moves we can arrange it so that some two variables that are next to each other in the second monomial are also next to each other (in the same order) in the first monomial. This reduces us to the first case. ∎

7

3 The Universal Algebra.

In this section we determine the basic structural properties of the universal G–graded algebra U_G and relate that algebra to the other algebras described in the introduction. The results are analogous to results in the theory of (non-graded) polynomial identities. We will see for example that U_G is a prime ring and that its algebra of central quotients, the universal G–graded algebra, is central simple.

We start with the following lemma.

LEMMA 5. *Let c be a nondegenerate cocycle on G and let $\mathbb{C}^c G$ be the twisted group algebra. Let L be a subfield of $\mathbb{C}$.*

(a) Let $\beta : G \times G \to L^\times$ be a two-cocycle and let $L^\beta G$ be the twisted group algebra. There is a homomorphism $\eta : L^\beta G \to \mathbb{C}^c G$ over L of G–graded algebras if and only if the cocycles c and β are cohomologous in $\mathbb{C}$. In particular $\mathrm{Im}(\eta)$ is a G–graded form of $\mathbb{C}^c G$.

(b) There exists a two cocycle $\beta : G \times G \to L^\times$ cohomologous to c over $\mathbb{C}$ if and only if L contains μ, the set of Proposition 1.

Proof. (a) If $\eta : L^\beta G \to \mathbb{C}^c G$ is a homomorphism of G-graded L–algebras then there are bases $\{v_\sigma\} \subset L^\beta G$, $\{u_\sigma\} \subset \mathbb{C}^c G$, and scalars $\{\lambda_\sigma\} \subset \mathbb{C}^\times$ such that $v_\sigma v_\tau = \beta(\sigma, \tau) v_{\sigma\tau}$, $u_\sigma u_\tau = c(\sigma, \tau) u_{\sigma\tau}$ and $\eta(v_\sigma) = \lambda_\sigma u_\sigma$. It follows that β and c are cohomologous over $\mathbb{C}$. A similar calculation shows the other direction.

(b) We have seen in Proposition 1 that $\phi([c]) = \mu$ where ϕ is the isomorphism between $H^2(G, \mathbb{C}^\times)$ and $\mathrm{Hom}(M(G), \mathbb{C}^\times)$. Hence if β is cohomologous to c over $\mathbb{C}$ then L must contain μ by part (d) of Proposition 1.

For the converse we may assume that $L = \mathbb{Q}(\mu)$. By the naturality of the Universal Coefficient Theorem we have a commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & Ext^1(G_{ab}, \mathbb{Q}(\mu)) & \longrightarrow & H^2(G, \mathbb{Q}(\mu)^\times) & \longrightarrow & \mathrm{Hom}(M(G), \mathbb{Q}(\mu)^\times) & \longrightarrow & 1 \\
& & & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \bar{\nu}} & & \\
& & & & H^2(G, \mathbb{C}^\times) & \xrightarrow{\phi} & \mathrm{Hom}(M(G), \mathbb{C}^\times) & &
\end{array}
$$

Clearly $\phi([c]) \in \mathrm{Hom}(M(G), \mathbb{C}^\times)$ is in $\mathrm{Im}(\bar{\nu})$ and hence $[c]$ is in $\mathrm{Im}(\nu)$. ∎

The first step in the analysis of U_G is the construction of a counterpart to the ring of generic matrices. Let G be a group of central type of order n^2 and let $c : G \times G \to \mathbb{C}^\times$ be a nondegenerate two-cocycle. For every $g \in G$ let t_{ig} for $i = 1, 2, 3, \ldots$ be indeterminates.

For each $g \in G$ let $t_g = t_{1g}$. We will assume $u_1 = 1$ (i.e. the cocycle c is normalized). Let k denote the field generated over $\mathbb{Q}$ by the indeterminates and the values of the cocycle. Note that by Lemma 5, k contains $\mathbb{Q}(\mu)$. Consider the twisted group algebra $k^c G$. Because the cocycle is nondegenerate $\mathbb{Q}(c(g,h))^c G$ is a central simple $\mathbb{Q}(c(g,h))$–algebra and hence $k^c G$ is a central simple k–algebra. Now let $\overline{U_G}$ denote the $\mathbb{Q}(\mu)$–subalgebra of $k^c G$ generated by the elements $t_{ig} u_g$ for all i and all $g \in G$. We want to describe $\overline{U_G}$. Let $\overline{Z}$ denote the center of $\overline{U_G}$.

PROPOSITION 6. *(a) The center $\overline{Z}$ of $\overline{U_G}$ is the $\mathbb{Q}(\mu)$–subalgebra of $\overline{U_G}$ generated by the following set of elements:*

$$\{t_{i_1 g_1} u_{g_1} t_{i_2 g_2} u_{g_2} \cdots t_{i_m g_m} u_{g_m} \mid g_1 g_2 \cdots g_m = 1\}$$

(b) We have $\overline{Z} \subseteq k$. In particular $\overline{Z}$ is an integral domain.

Proof. The ring $k\overline{U_G}$ contains the elements u_g for all $g \in G$ and so $k\overline{U_G} = k^c G$, which has center k. Hence the center of $\overline{U_G}$ is $\overline{U_G} \cap k$. But this intersection is precisely the $\mathbb{Q}(\mu)$ span of the given set of monomials. This proves both parts. ∎

Now let $s : G \times G \to k^\times$ be the function given by $s(g,h) = \frac{t_g t_h}{t_{gh}} c(g,h)$. This is a two-cocycle, cohomologous to c over k. Because c is nondegenerate, so is s and hence the algebra $k^s G$ is k–central simple. In fact it is convenient to view the algebra $k^s G$ as equal to $k^c G$. It is spanned over k by the elements $t_g u_g$.

Let Y be the subgroup of $k^\times$ generated by the values of s and let $\overline{S} = \mathbb{Q}(\mu)[Y]$, a subring of k. The $\overline{S}$–subalgebra of $k^s G$ generated by the elements $t_g u_g$ is the twisted group algebra $\overline{S}^s G$. Let $\overline{M} = \{t_{g_1} u_{g_1} t_{g_2} u_{g_2} \cdots t_{g_m} u_{g_m} \mid g_1 g_2 \cdots g_m = 1\}$. Let $\overline{F} \subseteq k$ denote the field of fractions of $\overline{Z}$.

PROPOSITION 7. *(a) The set $\overline{M}$ is a multiplicatively closed subset of $\overline{Z}$ and $\overline{M}^{-1} \overline{Z} = \overline{S}[\frac{t_{ig}}{t_g} \mid i \geq 1, g \in G]$.*

(b) We have $\overline{U_G}(\overline{M}^{-1} \overline{Z}) = \overline{S}^s G(\overline{M}^{-1} \overline{Z})$. Moreover $\overline{U_G} \overline{F} = \overline{S}^s G \overline{F} = \overline{F}^s G$.

(c) $\overline{F}^s G$ is a central simple $\overline{F}$–algebra.

(d) The ring $\overline{U_G}$ is a prime ring with ring of central quotients $Q(\overline{U_G})$ isomorphic to the central simple algebra $\overline{F}^s G$.

Proof. (a) We first show $\overline{M}^{-1} \overline{Z} \supseteq \overline{S}[\frac{t_{ig}}{t_g} \mid i \geq 1, g \in G]$. Observe that t_e and $t_g t_{g^{-1}} c(g, g^{-1})$ are in $\overline{M}$ and hence $t_{ig} t_{g^{-1}} c(g, g^{-1}) / t_g t_{g^{-1}} c(g, g^{-1}) = t_{ig}/t_g$ is in $\overline{M}^{-1} \overline{Z}$. Next, the elements $s(g,h)^{\pm 1}$ are in $\overline{M}^{-1} \overline{Z}$ because the elements

$t_g u_g t_h u_h t_{(gh)^{-1}} u_{(gh)^{-1}} = t_g t_h t_{(gh)^{-1}} c(g,h) c(gh, (gh)^{-1})$ and

$t_{gh} u_{gh} t_{(gh)^{-1}} u_{(gh)^{-1}} = t_{gh} t_{(gh)^{-1}} c(gh, (gh)^{-1})$ are in $\overline{M}$. Similar calculations show that the opposite inclusion also holds.

(b) This follows easily from part (a).

(c) We have $\overline{F}^s G \otimes_{\overline{F}} k = k^s G$ which is k–central simple and so $\overline{F}^s G$ is $\overline{F}$–central simple.

(d) By part (b) we have $\overline{U_G F} = \overline{S}^s G \overline{F} = \overline{F}^s G$ and $\overline{F}^s G$ is $\overline{F}$–central simple, the ring $\overline{U_G}$ is prime and because $\overline{F}$ is the ring of fractions of $\overline{Z}$ we see that the ring of central quotients $Q(\overline{U_G})$ is isomorphic to $\overline{F}^s G$. ∎

Let ψ denote the $\mathbb{Q}(\mu)$–algebra homomorphism from the free algebra $\Sigma(\mathbb{Q}(\mu))$ to $k^c G$ given by $\psi(x_{ig}) = t_{ig} u_g$. The image of this map is clearly $\overline{U_G}$.

PROPOSITION 8. *The kernel of ψ is the ideal $T(\mathbb{Q}(\mu))$ of graded identities of $\mathbb{C}^c G$ and hence ψ induces a $\mathbb{Q}(\mu)$–algebra isomorphism from U_G to $\overline{U_G}$. In particular the universal algebra U_G is a prime ring with center isomorphic to $\overline{Z}$ and and its ring of central quotients, that is $Q(U_G)$ is isomorphic to $\overline{F}^s G$.*

Proof. This is clear since a polynomial $p(x_{ig}) \in \Sigma(\mathbb{Q}(\mu))$ is an identity of $\mathbb{C}^c G$ if and only if $p(\lambda_{ig} u_g) = 0$ for any $\lambda_{ig} \in \mathbb{C}$ and this is equivalent to $p(t_{ig} u_g) = 0$ where $\{t_{ig}\}$ are central indeterminates. ∎

Because of this isomorphism we will from now on drop the bars on $\overline{Z}$, $\overline{F}$, etc.

We want to say more about F, the field of fractions of the center of the universal algebra U_G. The group Y is finitely generated and hence of the form $Y_t Y_f$ where Y_t is the torsion subgroup of Y and Y_f is a finitely generated free abelian group. Because Y is a subgroup of $k^\times$ it follows that Y_t is cyclic. We will see in the next proposition that in fact $Y_t = \mu$. Let $y_1, y_2, \ldots, y_m \in k$ denote a basis for Y_f. (We will see later - in Proposition 14 - that the rank of Y_f is equal to n, the order of G, and so $m = n$.)

PROPOSITION 9. *(a) We have $Y_t = Y \cap \mathbb{Q}(\{c(g,h) \mid g,h \in G\}) = \mu$.*

(b) We have $S = \mathbb{Q}(\mu)[Y] = \mathbb{Q}(\mu)[y_1^{\pm 1}, y_2^{\pm 1}, \ldots, y_m^{\pm 1}]$, the ring of Laurent polynomials in $y_1, \ldots, y_m$ over $\mathbb{Q}(\mu)$.

(c) We have $F = \mathbb{Q}(\mu)(y_1, \ldots, y_m)(\frac{t_{ig}}{t_g} \mid i \geq 1, g \in G)$. In particular F is isomorphic to the field of rational functions in countably many variables over $\mathbb{Q}(\mu)$.

Proof. To prove (a) we prove the following inclusions: (i) $Y_t \subseteq Y \cap \mathbb{Q}(\{c(g,h) \mid g,h \in G\})$, (ii) $\mu \subseteq Y_t$, and (iii) $Y \cap \mathbb{Q}(\{c(g,h) \mid g,h \in G\}) \subseteq \mu$. To prove (i) consider an arbitrary

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element z in Y, $z = \prod_{i=1}^{n} s(g_i, h_i)^{\epsilon_i} = \prod_{i=1}^{n} (\frac{t_{g_i} t_{h_i}}{t_{g_i h_i}} c(g_i, h_i))^{\epsilon_i}$ where ϵ_i are ± 1. If the product is not independent of the $t's$ (say t_g appears) it is clear that any positive power of z contains a power of t_g and hence does not equal 1. This shows (i). Next, let $\lambda \in \mu$ and let

$$x_{1g_1} x_{1g_2} \cdots x_{1g_k} - \lambda x_{1g_{\pi(1)}} x_{1g_{\pi(2)}} \cdots x_{1g_{\pi(k)}}$$

be an elementary identity. It follows that

$$t_{g_1} u_{g_1} t_{g_2} u_{g_2} \cdots t_{g_k} u_{g_k} = \lambda t_{g_{\pi(1)}} u_{g_{\pi(1)}} t_{g_{\pi(2)}} u_{g_{\pi(2)}} \cdots t_{g_{\pi(k)}} u_{g_{\pi(k)}}$$

and hence $s(g_1, g_2) s(g_1 g_2, g_3) \cdots s(g_1 g_2 \cdots g_{k-1}, g_k) t_{g_1 g_2 \cdots g_k} u_{g_1 g_2 \cdots g_k} =$

$\lambda s(g_{\pi(1)}, g_{\pi(2)}) s(g_{\pi(1)} g_{\pi(2)}, g_{\pi(3)}) \cdots s(g_{\pi(1)} g_{\pi(1)} \cdots g_{\pi(k-1)}, g_{\pi(k)}) t_{g_{\pi(1)} g_{\pi(2)} \cdots g_{\pi(k)}} u_{g_{\pi(1)} g_{\pi(2)} \cdots g_{\pi(k)}}$.

But $g_1 g_2 \cdots g_k = g_{\pi(1)} g_{\pi(2)} \cdots g_{\pi(k)}$ in G and so λ lies in Y. Because λ is a root of unity, we get $\lambda \in Y_t$. This proves (ii).

To prove (iii) let λ be an element of $Y \cap \mathbb{Q}(\{c(g, h) \mid g, h \in G\})$. We can write $\lambda = s(g_1, h_1)^{\epsilon_1} s(g_2, h_2)^{\epsilon_2} \cdots s(g_k, h_k)^{\epsilon_k} =$

$(\frac{t_{g_1} t_{h_1}}{t_{g_1 h_1}} c(g_1, h_1))^{\epsilon_1} (\frac{t_{g_2} t_{h_2}}{t_{g_2 h_2}} c(g_2, h_2))^{\epsilon_2} \cdots (\frac{t_{g_k} t_{h_k}}{t_{g_k h_k}} c(g_k, h_k))^{\epsilon_k}$.

We use the notation and set-up of part (b) of Proposition 1. Let d be the element of the free group F given by $d = x_{g_1} x_{h_1} (x_{g_1 h_1})^{-1} x_{g_2} x_{h_2} (x_{g_2 h_2})^{-1} \cdots x_{g_k} x_{h_k} (x_{g_k h_k})^{-1}$. Clearly d lies in R. Because λ lies in $\mathbb{Q}(\{c(g, h) \mid g, h \in G\})$, the variables t_g in the expression for λ must all cancel and so $d \in [F, F]$ and $\lambda = c(g_1, h_1))^{\epsilon_1} c(g_2, h_2))^{\epsilon_2} \cdots c(g_k, h_k))^{\epsilon_k}$. In particular we may consider the element $\overline{d}$ in $M(G)$. Then again as in part (b) of Proposition 1 we have $\phi([c])(\overline{d}) = \gamma(\overline{d}) = c(g_1, h_1))^{\epsilon_1} c(g_2, h_2))^{\epsilon_2} \cdots c(g_k, h_k))^{\epsilon_k} = \lambda$ and so λ lies in μ.

Part (b) is clear. Part (c) follows from part (b) and from the fact that $M^{-1} Z = S[\frac{t_{ig}}{t_g} \mid i \geq 1, g \in G]$. ∎

Our next goal is to study the algebra $S^s G$. We will see that it is Azumaya and that its simple images are the graded forms of $\mathbb{C}^c G = M_n(\mathbb{C})$.

PROPOSITION 10. *(a) Let m be a maximal ideal of S and let $y \to \overline{y}$ denote the canonical homomorphism from S to $\tilde{S} = S/m$. Consider the two-cocycle $\tilde{s} : G \times G \longrightarrow S/m$ induced by s and let $\tilde{S}^{\tilde{s}} G$ be the corresponding twisted group algebra. Then the algebra $S^s G / m S^s G$ is isomorphic to $\tilde{S}^{\tilde{s}} G$ and is a central simple $\tilde{S}$–algebra.*

(b) The ring $S^s G$ is Azumaya over S.

Proof. (a) We first show that the two-cocycle $\tilde{s}$ is nondegenerate. We use the following characterization of nondegeneracy (see Isaacs [13, Problem 11.8]): a two-cocycle $\delta : G \times$

$G \to \mathbb{C}^\times$ is nondegenerate if and only if for every element $g \in G$ there is an element $h \in G$ such that g and h commute but in the algebra $\mathbb{C}^\delta G$ the elements u_g and u_h do not commute. Let $g \in G$. Because c is nondegenerate there is an element $h \in G$ that centralizes g and such that $c(g,h)c(h,g)^{-1} \neq 1$. Moreover $c(g,h)c(h,g)^{-1} \in \mu$ and $s(g,h)s(h,g)^{-1} = c(g,h)c(h,g)^{-1}$. Because S contains the field $\mathbb{Q}(\mu)$, the restriction of the canonical homomorphism from S to $\tilde{S}$ is injective on μ and so $\tilde{s}(g,h)\tilde{s}(h,g)^{-1} \neq 1$. Hence $\tilde{s}$ is nondegenerate. It follows that $\tilde{S}^{\tilde{s}}G$ is a central simple $\tilde{S}$–algebra. It is clear that the canonical homomorphism from S^sG to $\tilde{S}^{\tilde{s}}G$ is surjective with kernel mS^sG.

(b) This follows from part (a) (see DeMeyer and Ingraham [8, Theorem 7.1]). ∎

Let L be any subfield of $\mathbb{C}$. Recall that every G–graded form of $\mathbb{C}^cG$ over L is a twisted group algebra $L^\beta G$ where β is a two-cocycle cohomologous to c over $\mathbb{C}$.

We will show now that S^sG is universal with respect to such forms. The following observation about the Hopf formula $M(G) = [F,F] \cap R/[F,R]$ will be useful (and is probably well known). We use the notation and discussion preceding Proposition 1.

LEMMA 11. *Every element in $[F,F] \cap R$ is equivalent modulo $[F,R]$ to an element of the form $x_{g_1} \cdots x_{g_k}(x_{g_{\pi(1)}} \cdots x_{g_{\pi(k)}})^{-1}$ where $\pi \in Sym(k)$ and $g_1 \cdots g_k = g_{\pi(1)} \cdots g_{\pi(k)}$.*

Proof. Let $y \in [F,F] \cap R$. We can write $y = x_{g_1}^{\epsilon_1} x_{g_2}^{\epsilon_2} \cdots x_{g_m}^{\epsilon_m}$ where $\epsilon_i = \pm 1$ for $1 \leq i \leq m$ and where for each group element g that appears the elements x_g and x_g^{-1} appear the same number of times (and so m is even) and where the product $g_1^{\epsilon_1} g_2^{\epsilon_2} \cdots g_m^{\epsilon_m}$ equals one. Let $m = 2r$. We <u>claim</u> that we may assume $\epsilon_i = 1$ if $1 \leq i \leq r$ and $\epsilon_i = -1$ if $r+1 \leq i \leq m$. The lemma follows easily from this claim. To prove the claim, suppose $\epsilon_i = -1$ for some i, $1 \leq i \leq r$. We consider $y = x_{g_1}^{\epsilon_1} x_{g_2}^{\epsilon_2} \cdots x_{g_m}^{\epsilon_m} = x_{g_1}^{\epsilon_1} x_{g_2}^{\epsilon_2} \cdots x_{g_r}^{\epsilon_r}(x_{g_i} x_{g_i^{-1}}(x_{g_i^{-1}})^{-1}(x_{g_i})^{-1})x_{g_{r+1}}^{\epsilon_{r+1}} \cdots x_{g_m}^{\epsilon_m}$. Now for any element $g \in G$ the element $x_g x_{g^{-1}}$ commutes modulo $[F,R]$ with every element of F. Hence we can move the element $x_{g_i} x_{g_i^{-1}}$ in y and obtain an element that is equivalent modulo $[F,R]$ to y and for which the element $x_{g_i}^{-1}$ is replaced by $x_{g_i^{-1}}$. Continuing in this way proves the claim. ∎

PROPOSITION 12. *Let L be a subfield of $\mathbb{C}$ which contains $\mathbb{Q}(\mu)$. Let $\varphi : S \longrightarrow L$ be a $\mathbb{Q}(\mu)$–algebra homomorphism. Let $\beta(g,h) = \varphi(s(g,h))$. Then φ induces a G–graded homomorphism $\hat{\varphi} : S^sG \longrightarrow L^\beta G$ and $L^\beta G$ is a G–graded form of $\mathbb{C}^cG$. In particular β is cohomologous to c over $\mathbb{C}$.*

Conversely if $L^\beta G$ is a G–graded form of $\mathbb{C}^cG$, then there is homomorphism φ of S into L such that $\gamma = \varphi(s(g,h))$ is a cocycle on G cohomologous to β over L and φ induces a homomorphism from S^sG onto $\varphi(S)^\gamma G$, a G–graded form of $L^\beta G$.

Proof. Let $\varphi : S \longrightarrow L$ be a $\mathbb{Q}(\mu)$–algebra homomorphism and let $\beta(g, h) = \varphi(s(g, h))$. We have seen in Proposition 10 that β is a nondegenerate cocycle on G and hence that $L^\beta G$ is a central simple L–algebra. Clearly φ induces a homomorphism $\hat{\varphi} : S^s G \longrightarrow L^\beta G$ and if A denotes the image of $S^s G$ and A has center R (say), then $A \otimes_R L = L^\beta G$. We are left with showing $L^\beta G$ is a form of $\mathbb{C}^c G$, that is we need to show β and c are cohomologous over $\mathbb{C}$. It suffices to show that they determine the same element in $\mathrm{Hom}(M(G), \mathbb{C}^\times)$, that is that $\phi([c]) = \phi([\beta])$, where ϕ is the isomorphism from $H^2(G, \mathbb{C}^\times)$ to $\mathrm{Hom}(M(G), \mathbb{C}^\times)$ (see the discussion preceding Proposition 1). By the lemma it suffices to show that if $g_1 \cdots g_k = g_{\pi(1)} \cdots g_{\pi(k)}$ then $\phi([\beta])(y) = \phi([c])(y)$ where $y = (x_{g_1} \cdots x_{g_k})(x_{g_{\pi(1)}} \cdots x_{g_{\pi(k)}})^{-1}$. For the purposes of the proof it will be useful to use the following notation: If α is any two-cocycle then we denote the expression

$$\frac{\alpha(g_1, g_2)\alpha(g_1 g_2, g_3) \cdots \alpha(g_1 g_2 \ldots g_{k-1}, g_k)}{\alpha(g_{\pi(1)}, g_{\pi(2)})\alpha(g_{\pi(1)} g_{\pi(2)}, g_{\pi(3)}) \cdots \alpha(g_{\pi(1)} g_{\pi(2)} \cdots g_{\pi(k-1)}, g_{\pi(k)})}$$

by $\alpha(g_1, g_2, \ldots, g_k)/\alpha(g_{\pi(1)}, g_{\pi(2)}, \ldots, g_{\pi(k)})$.

We have $\phi([\beta])(y) = \beta(g_1, g_2, \ldots, g_k)/\beta(g_{\pi(1)}, g_{\pi(2)}, \ldots, g_{\pi(k)}) =$

$\varphi(s(g_1, g_2, \ldots, g_k)/s(g_{\pi(1)}, g_{\pi(2)}, \ldots, g_{\pi(k)})) =$

$\varphi((t_{g_1} t_{g_2}/t_{g_1 g_2})(t_{g_1 g_2} t_{g_3}/t_{g_1 g_2 g_3}) \cdots (t_{g_1 g_2 \cdots g_{k-1}} t_{g_k}/t_{g_1 g_2 \cdots g_k})/$

$(t_{g_{\pi(1)}} t_{g_{\pi(2)}}/t_{g_{\pi(1)} g_{\pi(2)}}) \cdots (t_{g_{\pi(1)} g_{\pi(2)} \cdots g_{\pi(k-1)}} t_{g_{\pi(k)}}/t_{g_{\pi(1)} g_{\pi(2)} \cdots g_{\pi(k)}})$

$c(g_1, g_2, \ldots, g_k)/c(g_{\pi(1)}, g_{\pi(2)}, \ldots, g_{\pi(k)})) =$

$\varphi((t_{g_1 g_2 \cdots g_k}/t_{g_{\pi(1)} g_{\pi(2)} \cdots g_{\pi(k)}})c(g_1, g_2, \ldots, g_k)/c(g_{\pi(1)}, g_{\pi(2)}, \ldots, g_{\pi(k)})) =$

$\varphi(c(g_1, g_2, \ldots, g_k)/c(g_{\pi(1)}, g_{\pi(2)}, \ldots, g_{\pi(k)}))$, because $g_1 \cdots g_k = g_{\pi(1)} \cdots g_{\pi(k)}$. But this last expression is exactly $\phi([c])(y)$, as desired.

For the converse assume $L^\beta G$ is a G–graded form of $\mathbb{C}^c G$. Then $L^\beta G$ satisfies the graded identities of $M_n(\mathbb{C})$ and so there is a graded homomorphism from U_G sending each x_{ig} to u_g in $L^\beta G$. But under this map the images of the elements of M are nonzero elements in L and so we get an induced homomorphism from $S^s G$ that takes S into L. The result follows. $\blacksquare$

We can use this result to obtain a parametrization of the G–graded forms over L. We begin with a homomorphism φ from S to L and the two-cocycle β given by $\beta(g, h) = \varphi(s(g, h))$. In the previous proposition we say that β is cohomologous to c over $\mathbb{C}$ and so $L^\beta G$ is a form of $M_n(\mathbb{C})$. We will now show how to produce all other forms over L. Let V the free abelian group (of rank $\mathrm{ord}(G)$) on symbols $r_g, g \in G$. Let U be the subgroup generated by the elements $r_g r_h/r_{gh}$. Note that V/U is a finite group, isomorphic to $G/G' = G_{ab}$ via the map that sends r_g to gG'. For each element ψ in $\mathrm{Hom}(U, L^\times)$ let β_ψ be given by $\beta_\psi(g, h) = \beta(g, h)\psi(r_g r_h/r_{gh})$. Note that there is a canonical homomor-

phism from $\mathrm{Hom}(V, L^\times)$ to $\mathrm{Hom}(U, L^\times)$. We will let $\mathrm{Im}((\mathrm{Hom}(V, L^\times))$ denote its image in $\mathrm{Hom}(U, L^\times)$.

PROPOSITION 13. *The following hold:*

(a) For every $\psi \in \mathrm{Hom}(U, L^\times)$ the function β_ψ is a two-cocycle cohomologous to β over $\mathbb{C}$. In particular $L^{\beta_\psi}G$ is a G–graded form of $M_n(\mathbb{C})$.

(b) If $L^\gamma G$ is a G–graded form of $M_n(\mathbb{C})$ there is a homomorphism $\psi \in \mathrm{Hom}(U, L^\times)$ such that $\gamma = \beta_\psi$. In particular $L^\gamma G$ is isomorphic to $L^{\beta_\psi}G$ as a G–graded algebra.

(c) Two forms $L^{\beta_\psi}G$ and $L^{\beta_{\psi'}}G$ are G–graded isomorphic if and only if $\beta_\psi = f(\alpha)\beta_{\psi'}$ for some α in $\mathrm{Im}((\mathrm{Hom}(V, L^\times))$.

(d) The function $\psi \to \beta_\psi$ induces a one-to-one correspondence between the group $\mathrm{Hom}(U, L^\times)/\mathrm{Im}(\mathrm{Hom}(V, L^\times)$ and the set of G–graded isomorphism classes of G–graded forms of $\mathbb{C}^c G$ over L.

Proof. (a) It is clear that β_ψ is a cocycle. To show that β and β_ψ are cohomologous over $\mathbb{C}$ it suffices to show that they determine the same element in $\mathrm{Hom}(M(G), \mathbb{C}^\times)$, that is that $\phi([\beta_\psi]) = \phi([\beta])$, where ϕ is the isomorphism from $H^2(G, \mathbb{C}^\times)$ to $\mathrm{Hom}(M(G), \mathbb{C}^\times)$. The argument is quite similar to the proof of the corresponding fact in Proposition 12, so we will omit it.

(b) If $L^\gamma G$ is a G–graded form of $M_n(\mathbb{C})$ then we have seen that γ is cohomologous to β over $\mathbb{C}$ so there is a cochain $\lambda : G \to \mathbb{C}^\times$ such that for all $g, h \in G$, we have $\gamma(g, h) = (\lambda(g)\lambda(h)/\lambda(gh))\beta(g, h)$. It follows that for all $g, h \in G$, $\lambda(g)\lambda(h)/\lambda(gh)$ is in L. The cochain λ clearly determines a homomorphism from V to $\mathbb{C}^\times$ (sending r_g to $\lambda(g)$) and this homomorphism restricts to a homomorphism from ψ from U to $L^\times$ (given by $\psi(r_g r_h/r_{gh}) = \lambda(g)\lambda(h)/\lambda(gh)$) which satisfies $\gamma = \beta_\psi$.

(c) If $L^{\beta_\psi}G$ and $L^{\beta_{\psi'}}G$ are G–graded isomorphic then β_ψ is cohomologous to $\beta_{\psi'}$ over L. As in the proof of part (b) this means there is a homomorphism α from V to $L^\times$ such that $\beta_\psi(g, h) = (\alpha(g)\alpha(h)/\alpha(gh))\beta_{\psi'}$. It follows that $\psi = \alpha\psi'$, as desired.

(d) This follows from the first three parts. ∎

We can connect this parametrization with cohomology as follows. We have a short exact sequence of groups

$$1 \longrightarrow U \longrightarrow V \longrightarrow G_{ab} \longrightarrow 1$$

Applying the functor $\mathrm{Hom}(-, L^\times)$ we obtain a long exact sequence in cohomology

$$\ldots \longrightarrow \mathrm{Hom}(V, L^\times) \longrightarrow \mathrm{Hom}(U, L^\times) \longrightarrow Ext(G_{ab}, L^\times) \longrightarrow 1$$

where the surjectivity of the last map follows from the fact that $Ext(V, L^\times) = 0$ because V is a free abelian group. We also have the universal coefficients sequence:

$$1 \to Ext^1(G_{ab}, L^\times) \to H^2(G, L^\times) \to \mathrm{Hom}(M(G), L^\times) \to 1$$

Combining these equations shows that $\mathrm{Hom}(U, L^\times)/\mathrm{Im}(\mathrm{Hom}(V, L^\times))$ is isomorphic to $Ext^1(G_{ab}, L^\times)$, the kernel of the map from $H^2(G, L^\times)$ to $\mathrm{Hom}(M(G), L^\times)$.

The last result in this section is the determination of the rank of Y_f, the free part of the subgroup of $k^\times$ generated by the values of s. The computation involves the groups U and V. We first remark that because V/U is finite, the rank of U is equal to the rank of V, which is clearly n, the order of G.

PROPOSITION 14. *The group Y_f is isomorphic to U. In particular the rank of Y_f is n, the order of G.*

Proof. Recall that Y denotes the subgroup of $k^\times$ generated by the values of the cocycle s. Let H denote the subgroup of $k^\times$ generated by the values of the cocycle c. Then by Proposition 9 $Y \cap H = Y \cap \mathbb{C}^\times = \mu$. Let $\tilde{U}$ be the subgroup of $k^\times$ generated by set $\{t_g t_h/t_{gh} \mid g, h \in G\}$. Clearly the group $\tilde{U}$ is isomorphic to U. Now we have $Y_f \cong Y/Y_t \cong Y/Y \cap H \cong YH/H$. But $YH = \tilde{U}H$. Hence $YH/H = \tilde{U}H/H \cong \tilde{U}/\tilde{U} \cap H \cong \tilde{U}$ because $\tilde{U} \cap H = 1$. Hence $Y_f \cong \tilde{U} \cong U$, as desired. ∎

COROLLARY 15. *The field of fractions of S has transcendence degree n over $\mathbb{Q}(\mu)$.*

Proof. This follows from the description of S given in Proposition 9. ∎

4 The universal G–graded central simple algebra.

We have seen that each fine grading by a group G on $M_n(\mathbb{C})$ gives rise to a non-degenerate two-cocycle c on the group G (which must be of central type) such that $M_n(\mathbb{C})$ is graded isomorphic to $\mathbb{C}^c G$. We have also seen that the universal G–graded algebra U_G is a prime ring with ring of central quotients $Q(U_G)$ graded isomorphic to the F–central simple algebra $F^s G$ where s is the generic cocycle obtained from c and

$F = \mathbb{Q}(\mu)(y_1, \ldots, y_m)(\frac{t_{ig}}{t_g} \mid i \geq 1, g \in G)$ (see Proposition 9). In this section we obtain information about the index of $Q(U_G)$ and about the dependence of $Q(U_G)$ on the given cocycle c.

PROPOSITION 16. *The index of $Q(U_G)$ is equal to the maximum of the indices of the G–graded forms $L^\beta G$ where L varies over all subfields of $\mathbb{C}$ that contain $\mathbb{Q}(\mu)$ and β varies over all nondegenerate cocycles over L cohomologous to c over $\mathbb{C}$.*

Proof. Let M denote the maximum of the indices of the forms. We first claim that $\mathrm{ind}(Q(U_G)) \leq M$: In fact we can specialize the universal algebra U_G by sending x_g to $r_g u_g$ where the elements $\{r_g \mid g \in G\}$ are algebraically independent complex numbers. The resulting algebra $\overline{U}$ is isomorphic to U_G and tensoring with the field of fractions of its center gives a central simple algebra that is isomorphic to $Q(U_G)$. Clearly the index of this form equals the index of $Q(U_G)$ and so we get the inequality. We proceed to show that the index of $Q(U_G) = F^s G$ is greater than or equal to M. Recall (Proposition 9) that the center S of $S^s G$ is a ring of Laurent polynomials over the field $\mathbb{Q}(\mu)$. If M is any maximal ideal of S then S_M is a regular local ring. By Proposition 12 it suffices to show that if M is any maximal ideal of S then the index of the algebra $S^s G \otimes_S F = F^s G$ is greater than or equal to the index of the residue algebra $S^s G / M S^s G$. This is a well known fact. We outline a proof: Let $r_1, r_2, \ldots, r_m$ be a system of parameters of the regular local ring $R = S_M$ and let $F^s G = M_t(D)$ where D is an F–central division algebra (so $\mathrm{ind}(F^s G) = \mathrm{ord}(G)^{1/2}/t$). The localization $R_{(r_1)}$ is a discrete valuation ring with residue field the field of fractions of the regular local ring $R_1 = R/(r_1)$. The ring $R^s_{(r_1)} G$ is an Azumaya algebra over the discrete valuation ring $R_{(r_1)}$ and hence is isomorphic to $M_t(A)$ where A is an Azumaya algebra such that $A \otimes F = D$ (see Reiner [16, Theorem 21.6]). It follows that $R^s_1 G$ is an Azumaya algebra and if we let $Q(R_1)$ denote the field of fractions of R_1, then $R^s_1 G \otimes Q(R_1)$ is isomorphic to $M_t(D_1)$ where D_1 a central simple $Q(R_1)$–algebra, the residue algebra of D. The image of M in R_1 is generated by the images of $r_2, r_3, \ldots, r_m$ and these $m - 1$ elements form a system of parameters for R_1. We can therefore repeat the process and eventually obtain that $S^s_M G / M S^s_M G$ is isomorphic to an algebra of the form $M_t(E)$ for some central simple S/M–algebra E. Hence the index of $S^s_M G / M S^s_M G$ is at most the index of $F^s G$. ∎

The question for which groups G of central type there is a cocycle c such that $Q(U_{G,c})$ is a division algebra was answered in Aljadeff, et al [2] and Natapov [15]: We consider the following list of p-groups, called Λ_p:

1. G is abelian of symmetric type, that is $G \cong \prod (C_{p^{n_i}} \times C_{p^{n_i}})$,

2. $G \cong G_1 \times G_2$ where
   $G_1 = C_{p^n} \rtimes C_{p^n} = \langle \pi, \sigma \mid \sigma^{p^n} = \pi^{p^n} = 1 \text{ and } \sigma \pi \sigma^{-1} = \pi^{p^s+1} \rangle$ where $1 \leq s < n$ and

$1 \neq s$ if $p = 2$, and

G_2 is an abelian group of symmetric type of exponent $\leq p^s$,

3. $G \cong G_1 \times G_2$ where

$$G_1 = C_{2^{n+1}} \rtimes (C_{2^n} \times Z_2) = \left\langle \pi, \sigma, \tau \;\middle|\; \begin{array}{l} \pi^{2^{n+1}} = \sigma^{2^n} = \tau^2 = 1, \sigma\tau = \tau\sigma, \\ \sigma\pi\sigma^{-1} = \pi^3, \tau\pi\tau^{-1} = \pi^{-1} \end{array} \right\rangle \text{ and}$$

G_2 is an abelian group of symmetric type of exponent ≤ 2.

We let Λ be the collection of nilpotent groups such that for any prime p, the Sylow-p subgroup is on the Λ_p.

PROPOSITION 17. *Let G be a group of central type of order n^2. The following conditions are equivalent:*

(a) There is a nondegenerate cocycle c on G for which the universal G–graded algebra $U_{G,c}$ is a domain.

(b) There is a nondegenerate cocycle c on G for which the universal central simple algebra $Q(U_{G,c})$ is a division algebra.

(c) There is a nondegenerate cocycle c on G such that the resulting G–graded algebra $M_n(\mathbb{C})$ has a G–graded form that is a division algebra.

(d) The group G is on the list Λ.

Proof. The equivalence of (a) and (b) follows from the fact that $F^s G$ is isomorphic to $Q(U_{G,c})$. The equivalence of (b) and (c) follows from the previous proposition. Finally the equivalence of (c) and (d) follows from Aljadeff et al [2, Corollary 3] and Natapov [15, Theorem 3]. ∎

Our main result in this section is that if G is a group on the list Λ then the universal central simple G–graded algebra $Q(U_{G,c})$ is independent (up to a non-graded isomorphism) of the cocycle c. In fact we will show that for groups G on the list the automorphism group of G acts transitively on the set of classes of nondegenerate cocycles.

To begin let G be any central type group (not necessarily on Λ) and let c be a nondegenerate two-cocycle on G with coefficients in $\mathbb{C}^\times$. Let φ be an automorphism of G and let $\varphi(c)$ be the two-cocycle defined by $\varphi(c)(\sigma, \tau) = c(\varphi^{-1}(\sigma), \varphi^{-1}(\tau))$. It is clear that $\varphi(c)$ is also nondegenerate.

THEOREM 18. *If G is a group on the list Λ and c and c' are nondegenerate cocycles on G with values in $\mathbb{C}^\times$, then there is an automorphism φ of G such that $\varphi(c)$ is cohomologous over $\mathbb{C}$ to c'.*

Proof. Let G be a group on the list Λ. Because G is necessarily nilpotent we can assume that G lies on Λ_p for some prime p. In the course of the proof whenever we refer to basis elements $\{u_g\}_{g \in G}$ in the twisted group algebra $\mathbb{C}^c G$, we assume they satisfy $u_g u_h = c(g,h)u_{gh}$.

The strategy is as follows: For each group $G \in \Lambda_p$ we fix a set of generators Φ. Then we consider the family of sets of generators $\Phi' = \varphi(\Phi)$ that are obtained from Φ via an automorphism φ of G. We say that Φ and Φ' are equivalent. Next, we exhibit a certain nondegenerate cohomology class $\alpha \in H^2(G, \mathbb{C}^\times)$ by setting a set of equations (denoted by E_G) satisfied by elements $\{u_g\}_{g \in \Phi}$ in $\mathbb{C}^{c_0}G$, where c_0 is a two-cocycle representing α. We say that the cohomology class α (or by abuse of language, the two-cocycle c_0) is of "standard form" with respect to Φ. It is indeed abuse of language since in general the equations do not determine the two-cocycle c_0 uniquely but only its cohomology class. Finally we show that for any nondegenerate two-cocycle c with values in $\mathbb{C}^\times$ there is a set of generators $\Phi' = \varphi(\Phi)$ with respect to which c is of standard form. The desired automorphism of the group G will be determined by compositions of "elementary" automorphisms, that is automorphisms that are defined by replacing some elements of the generating set. We start with $\Phi^{(0)} = \Phi$ and denote the updated generating sets by $\Phi^{(r)} = \{g_1^{(r)}, \ldots, g_n^{(r)}\}, r = 1, 2, 3, \ldots$. Elements in the generating set that are not mentioned remain unchanged, that is $g_i^{(r)} = g_i^{(r-1)}$. In all steps it will be an easy check that the map defined is indeed an automorphism of G.

(I) We start with abelian groups on Λ_p. Let $G = C_{p^{r_1}} \times C_{p^{r_1}} \times C_{p^{r_2}} \times C_{p^{r_2}} \times \cdots \times C_{p^{r_m}} \times C_{p^{r_m}}$ and let $\Phi = \{\gamma_1, \gamma_2, \ldots, \gamma_{2m}\}$ be an ordered set of generators.

Fixing a primitive p^n-th root of unity ε, where $n \geq \max(r_k)$, $1 \leq k \leq m$, we say a two-cocycle c_0 is of standard form with respect to Φ if there are elements $\{u_g\}_{g \in \Phi}$ in $\mathbb{C}^{c_0}G$ that satisfy

$$\begin{cases} (u_{\gamma_{2k-1}}, u_{\gamma_{2k}}) = \varepsilon^{p^{n-r_k}} & \text{for all } 1 \leq k \leq m, \\ \text{all other commutators of } u_{\gamma_i}\text{'s are trivial.} \end{cases} \tag{1}$$

Note that $[c_0]$ is determined by (1). Given any nondegenerate two-cocycle c it is known that there is a set of generators $\Phi' = \varphi(\Phi)$ with respect to which c is of standard form (see, for example, Aljadeff and Sonn [5, Theorem 1.1]). Hence we are done in this case.

(II) Next we consider the group $G = C_{p^n} \rtimes C_{p^n} \times C_{p^{r_1}} \times C_{p^{r_1}} \times \cdots \times C_{p^{r_m}} \times C_{p^{r_m}}$ with a set of generators $\Phi = \{\pi, \sigma, \gamma_1, \ldots, \gamma_{2m}\}$ where π, σ satisfy $\sigma(\pi) = \pi^{p^s+1}$ for some $s = 1, \ldots, n-1$ if p is odd and for some $s = 2, \ldots, n-1$ if $p = 2$, and $r_k \leq s$ for all $1 \leq k \leq m$. We write $G = G_1 \times G_2$ where $G_1 = C_{p^n} \rtimes C_{p^n}$.

Fix a primitive p^n-th root of unity ε. There exist a two-cocycle $c_1 \in Z^2(G_1, \mathbb{C}^\times)$ and elements $\{u_g\}_{g \in \{\pi, \sigma\}}$ in $\mathbb{C}^{c_1} G_1$ that satisfy

$$u_\pi^{p^n} = u_\sigma^{p^n} = 1, \, u_\sigma u_\pi u_\sigma^{-1} = \varepsilon u_\pi^{p^{s+1}}. \tag{2}$$

Moreover the class $[c_1] \in H^2(G_1, \mathbb{C}^\times)$ is uniquely determined by (2) (see, for example, the proof of Karpilovsky [14, Theorem 10.1.25]).

Now, by Karpilovsky ([14, Proposition 10.6.1]) we have $M(G) \cong M(G_1) \times M(G_2) \times (G_1^{ab} \otimes G_2^{ab})$ (here G^{ab} denotes the abelianization of G) and hence there exist a two-cocycle $c_0 \in Z^2(G, \mathbb{C}^\times)$ and elements $\{u_g\}_{g \in \Phi}$ in $\mathbb{C}^{c_0} G$ that satisfy equations (2), (1) and $u_\pi u_{\gamma_i} = u_{\gamma_i} u_\pi$, $u_\sigma u_{\gamma_i} = u_{\gamma_i} u_\sigma$. We denote this set of equations by E_G. The class $[c_0] \in H^2(G, \mathbb{C}^\times)$ is uniquely determined by E_G and is easily seen to be nondegenerate. We say that a class $[c_0]$ is of standard form with respect to Φ if there are elements $\{u_g\}_{g \in \Phi}$ in $\mathbb{C}^{c_0} G$ that satisfy E_G.

Now, let $c \in Z^2(G, \mathbb{C}^\times)$ be any nondegenerate two-cocycle, and let u_g be representatives of elements of G in $\mathbb{C}^c G$. As explained above we set $\Phi^{(0)} = \{\pi^{(0)}, \sigma^{(0)}, \gamma_1^{(0)}, \ldots, \gamma_{2m}^{(0)}\} = \Phi = \{\pi, \sigma, \gamma_1, \ldots, \gamma_{2m}\}$. We are to exhibit a sequence of automorphisms such that their composition applied to $\Phi^{(0)}$ yields a generating set with respect to which the cocycle c is of standard form. First, note that we may assume (by passing to an equivalent cocycle, if necessary) that $u_{\pi^{(0)}}^{p^n} = u_{\sigma^{(0)}}^{p^n} = 1$. Let $\alpha \in \mathbb{C}^\times$ be determined by $u_{\sigma^{(0)}} u_{\pi^{(0)}} u_{\sigma^{(0)}}^{-1} = \alpha u_{\pi^{(0)}}^{p^{s+1}}$. It follows that $u_{\sigma^{(0)}} u_{\pi^{(0)}}^{p^{n-1}} u_{\sigma^{(0)}}^{-1} = \alpha^{p^{n-1}} u_{\pi^{(0)}}^{p^{n-1}}$ and hence, $\alpha^{p^{n-1}}$ is a p-th root of unity. Next, observe that since c is nondegenerate, $u_{\sigma^{(0)}}$ cannot commute with $u_{\pi^{(0)}}^{p^{n-1}}$ (for otherwise, $u_{\pi^{(0)}}^{p^{n-1}}$ is contained in the center of the algebra) and hence α is a primitive p^n-th root of unity. Now, replacing $\pi^{(0)}$ by a suitable prime to p power of $\pi^{(0)}$ (that is $\pi^{(1)} = (\pi^{(0)})^l$, l prime to p), we may assume that $\alpha = \varepsilon$. Leaving $\sigma^{(0)}$ and all other generators unchanged, that is $\sigma^{(1)} = \sigma^{(0)}, \gamma_1^{(1)} = \gamma_1^{(0)}, \ldots, \gamma_{2m}^{(1)} = \gamma_{2m}^{(0)}$, we obtain a generating set $\Phi^{(1)} = \{\pi^{(1)}, \sigma^{(1)}, \gamma_1^{(1)}, \ldots, \gamma_{2m}^{(1)}\}$ of G, equivalent to Φ, such that the elements $u_{\pi^{(1)}}, u_{\sigma^{(1)}}$ in $\mathbb{C}^c G_1$ satisfy (2).

Assume $(u_{\gamma_i^{(1)}}, u_{\pi^{(1)}}) = \xi_i \neq 1$ (clearly ξ_i is a root of unity of order $p^r (\leq \mathrm{ord}(\gamma_i^{(1)}) \leq p^s)$). Now, by the nondegeneracy of c, the root of unity ζ determined by the equation $u_{\sigma^{(1)}}^{p^{n-s}} u_{\pi^{(1)}} u_{\sigma^{(1)}}^{-p^{n-s}} = \zeta u_{\pi^{(1)}}$ is of order p^s and hence $\xi_i = \zeta^l$ for some l. Observe that $\mathrm{ord}(((\sigma^{(1)})^{p^{n-s}})^l) \leq p^r$ and hence we may put $\gamma_i^{(2)} = \gamma_i^{(1)}(\sigma^{(1)})^{-lp^{n-s}}$ and get $(u_{\gamma_i^{(2)}}, u_{\pi^{(1)}}) = 1$. Performing this process for all $1 \leq i \leq 2m$ (and leaving $\pi^{(1)}$ and $\sigma^{(1)}$ unchanged) we get a set of generators $\Phi^{(2)}$ such that $(u_{\gamma_i^{(2)}}, u_{\pi^{(2)}}) = 1$ for all i.

Next, if $(u_{\gamma_i^{(2)}}, u_{\sigma^{(2)}}) \neq 1$ we set $\gamma_i^{(3)} = \gamma_i^{(2)}(\pi^{(2)})^{-tp^{n-s}}$ for an appropriate positive integer t and get $(u_{\gamma_i^{(3)}}, u_{\sigma^{(2)}}) = 1$. Performing this process for all $1 \leq i \leq 2m$ we get a set of

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generators $\Phi^{(3)}$ such that $(u_{\gamma_i^{(3)}}, u_{\sigma^{(3)}}) = 1$ for all i.

Finally, we may proceed as in Case I and obtain a generating set $\Phi^{(4)}$, equivalent to Φ, with respect to which c is of standard form.

(III) Next we consider the group $G = C_{2^{n+1}} \rtimes (C_{2^n} \times C_2) \times C_2 \times C_2 \ldots \times C_2 \times C_2$, $n \geq 2$, with a set of generators $\Phi = \{\pi, \sigma, \tau, \gamma_1, \ldots, \gamma_{2m}\}$, where $\sigma\pi\sigma^{-1} = \pi^3$ and $\tau\pi\tau^{-1} = \pi^{-1}$. We write $G = G_1 \times G_2$ where $G_1 = C_{2^{n+1}} \rtimes (C_{2^n} \times C_2)$ and G_2 is elementary abelian of rank $2m$.

We first exhibit a construction of a field K and a cocycle $\beta \in Z^2(G_1, K^\times)$ such that the algebra $D \cong K^\beta G_1$ is K–central simple (in fact a K–central division algebra), which is analogous to that in Natapov [15]. Let $K = \mathbb{Q}(s,t)$ be the subfield of $\mathbb{C}$ generated by algebraically independent elements s and t. Let $L = K(v_\pi)/K$ be a cyclotomic extension where $v_\pi^{2^{n+1}} = -1$. The Galois action of $Gal(L/K) \cong Z_{2^n} \times Z_2 = \langle \sigma, \tau \rangle$ on L is given by

$$\sigma(v_\pi) = v_\pi^3 \quad \text{and} \quad \tau(v_\pi) = v_\pi^{-1}.$$

The algebra D is the crossed product $D = (L/K, \langle \sigma, \tau \rangle)$ determined by the following relations

$$v_\sigma^{2^n} = s, \ v_\tau^2 = t, \ (v_\sigma, v_\tau) = 1,$$

where v_σ and v_τ represent σ and τ in D. It is easy to see that D is isomorphic to a twisted group algebra of the form $K^\beta G_1$.

Fix a primitive 2^{n+1}-th root of unity ε, and let ξ be a square root of ε. Let $\widetilde{s}$ and $\widetilde{t}$ be elements of $\mathbb{C}$ such that $\widetilde{s}^{2^n} = s^{-1}$ and $\widetilde{t}^2 = t^{-1}$. Then the elements $u_{\pi^i \sigma^j \tau^k} = (\xi v_\pi)^i (\widetilde{s} v_\sigma)^j (\widetilde{t} v_\tau)^k$ in $\mathbb{C}^\beta G_1 \cong \mathbb{C} \otimes_K D$ satisfy

$$u_\pi^{2^{n+1}} = u_\sigma^{2^n} = u_\tau^2 = 1, \ u_\sigma u_\pi = \varepsilon u_\pi^3 u_\sigma, \ u_\tau u_\pi = \varepsilon^{-1} u_\pi^{-1} u_\tau, \ u_\tau u_\sigma = u_\sigma u_\tau. \tag{3}$$

Clearly, there exists a two-cocycle $c_1 \in Z^2(G_1, \mathbb{C}^\times)$, cohomologous to β, such that $u_{g_1} u_{g_2} = c_1(g_1, g_2) u_{g_1 g_2}$ for all $g_1, g_2 \in G_1$ and moreover the class $[c_1]$ is determined by (3).

Now, by Karpilovsky [14, Proposition 10.6.1] there exist a two-cocycle $c_0 \in Z^2(G, \mathbb{C}^\times)$ and elements $\{u_g\}_{g \in \Phi}$ in $\mathbb{C}^{c_0} G$ that satisfy (3), (1) and $u_g u_{\gamma_i} = u_{\gamma_i} u_g$ for $g \in \{\pi, \sigma, \tau\}$. We denote this set of equations by E_G. As in the previous cases, also here, $[c_0]$ is determined by E_G, and we say that the cocycle c_0 is of standard form with respect to Φ.

Now, let $c \in Z^2(G, \mathbb{C}^\times)$ be a nondegenerate two-cocycle, and consider the algebra $\mathbb{C}^c G = \oplus_{g \in G} \mathbb{C} u_g$. Let $\Phi^{(0)} = \{\pi^{(0)}, \sigma^{(0)}, \tau^{(0)}, \gamma_1^{(0)}, \ldots, \gamma_{2m}^{(0)}\} = \Phi = \{\pi, \sigma, \tau, \gamma_1, \ldots, \gamma_{2m}\}$. We may assume (by passing to an equivalent cocycle, if necessary) that $u_{\pi^{(0)}}^{2^{n+1}} = u_{\sigma^{(0)}}^{2^n} = u_{\tau^{(0)}}^2 = 1$. Let $\alpha \in \mathbb{C}^\times$ be determined by the equation $u_{\sigma^{(0)}} u_{\pi^{(0)}} u_{\sigma^{(0)}}^{-1} = \alpha u_{\pi^{(0)}}^3$. A straight-

forward calculation shows that for any k in $\mathbb{N}$,

$$u^k_{\sigma(0)} u_{\pi(0)} u^{-k}_{\sigma(0)} = \alpha^{\frac{3^k-1}{2}} u^{3^k}_{\pi(0)}, \quad 2^k \mid \frac{3^{2^k}-1}{2} \text{ and } 2^{k+1} \nmid \frac{3^{2^k}-1}{2}. \tag{4}$$

We $\underline{\text{claim}}$ that α is a primitive 2^{n+1}-th root of unity. To see this note that $\sigma^{(0)}$ and $(\pi^{(0)})^{2^n}$ commute and therefore, $u_{\sigma(0)} u^{2^n}_{\pi(0)} u^{-1}_{\sigma(0)} = \pm u^{2^n}_{\pi(0)}$. We need to show that $u_{\sigma(0)} u^{2^n}_{\pi(0)} u^{-1}_{\sigma(0)} = -u^{2^n}_{\pi(0)}$. If not we have

$$u^{2^n}_{\pi(0)} = \left(u_{\sigma(0)} u_{\pi(0)} u^{-1}_{\sigma(0)}\right)^{2^n} = \left(\alpha u^3_{\pi(0)}\right)^{2^n} = \alpha^{2^n} u^{2^n}_{\pi(0)},$$

that is $\alpha^{2^n} = 1$. Then, by (4) we have

$$u^{2^{n-1}}_{\sigma(0)} u_{\pi(0)} u^{-2^{n-1}}_{\sigma(0)} = \alpha^{\frac{3^{2^{n-1}}-1}{2}} u^{3^{2^{n-1}}}_{\pi(0)} = u_{\pi(0)}$$

and hence $u^{2^{n-1}}_{\sigma(0)}$ is in the center of $\mathbb{C}^c G$. This contradicts the nondegeneracy of the cocycle c and the claim is proved. It follows that if we replace $\pi^{(0)}$ by a suitable odd power of $\pi^{(0)}$ (that is $\pi^{(1)} = \pi^{(0)^l}$, l is odd) and leaving the other generators unchanged we obtain a generating set $\Phi^{(1)} = \{\pi^{(1)}, \sigma^{(1)}, \tau^{(1)} \gamma^{(1)}_1, \ldots, \gamma^{(1)}_{2m}\}$ of G, equivalent to Φ, such that $u_{\sigma(1)} u_{\pi(1)} u^{-1}_{\sigma(1)} = \varepsilon u^3_{\pi(1)}$.

Next, since $\sigma^{(1)}$ and $\tau^{(1)}$ commute and $\tau^{(1)}$ is of order 2 we have that $(u_{\sigma(1)}, u_{\tau(1)}) = \pm 1$. If the commutator is -1, we put $\tau^{(2)} = \tau^{(1)} \pi^{(1)^{2^n}}$ and $\sigma^{(2)} = \sigma^{(1)}$. Otherwise we leave both unchanged and, in either case, get an equivalent generating set $\Phi^{(2)}$ with $(u_{\sigma(2)}, u_{\tau(2)}) = 1$.

Let $\beta \in \mathbb{C}^\times$ be determined by the equation $u_{\tau(2)} u_{\pi(2)} u^{-1}_{\tau(2)} = \beta u^{-1}_{\pi(2)}$. Then

$$u_{\sigma(2)} u_{\tau(2)} u_{\pi(2)} u^{-1}_{\tau(2)} u^{-1}_{\sigma(2)} = \beta \varepsilon^{-1} u^{-3}_{\pi(2)}, \text{ and } u_{\tau(2)} u_{\sigma(2)} u_{\pi(2)} u^{-1}_{\sigma(2)} u^{-1}_{\tau(2)} = \varepsilon \beta^3 u^{-3}_{\pi(2)}$$

and by equality of the left hand sides we have that $\beta = \pm \varepsilon^{-1}$. Now put $\tau^{(3)} = \tau^{(2)} (\sigma^{(2)})^{2^{n-1}}$ if $\beta = -\varepsilon^{-1}$ and $\tau^{(3)} = \tau^{(2)}$ otherwise. Leaving the other generators unchanged we obtain an equivalent set of generators $\Phi^{(3)}$ such that the elements $u_{\tau(3)}, u_{\pi(3)} \in \mathbb{C}^c G$ satisfy $u_{\tau(3)} u_{\pi(3)} u^{-1}_{\tau(3)} = -\beta u^{-1}_{\pi(3)} = \varepsilon^{-1} u^{-1}_{\pi(3)}$. Thus we have shown there exist $\{u_g\}_{g \in \{\pi^{(3)}, \sigma^{(3)}, \tau^{(3)}\}}$ in $\mathbb{C}^c G_1$ that satisfy (3).

Next, we may proceed as in Aljadeff, et al ([2, Proposition 13]) to get a generating set $\Phi^{(4)}$ of G, equivalent to Φ, such that (a) (3) is satisfied (b) $(u_{\gamma^{(4)}_{2k-1}}, u_{\gamma^{(4)}_{2k}}) = \pm 1$ for all $1 \le k \le m$ and (c) all other commutators of the $u_{\gamma^{(4)}_i}$'s are trivial.

The next step is to add the condition $u_g u_{\gamma_i} = u_{\gamma_i} u_g$ for $g \in \{\pi, \sigma, \tau\}$:

For each $\gamma^{(4)}_i \in \{\gamma^{(4)}_1, \ldots, \gamma^{(4)}_{2m}\}$, if $(u_{\gamma^{(4)}_i}, u_{\pi^{(4)}}) = -1$ then put $\gamma^{(5)}_i = \gamma^{(4)}_i (\sigma^{(4)})^{2^{n-1}}$, otherwise leave $\gamma^{(4)}_i$ unchanged. Leaving $\pi^{(4)}, \sigma^{(4)}, \tau^{(4)}$ unchanged we obtain a generating

set $\Phi^{(5)}$ such that $(u_{\gamma_i^{(5)}}, u_{\pi^{(5)}}) = 1$ for all i. Next, if $(u_{\gamma_i^{(5)}}, u_{\sigma^{(5)}}) = -1$ put $\gamma_i^{(6)} = \gamma_i^{(5)}(\pi^{(5)})^{2^n}$, otherwise leave $\gamma_i^{(5)}$ unchanged. We obtain a generating set $\Phi^{(6)}$ such that $(u_{\gamma_i^{(6)}}, u_{\sigma^{(6)}}) = 1$ for all i.

At this point we have for each $1 \le k \le m$, $(u_{\gamma_{2k-1}^{(6)}}, u_{\gamma_{2k}^{(6)}}) = \pm 1$. In fact we <u>claim</u> that $(u_{\gamma_{2k-1}^{(6)}}, u_{\gamma_{2k}^{(6)}}) = -1$. If not, by the preceding steps we have that the element $u_{\gamma_{2k-1}^{(6)}}$ centralizes all the $u_{\gamma_i^{(6)}}$'s as well as $u_{\pi^{(6)}}$ and $u_{\sigma^{(6)}}$. It follows that $u_{\gamma_{2k-1}^{(6)}}$ does not centralize $u_{\tau^{(6)}}$, for otherwise we get a contradiction to the nondegeneracy of the two-cocycle c. A similar argument shows that $u_{\gamma_{2k}^{(6)}}$ does not centralize $u_{\tau^{(6)}}$, but then the product $u_{\gamma_{2k-1}^{(6)}} u_{\gamma_{2k}^{(6)}}$ is central, a contradiction. This proves the claim. We refer to the pair of elements $\gamma_{2k-1}, \gamma_{2k}$ as *partners*. We proceed now as follows: If $(u_{\gamma_1^{(6)}}, u_{\tau^{(6)}}) = -1$, put $\tau^{(7)} = \tau^{(6)}\gamma_2^{(6)}$ (i.e. multiply $\tau^{(6)}$ by the partner of $\gamma_1^{(6)}$), and leave $u_{\gamma_1^{(6)}}$ and all the other generators unchanged. We continue in a similar way with $\gamma_2^{(6)}, \gamma_3^{(6)}, \ldots, \gamma_{2m}^{(6)}$.

We now have a generating set $\Phi^{(r)}$ (some r), equivalent to Φ, such that $\{u_g\}_{g \in \Phi^{(r)}}$ in $\mathbb{C}^c G$ satisfy (a) equations (3) (b) $(u_{\gamma_{2k-1}^{(r)}}, u_{\gamma_{2k}^{(r)}}) = -1$ for all $1 \le k \le m$ (c) all other commutators of the $u_{\gamma_i^{(r)}}$'s are trivial and (d) $u_{\gamma_i^{(r)}}$ centralizes the subalgebra $\mathbb{C}(u_{\pi^{(r)}}, u_{\sigma^{(r)}}, u_{\tau^{(r)}})$ for all i. It follows that the cocycle c is of standard form with respect to $\Phi^{(r)}$.

(IV) In the last step we consider the group $G = C_4 \rtimes (C_2 \times C_2) \times C_2 \times C_2 \ldots \times C_2 \times C_2$ with a set of generators $\Phi = \{\pi, \sigma, \tau, \gamma_1, \ldots, \gamma_{2m}\}$, where $\sigma\pi\sigma^{-1} = \pi^3$ and $\tau\pi = \pi\tau$. We write $G = G_1 \times G_2$ where $G_1 = C_4 \rtimes (C_2 \times C_2)$ and G_2 is elementary abelian of rank $2m$.

As in the previous case, a construction of a K–central division algebra of the form $K^\beta G_1$, analogous to that in Natapov [15], yields a 4×4 matrix algebra isomorphic to $\mathbb{C}^{c_1} G_1$ with a basis $\{u_g\}_{g \in G_1}$ such that $u_{\pi^i \sigma^j \tau^k} = u_\pi^i u_\sigma^j u_\tau^k$ and

$$u_\pi^4 = u_\sigma^2 = u_\tau^2 = 1, \ u_\sigma u_\pi = \varepsilon u_\pi^3 u_\sigma, \ u_\tau u_\pi = -u_\pi u_\tau, \ u_\tau u_\sigma = u_\sigma u_\tau, \tag{5}$$

where ε is a 4-th root of unity. Clearly, the class $[c_1]$ is determined by (5).

Now, by Karpilovsky [14, Proposition 10.6.1] there exist a two-cocycle $c_0 \in Z^2(G, \mathbb{C}^\times)$ and elements $\{u_g\}_{g \in \Phi}$ in $\mathbb{C}^{c_0} G$ that satisfy (5), (1), and $u_g u_{\gamma_i} = u_{\gamma_i} u_g$ for $g \in \{\pi, \sigma, \tau\}$. We denote these equations by E_G. Note that $[c_0]$ is determined by E_G. As usual we will say the cocycle c_0 is of standard form with respect to Φ.

Now, let $c \in Z^2(G, \mathbb{C}^\times)$ be a nondegenerate two-cocycle and write $\mathbb{C}^c G = \oplus_{g \in G} \mathbb{C} u_g$. As in previous steps put $\Phi^{(0)} = \{\pi^{(0)}, \sigma^{(0)}, \tau^{(0)}, \gamma_1^{(0)}, \ldots, \gamma_{2m}^{(0)}\} = \Phi = \{\pi, \sigma, \tau, \gamma_1, \ldots, \gamma_{2m}\}$. We may assume (by passing to an equivalent cocycle, if necessary) that $u_{\pi^{(0)}}^4 = u_{\sigma^{(0)}}^2 = u_{\tau^{(0)}}^2 = 1$.

We $\underline{\text{claim}}$ that the element $\alpha \in \mathbb{C}^\times$ determined by the equation $u_{\sigma^{(0)}} u_{\pi^{(0)}} u_{\sigma^{(0)}}^{-1} = \alpha u_{\pi^{(0)}}^3$ is a root of unity of order 4. To see this note that $(u_g, u_{\pi^{(0)}}^2) = 1$ for any element g in G of order 2 that commutes with $\pi^{(0)}$. It follows that $(u_{\sigma^{(0)}}, u_{\pi^{(0)}}^2) = -1$, for otherwise $u_{\pi^{(0)}}^2$ is contained in the center of the algebra. This proves the claim. Therefore, we may set $\pi^{(1)} = \pi^{(0)^3}$ if necessary and we obtain a generating set $\Phi^{(1)} = \{\pi^{(1)}, \sigma^{(1)}, \tau^{(1)} \gamma_1^{(1)}, \ldots, \gamma_{2m}^{(1)}\}$ of G, equivalent to Φ, such that $u_{\sigma^{(1)}} u_{\pi^{(1)}} u_{\sigma^{(1)}}^{-1} = \varepsilon u_{\pi^{(1)}}^3$. By the nondegeneracy of c there is an element $h^{(1)} \in \{\tau^{(1)}, \gamma_1^{(1)}, \ldots, \gamma_{2m}^{(1)}\}$ (these elements generate $C_G(\pi^{(1)})$, the centralizer of $\pi^{(1)}$ in G) such that $(u_{h^{(1)}}, u_{\pi^{(1)}}) = -1$. We set $\tau^{(2)} = h^{(1)}$ and $h^{(2)} = \tau^{(1)}$ and leave the other generators unchanged. Next, arguing as in the previous case we may assume that $(u_{\tau^{(2)}}, u_{\sigma^{(2)}}) = 1$. Thus we have a generating set $\Phi^{(2)}$ such that $u_{\pi^{(2)}}, u_{\tau^{(2)}}$ and $u_{\sigma^{(2)}}$ satisfy equation (5). Finally, as in the previous case we obtain a generating set $\Phi^{(r)} = \{\pi^{(r)}, \sigma^{(r)}, \tau^{(r)}, \gamma_1^{(r)}, \ldots, \gamma_{2m}^{(r)}\}$ (some r), whose elements satisfy (a) $u_{\gamma_i^{(r)}}$ centralize the subalgebra $\mathbb{C}(u_{\pi^{(r)}}, u_{\sigma^{(r)}}, u_{\tau^{(r)}})$ for all i (b) $(u_{\gamma_{2k-1}^{(r)}}, u_{\gamma_{2k}^{(r)}}) = -1$ for all $1 \leq k \leq m$, and (c) all other commutators of the $u_{\gamma_i^{(r)}}$'s are trivial. Therefore c is of standard form with respect to $\Phi^{(r)}$. This completes the proof. ∎

COROLLARY 19. *If G is a group on the list Λ and c and c' are nondegenerate cocycles on G with values in $\mathbb{C}^\times$, then the universal central simple algebras $Q(U_{G,c})$ and $Q(U_{G,c'})$ are isomorphic.*

Proof. By Theorem 18 there is an automorphism φ of G such that $\varphi(c)$ and c' are cohomologous two-cocycles. In particular it follows that $Q(U_{G,c})$ and $Q(U_{G,c'})$ have the same field of definition $\mathbb{Q}(\mu)$ and φ induces an automorphism φ_* of $\mathbb{Q}(\mu)$. Clearly, φ_* extends to a $\mathbb{Q}$ isomorphism of the free algebras $\Sigma_c(\mathbb{Q}(\mu)) \longrightarrow \Sigma_{\varphi(c)}(\mathbb{Q}(\mu))$ by putting $x_{ig} \longmapsto x_{i\varphi(g)}$. Furthermore it induces an isomorphism of the corresponding universal algebras $U_{G,c} = \Sigma_c(\mathbb{Q}(\mu))/T_c(\mathbb{Q}(\mu))$ and $U_{G,\varphi(c)}$, and therefore an isomorphism of their central localization, namely the universal central simple G–graded algebras. ∎

COROLLARY 20. *Let G be a group of central type and let c be a nondegenerate cocycle on G. The universal central simple algebra $Q(U_{G,c})$ is a division algebra if and only if G is on the list Λ.*

Proof. This follows from the the previous corollary and Proposition 17. ∎

Recall from the introduction that for a group G of central type we let $\mathrm{ind}(G)$ denote the maximum over all nondegenerate cocycles c of the indices of the simple algebras $Q(U_{G,c})$. We have seen that if G is not on the list, then the universal central simple algebra $Q(U_G)$ is not a division algebra and so the index of G is strictly less than $\mathrm{ord}(G)^{1/2}$. In fact

it is shown in Aljadeff and Natapov [4] that the groups on the list are the only groups responsible for the index of $Q(U_G)$. Here is the precise result.

THEOREM 21. *Let P be a p-group of central type and let $Q(U_{P,c})$ be the universal central simple P–graded algebra for some nondegenerate two-cocycle c on P. Then there is a sub-quotient group H of P on the list Λ, such that $Q(U_{P,c}) \cong M_{p^r}(Q(U_{H,\alpha}))$, where the cocycle α on H is obtained in a natural way from c, namely there is a subgroup $\hat{H}$ of P such that $\hat{H}/N \cong H$ and $\inf([\alpha]) = \mathrm{res}([c])$ on $\hat{H}$.*

COROLLARY 22. *Let G be a central type p-group. Consider all central type groups H that are isomorphic to sub-quotients of G and are members of Λ. Then $\mathrm{ind}(G) \leq \sup(\mathrm{ord}(H)^{1/2})$.*

DeMeyer and Janusz prove in [9, Corollary 4] that for an arbitrary group of central type G and a nondegenerate two-cocycle $c \in Z^2(G, \mathbb{C}^\times)$ any Sylow-p subgroup G_p of G is of central type, and furthermore the restriction of c to G_p is nondegenerate. Using this result Geva in his thesis proved that for any subfield k of $\mathbb{C}$ the twisted group algebra $k^c G$ is (non-graded) isomorphic to $\bigotimes_{p | \mathrm{ord}(G)} k^c G_p$. It follows that for each prime p dividing the order of G there is a p-group H_p on the list Λ which is isomorphic to a sub-quotient of G and such that $F^s G \cong \bigotimes_p M_{p^r}(F^{\alpha_p} H_p)$. Combining this with the preceding Corollary we obtain:

COROLLARY 23. *Let G be a group of central type, and for each prime p, G_p be a Sylow-p subgroup of G. Consider all central type groups H_p that are isomorphic to sub-quotients of G_p and are members of Λ_p. Then $\mathrm{ind}(G) \leq \prod_p \sup(\mathrm{ord}(H_p)^{1/2})$.*

It follows from Proposition 17 that for any group G of central type not on the list and any G–grading on $M_n(\mathbb{C})$, the universal central simple algebra is not a division algebra. That means we can find nonidentity polynomials $p(x_{ig})$ and $q(x_{ig})$ over the field of definition $\mathbb{Q}(\mu)$ such that $p(x_{ig})q(x_{ig})$ is a graded identity. In fact there will exist a nonidentity polynomial $p(x_{ig})$ over $\mathbb{Q}(\mu)$ such that $p(x_{ig})^r$ is a graded identity for some positive integer r. This is clearly equivalent to saying that under every substitution the value of p in $M_n(\mathbb{C})$ is a nilpotent matrix. We will refer to such a polynomial as a *nilpotent polynomial*.

We present an explicit example of a nilpotent polynomial. We let S_3 be the permutation group on three letters, and $C_6 = \langle z \rangle$ be a cyclic group of order 6. Let $\sigma = (123)$ and $\tau = (12)$ be generators of S_3 and define an action of S_3 on C_6 by $\tau(z) = z^{-1}$ and $\sigma(z) = z$. We consider the group $G = S_3 \ltimes C_6$. Note that G is not nilpotent and hence not on the list Λ.

Let $C_6^\vee = \operatorname{Hom}(C_6, \mathbb{C}^\times) = \langle \chi_z \rangle$ denote the dual group of C_6 and let $\langle \ , \ \rangle$ denote the usual pairing between C_6 and $C_6^\vee$ (i.e. $\langle a, \chi \rangle = \chi(a)$ for all $a \in C_6$, $\chi \in C_6^\vee$). It is well known ([14, Corollary 10.10.2]) that $H^1(S_3, C_6^\vee) \cong H^2(S_3 \ltimes C_6, \mathbb{C}^\times)$. The isomorphism may be given by

$$\pi \mapsto c \ : \ c(h_1 a_1, h_2 a_2) = \langle h_2 \cdot a_1, \pi(h_2) \rangle, \tag{6}$$

for all $h_1 a_1, h_2 a_2 \in S_3 \ltimes C_6$. Note that the restriction of c on S_3, as well as on C_6, is trivial.

Moreover, it is shown in Etingof and Gelaki [10] that there exists a bijective one-cocycle $\pi : S_3 \to C_6^\vee$ and, because of the bijectivity, the corresponding two-cocycle c is nondegenerate. In particular this proves that G is of central type (see Etingof and Gelaki [11]). One may define a bijective π as follows:

$$\begin{array}{ll} \pi(id) = \chi_1 & \pi(12) = \chi_z \\ \pi(123) = \chi_z^2 & \pi(23) = \chi_z^5 \\ \pi(132) = \chi_z^4 & \pi(13) = \chi_z^3 \end{array}$$

Let c denote the nondegenerate two-cocycle which corresponds to π. Then the twisted group algebra $\mathbb{C}^c G$ is isomorphic to $M_6(\mathbb{C})$ or, equivalently, $M_6(\mathbb{C})$ is G–graded with the class $[c]$. The group G is not on the list Λ and hence the corresponding universal algebra U_G is not a domain. Consider the subalgebra $\mathbb{C}(u_\sigma, u_y)$ where $y = z^2$. It is straightforward to verify that the cocycle c satisfies $c(y, \sigma) = \omega^2$, where $\omega = e^{\frac{2\pi i}{3}}$ is a primitive third root of unity in $\mathbb{C}$, and hence $\mathbb{C}(u_\sigma, u_y)$ is a symbol algebra. It follows that the corresponding subalgebra $F(t_\sigma u_\sigma, t_y u_y)$ of the universal central simple algebra $Q(U_{G,c}) = F^{s(c)}G$ is a symbol algebra $(a, b)_3$, and by Aljadeff, et al [3, Lemma 6] a and b are roots of unity. It follows that $F(t_\sigma u_\sigma, t_y u_y)$ is split, that is isomorphic to $M_3(F)$. Thus we know that there exists a degree three nilpotent element in $U_{G,c}$. To construct it we consider the element $u_\sigma - u_y u_\sigma$ in $\mathbb{C}^c G$. Note that this is not zero since u_σ and $u_y u_\sigma$ are linearly independent over $\mathbb{C}$. Its square

$$(u_\sigma - u_y u_\sigma)^2 = u_\sigma^2 - u_\sigma u_y u_\sigma - u_y u_\sigma^2 + (u_y u_\sigma)^2$$

is not zero as well, but using $u_y u_\sigma = \omega^2 u_\sigma u_y$ and $u_y^3 = 1$ one shows that its third power

$$(u_\sigma - u_y u_\sigma)^3 = u_\sigma^3 - u_\sigma^2 u_y u_\sigma - u_\sigma u_y u_\sigma^2 + u_\sigma^2 u_y u_\sigma u_y u_\sigma - u_y u_\sigma^3 + u_y u_\sigma^2 u_y u_\sigma + u_y u_\sigma u_y u_\sigma^2 - (u_y u_\sigma)^3$$

vanishes.

Now, as mentioned above, $res_{S_3}^G(c) = res_{C_6}^G(c) = 1$, hence we may assume that $u_g u_h = u_{gh}$ for all $g, h \in S_3$ or $g, h \in C_6$. In particular, we have $u_e = u_\tau^2 u_y^3$. Also, using the cocycle values as defined in (6), we obtain $u_y = \omega u_\tau u_y u_\tau u_y^2$.

Let $x_\tau = x_{1\tau}, x_\sigma = x_{1\sigma}, x_y = x_{1y}$ be indeterminates in $\Omega = \{x_{ig} : i \in \mathbb{N}, g \in G\}$. Let the polynomial $f(x_\tau, x_\sigma, x_y) \in \mathbb{Q}(\omega)\langle \Omega \rangle$ be given by

$$f(x_\tau, x_\sigma, x_y) = x_\sigma x_\tau^2 x_y^3 - \omega x_\tau x_y x_\tau x_y^2 x_\sigma.$$

It follows that the polynomial $f^3(x_\tau, x_\sigma, x_y)$ is a polynomial identity for $M_6(\mathbb{C})$ while f and f^2 are not.

We end this example by expressing f^3 explicitly as a $\mathbb{Q}(\omega)-$linear combination of elementary identities:

$$
\begin{aligned}
f^3(x_\tau, x_\sigma, x_y) \ = \ & (x_\sigma x_\tau^2 x_y^3)^3 - (x_\tau x_y x_\tau x_y^2 x_\sigma)^3 \\
& + \ \omega^2 x_\tau x_y x_\tau x_y^2 x_\sigma (x_\sigma x_\tau^2 x_y^3)^2 - \omega (x_\sigma x_\tau^2 x_y^3)^2 x_\tau x_y x_\tau x_y^2 \\
& + \ x_\tau x_y x_\tau x_y^2 x_\sigma (x_\sigma x_\tau^2 x_y^3)^2 - \omega x_\sigma x_\tau^2 x_y^3 x_\tau x_y x_\tau x_y^2 x_\sigma x_\sigma x_\tau^2 x_y^3 \\
& + \ \omega^2 x_\sigma x_\tau^2 x_y^3 (x_\tau x_y x_\tau x_y^2 x_\sigma)^2 - (x_\tau x_y x_\tau x_y^2 x_\sigma)^2 x_\sigma x_\tau^2 x_y^3 \\
& + \ \omega^2 (x_\tau x_y x_\tau x_y^2 x_\sigma) x_\sigma x_\tau^2 x_y^3 (x_\tau x_y x_\tau x_y^2 x_\sigma) - \omega (x_\tau x_y x_\tau x_y^2 x_\sigma)^2 x_\sigma x_\tau^2 x_y^3.
\end{aligned}
$$

If G is on the list Λ and c is a nondegenerate two-cocycle on G we have seen that the algebra $Q(U_{G,c})$ is a division algebra of degree $n = \mathrm{ord}(G)^{1/2}$. In the next result we calculate the index of $Q(U_{G,c}) \otimes_{\mathbb{Q}(\mu)} \mathbb{C}$, where $\mathbb{Q}(\mu)$ is the field of definition for the graded identities.

PROPOSITION 24. *If G is on the list Λ and c is a nondegenerate two-cocycle on G then the index of $Q(U_{G,c}) \otimes_{\mathbb{Q}(\mu)} \mathbb{C}$ is $n/\mathrm{ord}(G')$. In particular $Q(U_{G,c}) \otimes_{\mathbb{Q}(\mu)} \mathbb{C}$ is a division algebra if and only if G is abelian of symmetric type.*

Proof. By Aljadeff and Haile [1] (see the discussion at the beginning of section two) the subalgebra $F^s G'$ of $Q(U_{G,c}) = F^s G$ is a cyclotomic extension of F. It follows that the index of $Q(U_{G,c}) \otimes_{\mathbb{Q}(\mu)} \mathbb{C}$ is at most $n/\mathrm{ord}(G')$. We proceed to prove the opposite inequality. By Natapov [15, proof of Theorem 6] there is a subfield K of $\mathbb{C}$ containing $\mathbb{Q}(\mu)$ and a nondegenerate cocycle β on G with values in K such that the algebra $K^\beta G \otimes_{\mathbb{Q}(\mu)} \mathbb{C}$ has index exactly $n/\mathrm{ord}(G')$. By a specialization argument it follows that the algebra $Q(U_{G,\beta})$ has index at least $n/\mathrm{ord}(G')$. By Corollary 19 $Q(U_{G,\beta})$ is isomorphic to $Q(U_{G,c})$ and hence the index of $Q(U_{G,c})$ is at least $n/\mathrm{ord}(G')$. $\blacksquare$

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